A filtered no arbitrage model for the term structures from noisy data

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Abstract

We consider the problem of pricing in financial markets when agents do not have access to full information. The particular problem concerns the pricing of non traded or illiquid bonds on the basis of the observations of the yields of traded zero-coupon bonds. The approach being used gives an example of how stochastic filtering techniques, in particular the Kalman filter, can be usefully applied to pricing under incomplete information.

1 Introduction

The context of our study are multifactor affine term structure models under the condition of absence of arbitrage. For a description of affine term structure models in general and absence of arbitrage we refer to [1] (see also [2] and [3], [4]).

We assume that the actually observed term structure does not correspond exactly to a theoretical arbitrage-free factor model. We thus let the observed bond prices or, equivalently, their yields correspond to a perturbed multifactor term structure. We assume, furthermore, that the factors cannot be reconstructed exactly from the observations and so they have to be filtered. The purpose is to derive a consistent, arbitrage-free pricing system to price illiquid and non traded bonds on the basis of the incomplete information coming from the observations of the traded bonds where the prices/yields of the latter correspond to the perturbed term structure model.

We show that this consistent pricing system can be defined via projections onto the subfiltration generated by the observations and this leads to a filtering problem that can be approached via Kalman filtering.

The present paper describes in a synthetic way the approach and the results of the full paper [6].

2 The perturbed factor model

2.1 Preliminaries

We start from a theoretical abstract factor model, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, Q)$, where the factors x_t form an $n-$variate Gaussian process satisfying

$$dx_t = A(t)x_t dt + B(t)dw_t \tag{2.1}$$

with (w_t) a Wiener process of given dimension m and x_0 a zero-mean Gaussian random variable. The factors are supposed to drive the term structure in the sense that the instantaneous (continuously compounded) forward rates $f(t,T)$ satisfy the affine model

$$f(t,T) = C(t,T)x_t + G(t,T)\,. \tag{2.2}$$

To prevent the possibility of arbitrage, the functions $C(t,T)$ and $G(t,T)$ cannot be specified arbitrarily but have to be such that there exists at least one equivalent martingale measure and this in turn excludes the possibility of arbitrage. By imposing, as is usually done, that the given measure Q is already a martingale measure one obtains the so-called Heath-Jarrow-Morton (HJM) condition (see [7]) implying that

$$C(t,T) = C(T)\exp\left\{\int_t^T A(u)du\right\}\,, \tag{2.3}$$

where $C(T)$ can be chosen arbitrarily and is supposed to be bounded on bounded intervals, while

$$G(t,T) = -C(0,T)x_0 + f^*(0,T) + \frac{1}{2}\int_0^t \beta_T(s,T)ds\,, \tag{2.4}$$

where $\beta(t,T) := \left|\left|\int_t^T C(t,u)B(t)du\right|\right|^2$ and $f^*(0,T)$ are the (observed) initial forward rates.

Purpose : *Derive a consistent pricing system to price illiquid and non traded bonds on the basis of the incomplete information coming from the observations of the prices/yields of a finite number N of traded bonds.*

Basic assumption : *Each of the N observations is accompanied by an additional uncertainty and the additional uncertainty sources together form a further factor ξ_t of dimension N.*

This is a realistic assumption and is satisfied e.g. in the case when a low-dimensional, parsimonious factor model describes well certain long-term, time-series features of the term structure but fails to achieve sufficient accuracy in fitting the current prices.

Notice that the basic assumption implies that the global factors (the original factors x and the further factors ξ) cannot be reconstructed exactly from the observations and have therefore to be filtered.

2.2 The perturbed model

We assume that the additional multivariate factors ξ_t form a Gaussian process as well and so we consider for the perturbed system a model of the form

$$\begin{cases} dx_t = A(t)x_t dt + B(t)dw_t \\ d\xi_t = A_\xi(t)\xi_t dt + B_\xi(t)dv_t \\ \tilde{f}(t,T) = C(t,T)x_t + C_\xi(t,T)\xi_t + \tilde{G}(t,T)\ ,\ t \leq T\ , \end{cases} \tag{2.5}$$

where (v_t) is an $N-$dimensional Wiener independent of (w_t), x_0 and ξ_0 are zero-mean Gaussian, and $\tilde{f}(t,T)$ denotes the perturbed instantaneous forward rate. Putting $\tilde{x}_t = [x_t, \xi_t]'$, the system (2.5) may be rewritten in compact form as

$$\begin{cases} d\tilde{x}_t = \tilde{A}(t)\tilde{x}_t dt + \tilde{B}(t)d\tilde{w}_t \\ \tilde{f}(t,T) = \tilde{C}(t,T)\tilde{x}_t + \tilde{G}(t,T)\,. \end{cases} \tag{2.6}$$

