Convex Invertible Cones, Nevalinna-Pick Interpolation and the Set of Lyapunov Solutions

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Abstract: For a real matrix A whose spectrum avoids the imaginary axis, it is shown that the three following, seemingly independent problems, are in fact equivalent.

- Characterizing the set of real symmetric solutions of the algebraic Lyapunov inclusion associated with A.
- The image of all Nevalinna-Pick interpolations associated with the spectrum of A.
- The structure of the Convex Invertible Cone generated by A.

The analogous result for the case where the matrix A is *complex* and the characterization is of the set of *Hermitian* solutions to the algebraic Lyapunov inclusion, is addressed as well.

The inertia of matrix A whose spectrum avoids the imaginary axis will be called *regular*. For such a matrix we explore three seemingly independent problems. Due to space limitations, we mostly consider the case where A is real.

(a) All Real Symmetric Lyapunov Solutions.

Denoting by $\mathbf{P}$ $(\overline{\mathbf{P}})$ the set of Hermitian positive (semi)-definite matrices, let us define the set $\mathbf{S}(A)$ of all real symmetric solutions to a Lyapunov inclusion, associated with A,

$$\mathbf{S}(A) := \{S \ : \ (SA + A^T S) \in \mathbf{P}\}.$$

In this context, there are (at least) two classical questions associated with a given pair of matrices A, B. First, under what conditions $\mathbf{S}(A) \bigcap \mathbf{S}(B)$ is not empty and second, under what conditions $\mathbf{S}(A) \subseteq \mathbf{S}(B)$. The first problem was explored in [2] and references therein. In practice, one uses the LMI approach to find a matrix S which belongs to the intersection $\mathbf{S}(A) \subseteq \mathbf{S}(B)$. Here we focus our attention on the second problem.

Relatively little is known on the structure of the set $\mathbf{S}(A)$. Clearly,

$$\mathbf{S}(\alpha A) = \mathbf{S}(A) = \mathbf{S}(A^{-1}), \tag{1}$$

for all $\alpha > 0$. In fact, $\mathbf{S}(A)$ is an open convex cone of non-singular real symmetric matrices, for more details see [2, sections III, IV]. Next, recall that the case of equality $\mathbf{S}(A) = \mathbf{S}(B)$ was characterized by R. Loewy.

Theorem 1 :[5]. *Let A and B be a pair of matrices with regular inertia. Then $\mathbf{S}(A) = \mathbf{S}(B)$, if and only if for some $\alpha > 0$ either $B = \alpha A$ or $B = \alpha A^{-1}$.*

In the sequel we shall use $\mathbf{H}(A)$, the complex analogous of the set $\mathbf{S}(A)$. Namely, for A which is not necessarily real, but with regular inertia, we denote by $\mathbf{H}(A)$ the set of all Hermitian solutions to the Lyapunov inclusion,

$$\mathbf{H}(A) := \{H \ : \ (HA + A^* H) \in \mathbf{P}\}.$$

We shall also find it convenient to define a set $\tilde{\mathbf{H}}(A)$ which strictly contains $\mathbf{H}(A)$,

$$\tilde{\mathbf{H}}(A) := \{H \ : \ HA + A^* H := Q \in \overline{\mathbf{P}}, \quad A^*, Q \text{ controllable}\}.$$

Clearly, all matrices in $\tilde{\mathbf{H}}(A)$ are non-singular and share the same inertia.

(b) Nevanlinna-Pick Interpolation.

First, we denote by $\mathcal{RPR}$, Rational Positive Real, the set of all scalar real rational functions that are analytic in the open right half of the complex plane and map the open right half onto its closure. In the context of electrical circuits, the driving point immittance of a lumped R-L-C one port, is an $\mathcal{RPR}$ function. Moreover, every $\mathcal{RPR}$ function may be realized as the driving point immittance of a lumped R-L-C circuit, see [3].

Let $\mathcal{RPRO}$, be the subset of all odd functions within $\mathcal{RPR}$, namely $f(s) \in \mathcal{RPR}$ and in addition $f(-s) = -f(s)$. In the context of electrical circuits they correspond to the driving point immittance of a lumped L-C circuits.

