Switched systems that are periodically stable may be unstable

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Abstract

We prove the existence of two 2×2 real matrices such that all periodic products of these matrices converge to zero but there exists an infinite product that does not. We outline implications of this result for the stability of switched linear systems, and for the finiteness conjecture.

Remark: This is a conference version of a paper submitted for publication. Proofs of the results can be found in the journal version [3].

1 Introduction

In this contribution, we prove the existence of switched linear systems that are periodically stable but are not absolutely stable. The *switched linear system* associated to the finite set of real matrices $\{A_p : p \in P\}$ is given by

$$x_{t+1} = A_{\sigma(t)} x_t.$$

Starting from the initial state x_0, the trajectory associated to the *switching function* $\sigma : \mathbb{N} \to P$ is given by

$$x_{t+1} = A_{\sigma(t)} \cdots A_{\sigma(0)} x_0$$

A switched linear system is *absolutely stable* if trajectories associated to arbitrary switching functions converge to the origin, and it is *periodically stable* if trajectories associated to

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periodic switching functions converge to the origin. The problems of determining if a given switched system is absolutely or periodically stable are both computationally intractable (NP-hard; see [18]). It is yet unknown whether these problems are decidable, see [4] for a discussion of this issue. The related problem of determining if all trajectories of a switched linear system are *bounded* is known to be undecidable [5]. For a discussion of various other issues related to switched linear systems[3]; see [1], [14], [15].

Absolute stability clearly implies periodic stability. In this contribution, we show with an example that the converse of this statement is not true. More specifically, we prove that there are uncountably many values of the real parameters a and b for which the switched linear system

$$x_{t+1} \in \left\{ a \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} x_t \tag{1.1}$$

is periodically stable but is not absolutely stable.

This result arises as a byproduct of a counterexample to the Lagarias-Wang *finiteness conjecture*. This conjecture was introduced in [13] in connection with problems related to spectral radius computation of finite sets of matrices. Let $\rho(A)$ be the spectral radius[4] of the matrix A and let Σ be a finite set of matrices. The generalized spectral radius of Σ is defined by

$$\rho(\Sigma) = \limsup_{k \to +\infty} \max\{\rho(A_1 \cdots A_k)^{1/k} : A_i \in \Sigma,\ i = 1, \dots, k\}$$

This quantity was introduced in [7] (see [8] for a corrigendum/addendum). The generalized spectral radius is known to coincide (see [2]) with the earlier defined joint spectral radius [16], the notion appears in a wide range of contexts and has led to a number of recent contributions (see, e.g., [4, 5, 9, 8, 12, 18, 19, 20]); a list of over hundred related contributions is given in [17]. It is known that

$$\rho(\Sigma) \geq \max\{\rho(A_1 \cdots A_k)^{1/k} : A_i \in \Sigma,\ i = 1, \dots, k\}$$

for all $k \geq 0$. According to the finiteness conjecture, equality in this expression is always obtained for some finite k. The existence of a counterexample to the conjecture is proved in [6] by using iterated function systems, topical maps and sturmian sequences. The proof relies in part on a particular fixed point theorem known as Mañé's lemma. In this contribution, we provide an alternative proof that is self-contained and fairly elementary. From results in [9] relating spectral radius of sets of matrices and rate of growth of long products of matrices, it follows that our counterexample is equivalent to the existence of systems of the form (1.1) that are periodically stable but are not absolutely stable.

[3]Switched linear systems are also known as *discrete linear inclusions* [9].

[4]The spectral radius of a matrix is equal to the absolute value of its largest eigenvalue.

2 Proof outline

Let us now briefly outline our proof. We define

$$A_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and

$$A_0^\alpha = \frac{1}{\rho_\alpha} A_0, \quad A_1^\alpha = \frac{\alpha}{\rho_\alpha} A_1$$

with $\rho_\alpha = \rho(\{A_0, \alpha A_1\})$. Since $\rho(\lambda\Sigma) = |\lambda|\,\rho(\Sigma)$, the spectral radius of the set $\Sigma_\alpha = \{A_0^\alpha, A_1^\alpha\}$ is equal to one. Let $I = \{0, 1\}$ be a two letters alphabet and let

$$I^+ = \{0, 1, 00, 01, 10, 11, 000, \ldots\}$$

be the set of finite nonempty words. With the word $w = w_1 \ldots w_t \in I^+$ we associate the products $A_w = A_{w_1} \ldots A_{w_t}$ and $A_w^\alpha = A_{w_1}^\alpha \ldots A_{w_t}^\alpha$. A word $w \in I^+$ will be said *optimal* for some α if $\rho(A_w^\alpha) = 1$. We use J_w to denote the set of α's for which $w \in I^+$ is optimal. If the finiteness conjecture is true, the union of the sets J_w for $w \in I^+$ covers the real line. We show that this union does not cover the interval $[0, 1]$.

In Section 4, we show that if two words $u, v \in I^+$ are essentially equal, then $J_u = J_v$. Two words $u, v \in I^+$ are *essentially equal* if the periodic infinite words $U = uu \ldots$ and $V = vv \ldots$ can be decomposed as $U = xww \ldots$ and $V = yww \ldots$ for some $x, y, w \in I^+$. Words that are not essentially equal are *essentially different*. Obviously, if u and v are essentially different, then so are also arbitrary cyclic permutations of u and v.