We shall assume that also this perturbed term structure model does not allow for the possibility of arbitrage which, analogously to (2.3) and (2.4), leads to the requirement that

$$\begin{aligned} \tilde{C}(t,T) &= \tilde{C}(T)\exp\left[\int_t^T \tilde{A}(u)du\right] \\ \tilde{G}(t,T) &= -\tilde{C}(0,T)\,\tilde{x}_0 + \tilde{f}^*(0,T) + \tfrac{1}{2}\int_0^t \tilde{\beta}_T(s,T)ds\,. \end{aligned} \tag{2.7}$$

with $\tilde{\beta}(t,T) = \beta(t,T) + \left\| \int_t^T C_\xi(t,u)B_\xi(t)du \right\|^2$.

In what follows we shall denote by $\tilde{p}(t,T)$ the corresponding perturbed zero-coupon bond prices, i.e.

$$\tilde{p}(t,T) = \exp\left[-\int_t^T \tilde{f}(t,u)du\right] \tag{2.8}$$

and by $\tilde{M}_t$ the corresponding money market account, i.e.

$$\tilde{M}_t = \exp\left[\int_0^t \tilde{r}_s ds\right] \quad \text{with} \quad \tilde{r}_t = \tilde{f}(t,t)\,. \tag{2.9}$$

Having imposed the conditions (2.7), it results that Q is a martingale measure also for the perturbed model, i.e. the discounted bond prices $\tilde{M}_t^{-1}\tilde{p}(t,T)$ are $(Q,\mathcal{F}_t)-$martingales. According to standard usage, $\tilde{M}_t$ is called the *numeraire* and Q the corresponding martingale measure.

3 The projected price system

3.1 Preliminaries

Given the triple $(\tilde{M},Q,\mathcal{F})$, i.e the numeraire, the corresponding martingale measure Q, and the filtration $\mathcal{F}_t$, the corresponding arbitrage-free price system for an (integrable) claim $X \in \mathcal{F}_T$ is

$$\Pi_{t,T}(X;\tilde{M},Q,\mathcal{F}) := \tilde{M}_t\, E^Q\left\{\frac{X}{\tilde{M}_T} \mid \mathcal{F}_t\right\}. \tag{3.10}$$

If our information corresponds to a subfiltration $\hat{\mathcal{F}} \subset \mathcal{F}$, then one would naturally consider as pricing system the *projected system*

$$\Pi_{t,T}(X;\tilde{M},Q,\hat{\mathcal{F}}) := \tilde{M}_t\, E^Q\left\{\frac{X}{\tilde{M}_T} \mid \hat{\mathcal{F}}_t\right\}. \tag{3.11}$$

Having assumed that one can observe N traded zero-coupon bonds for maturities $T_1,\cdots,T_N$, in what follows we shall assume that the subfiltration $\hat{\mathcal{F}}$ is generated by the observed prices, i.e. $\hat{\mathcal{F}}_t = \sigma\{\tilde{p}(u,T_i)\,;\, u \le t\,,\ i=1,\cdots,N\}$.

Question : is (3.11) a good definition ?

If one assumes that $\tilde{M}_t$ is observed ($\tilde{M}$ is $\hat{\mathcal{F}}-$adapted), then in [5] it is shown that (3.11) is indeed justified and leads to

$$\hat{p}(t,T) := \Pi_{t,T}(X;\tilde{M},Q,\hat{\mathcal{F}}) = E^Q\left\{\tilde{p}(t,T)\,|\,\hat{\mathcal{F}}_t\right\} \tag{3.12}$$

thus motivating better the expression *projected price system*.

Since it is not very realistic to assume $\tilde{M} \in \hat{\mathcal{F}}$, in [6] this assumption is dropped and observable numeraires are considered instead. This is discussed in the next subsection.