Consider now the Nevanlinna-Pick interpolation Problem (NPP), where one is provided with a pair of sequences of complex numbers $\{\lambda_1 , \ldots , \lambda_n\}$, $\{\mu_1 , \ldots , \mu_n\}$; and seeks a function $f \in \mathcal{RPRO}$ so that $\mu_j = f(\lambda_j)$ for $j = 1, \ldots , n$. We also impose the two following restrictions on the data set. First,

$$\lambda_j^* + \lambda_k \neq 0 , \qquad n \geq j \geq k \geq 1. \tag{2}$$

Recall that if $A = V^{-1} diag\{\lambda_1 , \ldots , \lambda_n\} V$ for some non-singular V, (2) amounts to the non-singularity of the Kronecker sum of A and A^*, see e.g. [4, Theorem 4.4.5]. This in particular implies that inertia of A is regular.

In addition to (2), we assume that λ_j are all distinct. A set satisfying these conditions will be called *reduced*. Based on this data we shall construct the Pick matrix Π defined by,

$$[\Pi]_{jk} = \frac{\mu_j^* + \mu_k}{\lambda_j^* + \lambda_k} , \qquad n \geq j, k \geq 1. \tag{3}$$

We thus have the following, which is based on a result of Youla and Saito, [7]

Theorem 2 : [3]. *Let* $\{\lambda_1 , \ldots , \lambda_n\}$ *and* $\{\mu_1 , \ldots , \mu_n\}$ *be two given sequences of complex numbers, with* λ_j *being reduced. Construct the matrices*

$$A := diag\{\lambda_1 , \ldots , \lambda_n\}, \qquad B := diag\{\mu_1 , \ldots , \mu_n\}.$$

Then the following are equivalent

(i) *There exists* $f \in \mathcal{RPR}$ *so that* $B = f(A)$.

(ii) *There exists* $f \in \mathcal{RPRO}$ *so that* $B = f(A)$.

(iii) *The Pick matrix* Π, (3) *is positive semi-definite.*

Moreover, the minimal degree of the interpolating function f *is equal to the rank of the Pick matrix* Π.

We wish to point out that if a pair of scalars $\lambda,\ \mu$ satisfies $\mu = f(\lambda)$ for some $f \in \mathcal{RPRO}$, then also $-\mu = f(-\lambda)$, $\mu^* = f(\lambda^*)$ and $-\mu^* = f(-\lambda^*)$.

(c) The Structure of A Singly Generated CIC

For a square A matrix with regular inertia we define $\mathcal{C}(A)$, the Convex Invertible Cone (**cic**) generated by it through the following iterative process, where $j = 0,\ 1,\ \ldots$. Let $\mathbf{X}_o := A$ and $\mathbf{X}_{j+1}$ is obtained from $\mathbf{X}_j$ by taking all positive combinations of members in $\mathbf{X}_j$ and their inverses. Then $\mathcal{C}(A)$ is the union of the increasing sequences $\{\mathbf{X}_j\}$. The structure of a **cic** was first explored in [1]. In particular we know that all matrices in $\mathcal{C}(A)$ are non-singular, if and only if the inertia of A is regular.

Note that the introduction of $\mathcal{C}(A)$ is well motivated by Theorem 1 and the relations in (1). We now characterize a **cic** generated by a single matrix with regular inertia.

Theorem 3 :[3]. (i) *Every* $f \in \mathcal{RPRO}$ *admits the following Foster representation,*

$$f(s) = a_o s + \frac{b_o}{s} + \sum_j \left(a_j s + \frac{b_j}{s} \right)^{-1} , \qquad a_o, b_o \geq 0 \ , \quad a_j, b_j > 0,$$

for $j = 1,\ 2,\ \ldots$ *and if* $j = 0$ *then* $a_o + b_o > 0$.

(ii) *Let* A *be a matrix with regular inertia. Then*

$$\mathcal{C}(A) = \{ f(A) \ : \ f \in \mathcal{RPRO} \ \}.$$

The first part classical and is due to R. M. Foster (1924).

Main Result : *Let* A, B *be a pair of matrices with regular inertia. Then the following are equivalent.*

(i) $B \in \mathcal{C}(A)$.

(ii) $\mathcal{C}(B) \subseteq \mathcal{C}(A)$.