We show in the same section that the sets J_u and J_v are disjoint if u and v are essentially different. This part of the proof requires some properties of infinite words presented in Section 3. The proof is then almost complete. To conclude, we observe in Section 5 that the sets $J_w \cap [0, 1]$ are closed sub-intervals of $[0, 1]$. There are countably many words in I^+ and so $\cup_{w \in I^+}(J_w \cap [0, 1])$ is a countable union of disjoint closed sub-intervals of $[0, 1]$. Except for a trivial case that we can exclude here, there are always uncountably many points in $[0, 1]$ that do not belong to such a countable union. Each of these points provides a particular counterexample to the finiteness conjecture.

3 Palindromes in infinite words

The *length* of a word $w = w_1 \ldots w_t \in I^* = I^+ \cup \{\emptyset\}$ is equal to $t \geq 0$ and is denoted by $|w|$. The *mirror image* of w is the word $\tilde{w} = w_t \ldots w_1 \in I^*$. A *palindrome* is a word that is identical to its mirror image. For $u, v \in I^*$, we write $u > v$ if u is lexicographically larger than v, that is, $u_i = 1$, $v_i = 0$ for some $i \geq 1$ and $u_j = v_j$ for all $j < i$. This is only a partial order since, for example, 101000 and 1010 are not comparable. Let $F(U)$ denote the set of all finite factors of $U = uuu \ldots$.

Lemma 3.1. *Let $u, v \in I^+$ be two words that are essentially different. Then there exists a pair of words $0p0$ and $1p1$ in the set $F(U) \cup F(V)$ such that p is a palindrome.*

Corollary 3.2. *Let $u, v \in I^+$ be two essentially different words and let $U = uuu\dots$ and $V = vvv\dots$. Then there exist words $a, b, x, y \in I^+$ satisfying $|x| = |y|$, $x > y$, $\tilde{x} > \tilde{y}$, $x > \tilde{y}$, $\tilde{x} > y$, and a palindrome $p \in I^*$ such that*

$$U = apxpxp\dots \quad \text{and} \quad V = bpypyp\dots$$

or one of the words U and V, say U, can be decomposed as

$$U = apxpxp\dots = bpypyp\dots.$$

4 Optimal words are essentially equal

For a given word $w \in I^+$ we define $J_w = \{\alpha : \rho(A_w^\alpha) = 1\}$. Our goal in this section is to prove that J_u and J_v are equal when u and v are essentially equal, and have otherwise empty intersection.

Lemma 4.1. *Let $u, v \in I^+$ be two words that are essentially equal. Then $J_u = J_v$.*

Proof. Assume $u, v \in I^+$ are essentially equal. Then $U = uu\dots$ and $V = vv\dots$ can be written as $U = ss\dots$ and $V = tt\dots$ with $|s| = |t|$ and t a cyclic permutation of s. The spectral radius satisfies $\rho(AB) = \rho(BA)$ and so the spectral radius of a product of matrices is invariant under cyclic permutations of the product factors. From this it follows that $\rho(A_s^\alpha) = \rho(A_t^\alpha)$ and hence u is optimal whenever v is. □

We need two preliminary lemmas for proving the next result.

Lemma 4.2. *For any word $w \in I^+$ we have*

$$A_{\tilde{w}} - A_w = k(w)T,$$

where $k(w)$ is an integer and

$$T = A_0A_1 - A_1A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Moreover, $k(w)$ is positive if and only if $w > \tilde{w}$.

We say that a matrix A *dominates* B if $A \geq B$ componentwise and $\mathrm{tr}A > \mathrm{tr}B$ (tr denotes the trace). The eigenvalues of the 2×2 matrix A are given by $(\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}))/2$. For all words w, the matrix A_w satisfies $\det(A_w) = 1$ because it is a product of matrices A_0 and A_1 of determinant 1. It also satisfies $\mathrm{tr}A_w \geq 2$, because of the particular form of A_0 and A_1. So, the spectral radius of any matrix A_ω is an increasing monotone function of $\mathrm{tr}A_\omega$. We therefore have $\rho(A_u) > \rho(A_v)$ whenever A_u dominates A_v.

Lemma 4.3. *For any word of the form $w = psq$, where $s > \tilde{s}$ and $q < \tilde{p}$, the matrix $A_{w'}$ with $w' = p\tilde{s}q$ dominates A_w.*

Let $w = psq$. If $s > \tilde{s}$ and $q < \tilde{p}$, we say that $s \to \tilde{s}$ is a *dominating flip*. We are now ready to prove the main result of this section.

Lemma 4.4. *Let $u, v \in I^+$ be two words that are essentially different. Then $J_u \cap J_v = \emptyset$.*

5 Finiteness conjecture

We are now ready to prove the main result.

Theorem 5.1. *There are uncountably many values of the real parameter α for which the pair of matrices*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \alpha \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is periodically stable but not asymptotically stable and so, violates the finiteness conjecture.

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