3.2 Invariance with respect to the numeraire

The following proposition is shown in [6]

Proposition 3.1. *Let M^1, M^2, with corresponding martingale measures Q^1, Q^2, be numeraires such that $(M^1,Q^1,\mathcal{F})$ and $(M^2,Q^2,\mathcal{F})$ define the same price system. If $\hat{\mathcal{F}} \subset \mathcal{F}$ and M^1, M^2 are $\hat{\mathcal{F}}-$adapted, then*

$$\Pi_{t,T}(X;M^1,Q^1,\hat{\mathcal{F}}) = \Pi_{t,T}(X;M^2,Q^2,\hat{\mathcal{F}})\,. \tag{3.13}$$

As instance of observable numeraires one may take

$$M_t^i = \frac{\tilde{p}(t,T_i)}{\tilde{p}(0,T_i)} \quad i = 1,\cdots,N\,, \tag{3.14}$$

where $\tilde{p}(t,T_i)$ are the N observable/traded zero-coupon bond prices. The corresponding measures Q^i are then defined on $\mathcal{F}_T$ with $T \leq T_i$. As consequence of proposition 3.1 one has that $\Pi_{t,T}(1; M^i, Q^i, \hat{\mathcal{F}})$ and $\Pi_{t,T}(1; M^j, Q^j, \hat{\mathcal{F}})$ are equal for $t \leq T \leq \min(T_i, T_j)$. The projected price system (3.11) is thus a good definition in the sense that the prices according to (3.11) are independent of which of the traded zero-coupon bonds is chosen as numeraire. Since larger maturity bonds allow for a larger domain of definition, it is convenient to choose as numeraire the bond with largest maturity T_N and so the projected zero-coupon bond prices take the form

$$\hat{p}(t,T) = \Pi_{t,T}(1; M^N, Q^N, \hat{\mathcal{F}})\,, \quad t \leq T \leq T_N\,. \tag{3.15}$$

It appears convenient to extend this definition beyond T_N and this is the purpose of the next subsection.

3.3 The projected price system

The following result is shown in [6].

Proposition 3.2. *Given $\tilde{M}_t$ according to (2.9), let*

$$M_t^0 := \frac{1}{E^Q\left\{\frac{1}{\tilde{M}_t} |\, \hat{\mathcal{F}}_t\right\}}\,; \tag{3.16}$$

then for $i = 1,\cdots,N$ and $T \leq T_i$,

$$\Pi_{t,T}(X; M^i, Q^i, \hat{\mathcal{F}}) = \Pi_{t,T}(X; M^0, Q, \hat{\mathcal{F}}) \tag{3.17}$$

for all bounded $\hat{\mathcal{F}}_T-$measurable X.

Proposition 3.2 implies that Q is a martingale measure also for M^0 as numeraire and that, for $\hat{\mathcal{F}}_T-$claims, the triple $(M^0, Q, \hat{\mathcal{F}})$ is yet another way to represent the price system defined by either of the triples $(M^i, Q^i, \hat{\mathcal{F}})$.

Since $E^Q\left\{\frac{1}{\tilde{M}_T} |\, \hat{\mathcal{F}}_t\right\} = \tilde{p}(t,T)/\tilde{M}_t$, from (3.15) and (3.17) it follows that

$$\hat{p}(t,T) = \frac{E^Q\left\{\tilde{p}(t,T)/\tilde{M}_t |\, \hat{\mathcal{F}}_t\right\}}{E^Q\left\{1/\tilde{M}_t |\, \hat{\mathcal{F}}_t\right\}} \tag{3.18}$$

and we shall now take (3.18) as our **definition** of the **projected price system**. The immediate advantage of (3.18) with respect to (3.15) is that it allows to extend the definition also beyond T_N, more precisely, for $0 \leq t \leq T < \infty$. Formula (3.18) has however also the further advantage that it allows stochastic filtering to come into play to compute $\hat{p}(t,T)$ and this is the subject of the next section.

4 Computation of the projected prices by Kalman filtering

4.1 Kalman filter model

The subfiltration $\hat{\mathcal{F}}_t$, that was assumed to be generated by the N observed zero-coupon bond prices $\tilde{p}(t, T_i)$, can equivalently be assumed to be generated also by the cumulative yields $\tilde{y}(t, T_i)$ defined by

$$\tilde{y}(t,T) := -\log \tilde{p}(t,T) = \int_t^T \tilde{f}(t,u)du\,. \tag{4.19}$$

From (4.19) it follows on one hand (see (2.6)) that

$$\tilde{y}(t,T) = \left(\int_t^T \tilde{C}(t,u)du\right)\tilde{x}_t + \int_t^T \tilde{G}(t,u)du\,, \tag{4.20}$$

on the other hand that

$$\begin{array}{rcl} d\tilde{y}(t,T) & = & -\tilde{f}(t,t)dt + \int_t^T d\tilde{f}(t,s)ds \\ & = & -\tilde{C}(t)\tilde{x}_t dt - \tilde{G}(t)dt + \left(\int_t^T \tilde{C}(t,u)du\tilde{B}(t)\right)d\tilde{w}_t + \left(\int_t^T \tilde{G}_t(t,u)du\right)dt \end{array} \tag{4.21}$$

having put $\tilde{C}(t) := \tilde{C}(t,t)$, $\tilde{G}(t) := \tilde{G}(t,t)$.