(iii) *There exists* $f \in \mathcal{RPRO}$ *so that* $B = f(A)$.

If A and B are real, the above statements imply the following.

(iv) $\mathbf{S}(A) \subseteq \mathbf{S}(B)$.

If in addition A and B are co-diagonalizable, namely $A = V^{-1} diag\{\lambda_1 \ , \ \ldots \ , \ \lambda_n\} V$, $B = V^{-1} diag\{\mu_1 \ , \ \ldots \ , \ \mu_n\} V$ for some non-singular matrix V, then the converse implication is true as well.

Furthermore, if the scalars λ_j satisfy condition (2), *the four above statements are equivalent to the following.*

(v) *The Pick matrix Π (3) associated with A and B is positive semi-definite. Moreover the minimal degree of the interpolating function f in* (iii) *is equal to the rank of* Π.

(vi) *The Cauchy matrix H defined through the relation $[H]_{jk} := (\lambda_j^* + \lambda_k)^{-1}$, $n \geq j \geq k \geq 1$, belongs to $\tilde{\mathbf{H}}(A) \bigcap \tilde{\mathbf{H}}(B)$.*

We now show that without assuming co-digonalizability of A and B, (iv) need not imply (i).

Example 4 : Consider the matrices $A := \begin{pmatrix} 7 & 24 \\ -24 & 7 \end{pmatrix}$ and $B := \begin{pmatrix} 1 & 0 \\ 0 & 49 \end{pmatrix}$. In [1, Example 3.10] it was shown that $\mathbf{S}(B)$ strictly contains $\mathbf{S}(A)$. However, since all matrices in $\mathcal{C}(A)$ are of the form $\mathbf{R}_+ \begin{pmatrix} 7 & \alpha \\ -\alpha & 7 \end{pmatrix}$ where $\alpha \in [-24, 24]$, $B \notin \mathcal{C}(A)$. □

Real vs. Complex

Although results are quite similar, we briefly point out at differences between the real and the complex cases.

- First recall that in contrast to the real case described in (1), for a complex matrix A with regular inertia we have that

$$\mathbf{H}(\alpha A + irI) = \mathbf{H}(A) = \mathbf{H}(A^{-1}),$$

for all scalars $\alpha > 0$ and $r \in \mathbf{R}$. Thus the **cic** structure, implied by (1) is no longer suitable here. This observation is further supported by the following result analogous to Theorem 1.

Theorem 5 :[6]. *Let A and B be a pair of matrices with regular inertia. Then $\mathbf{H}(A) = \mathbf{H}(B)$, if and only if, $B = (aI + ibA)(icI + dA)^{-1}$ where the real scalars a, b, c, d satisfy $ad + bc > 0$.*

• Given matrices A, B, where A is diagonalizable. If $\mathbf{H}(A) \subseteq \mathbf{H}(B)$, then A and B are necessarily co-diagonalizable. In contrast, $\mathbf{S}(A) \subseteq \mathbf{S}(B)$ with A diagonalizable, does not imply co-diagonalizability of A and B, see Theorem 4 after (iv) and Example 5.

• In NPP the interpolating function f is still analytic outside the imaginary axis and maps each open half plane onto itself, but it is no longer confined to be real. Thus if $\mu = f(\lambda)$, the image under the same f of $-\lambda$, λ^* and $-\lambda^*$ need not be directly related to μ.

• **Example 6 :** Note that for A and B in Example 4, although $\mathbf{S}(B)$ strictly contains $\mathbf{S}(A)$, $\mathbf{H}(A)$ neither contains nor is contained in $\mathbf{H}(B)$. Indeed, take $H_1 := \begin{pmatrix} 1 & ir \\ -ir & 1 \end{pmatrix}$, where $r \in \mathbf{R}$ is a parameter. Then $H_1 \in \mathbf{H}(B)$ whenever $|r| < 0.28$, but $H_1 \in \mathbf{H}(A)$ for all $|r| < 1$. Take also $H_2 := B^{-1}$ then trivially $H_2 \in \mathbf{S}(B)$, but $H_2A + A^*H_2 = 2\begin{pmatrix} 7^3 & -24^2 \\ -24^2 & 7 \end{pmatrix}$ thus $H_2 \notin \mathbf{S}(A)$.

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