Observing $\tilde{y}(t, T_i)$ $(i = 1, \cdots, N)$ is in turn equivalent to observing the $N-$vector

$$\tilde{z}_t := \left[\tilde{y}(t,T_i) - \int_t^{T_i} \tilde{G}(t,u)du\right]_{i=1,\cdots,N}\,. \tag{4.22}$$

Defining also the $N-$column vectors

$$\begin{array}{rcl} C_t^e & := & \left[\tilde{C}(t), \cdots, \tilde{C}(t)\right]' \\ V_t & := & \left[\int_t^{T_i} \tilde{C}(t,u)du\,\tilde{B}(t)\right]_{i=1,\cdots,N} \end{array} \tag{4.23}$$

one obtains

$$d\tilde{z}_t = -C_t^e \tilde{x}_t dt + V_t d\tilde{w}_t\,. \tag{4.24}$$

The pair $(\tilde{x}_t, \tilde{z}_t)$ is now a partially observable system with $\tilde{x}_t$ the unobservable state component and $\tilde{z}_t$ the observations, that satisfies the linear Gaussian system given by the first relation in (2.6) and by (4.24), and to which one can thus apply the Kalman filter to obtain the conditional mean and covariance, i.e.

$$\left\{ \begin{array}{rcl} \hat{x}_t & = & E^Q\left\{\tilde{x}_t |\, \hat{\mathcal{F}}_t\right\} \\ P_t & = & E^Q\left\{(\tilde{x}_t - \hat{x}_t)(\tilde{x}_t - \hat{x}_t)' |\, \hat{\mathcal{F}}_t\right\}\,. \end{array} \right. \tag{4.25}$$

4.2 Computation of the projected prices

To compute the projected prices in (3.18), the following lemma is proved in [6].

Lemma 4.1. *The following representation holds*

$$\begin{aligned}\hat{p}(t,T) &= \frac{E^Q[\tilde{p}(t,T)/\tilde{M}_t|\hat{\mathcal{F}}_t]}{[1/\tilde{M}_t|\hat{\mathcal{F}}_t]} = \\ &= \exp\Big[E^Q\left\{-\tilde{y}(t,T)\,|\,\hat{\mathcal{F}}_t\right\} + \tfrac{1}{2}var\left\{E^Q\left\{\tilde{y}(t,T)\,|\,\hat{\mathcal{F}}_t\right\}\right\} \\ &\quad +cov\left\{E^Q\left\{\tilde{y}(t,T)\,|\,\hat{\mathcal{F}}_t\right\}, E^Q\left\{\int_0^t \tilde{f}(u,u)du\,|\,\hat{\mathcal{F}}_t\right\}\right\}\Big]\,.\end{aligned} \tag{4.26}$$

Using this lemma and the filter result of the previous subsection 4.1 together with (4.20), one finally obtains, after some further calculations (see always [6]), the following computable expression

$$\hat{p}(t,T) \qquad = \exp\left\{-\left(\int_t^T \tilde{C}(t,u)du\right)\hat{x}_t - \int_t^T \tilde{G}(t,u)du\right\} \tag{4.27}$$

$$\cdot\exp\left\{\frac{1}{2}\left[\int_t^T \tilde{C}(t,u)du\right]\hat{P}_t\left[\int_t^T \tilde{C}'(t,u)du\right]\right\} \tag{4.28}$$

$$\cdot\exp\left\{\left\{\int_0^t \tilde{C}(u,u)\hat{P}_u e^{\int_u^t \tilde{A}'(\tau)d\tau}du\right\}\int_t^T \tilde{C}'(t,u)du\right\}\,, \tag{4.29}$$

where

$$\hat{P}_t := \tilde{P}_t - P_t \tag{4.30}$$

with P_t according to (4.25) and $\tilde{P}_t$ satisfying

$$d\tilde{P}_t/dt = \tilde{A}(t)\tilde{P}_t + \tilde{P}_t\tilde{A}'(t) + \tilde{B}(t)\tilde{B}'(t)\,. \tag{4.31}$$

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