Composition of Dirac Structures and Control of Port-Hamiltonian Systems

A.J. van der Schaft*[1], J. Cervera**[2]
* University of Twente, Faculty of Mathematical Sciences,
P.O. Box 217, 7500 AE Enschede, The Netherlands
Email: a.j.vanderschaft@math.utwente.nl
** Universidad de Murcia, Departamento de Informática y Sistemas
30071 Campus de Espinardo, Murcia, Spain
Email: jcervera@um.es

Abstract

Key feature of Dirac structures (as opposed to Poisson or symplectic structures) is the fact that the standard composition of two Dirac structures is again a Dirac structure. In particular this implies that any power-conserving interconnection of port-Hamiltonian systems is a port-Hamiltonian system itself. This constitutes a fundamental property in the port-Hamiltonian approach to modeling, simulation and control of complex physical systems. Furthermore, the composed Dirac structure directly determines the algebraic constraints of the interconnected system, as well as its Casimir functions. Especially the Casimirs are of prime importance in the set-point regulation of port-Hamiltonian systems. It is therefore of importance to characterize the set of achievable Dirac structures when a given plant port-Hamiltonian system is interconnected with an arbitrary controller port-Hamiltonian system.

The set of achievable Dirac structures in a restricted sense has been recently characterized in [1, 2]. Here we extend this theorem to the present situation occurring in the interconnection of a plant and controller Hamiltonian system. Furthermore, we give an insightful procedure for the construction of the controller Dirac structure. This procedure works for the general case of (non-closed) Dirac structures on manifolds. In this way we also fully characterize the set of achievable Casimir functions of the interconnected ("closed-loop") system. This yields a fundamental limitation to the design of stabilizing controllers for underactuated mechanical systems by interconnection with a port-Hamiltonian controller.

1 Introduction

Network modeling of complex physical systems (possibly containing components from different physical domains) leads to a class of nonlinear systems, called *port-Hamiltonian systems*, see e.g. [3, 4, 5, 2, 6, 7, 8]. Port-Hamiltonian systems are defined by a Dirac structure

[1]Partially supported by the EU-project *GeoPlex*, IST-2001-34166

[2]Partially supported by *Fundación Séneca*, *Centro de Coordinación de la Investigación*, under program *Becas de Formación del Personal Investigador*, and by the EU-TMR *Nonlinear Control Network*

(formalizing the power-conserving interconnection structure of the system), an energy function (the Hamiltonian), and a resistive structure. Key property of Dirac structures is that the power-conserving composition of Dirac structures again defines a Dirac structure, see [2, 1]. This implies that any power-conserving interconnection of port-controlled Hamiltonian systems is also a port-controlled Hamiltonian system, with Dirac structure being the composition of the Dirac structures of its constituent parts and Hamiltonian the sum of the Hamiltonians. As a result power-conserving interconnections (in particular classical feedback interconnections) of port-Hamiltonian systems can be studied to a large extent in terms of the composition of their Dirac structures. In particular the feedback interconnection of a given plant port-Hamiltonian system with a yet to be specified port-Hamiltonian controller system can be studied from the point of view of the composition of a given plant Dirac structure with a controller Dirac structure. Preliminary results concerning the achievable "closed-loop" Dirac structures have been obtained in [1, 2]. Here we extend those results, and we also derive an explicit characterization of the obtainable Casimir functions of the closed-loop system, which is crucial for the passivity-based control of the port-Hamiltonian system plant system, see e.g. [9, 6, 7, 10].

2 Dirac structures and port-Hamiltonian systems

Port-based modeling (e.g. by bond-graphs) of complex lumped-parameter physical systems directly leads to models consisting of a power-conserving interconnection (generalized junction structure in bond-graph terminology), and the constitutive relations describing the energy-storing and energy-dissipating elements.

The key to geometrically formalize these models as port-Hamiltonian systems is to describe the power-conserving interconnection by the notion of a *Dirac structure*.

2.1 Dirac structures

We start with a space of *power variables* $\mathcal{V} \times \mathcal{V}^*$, for some linear space $\mathcal{V}$, with *power* defined by

$$P = < v^* \mid v >, \quad (v, v^*) \in \mathcal{V} \times \mathcal{V}^*, \tag{2.1}$$

where $< v^* \mid v >$ denotes the duality product, that is, the linear functional $v^* \in \mathcal{V}^*$ acting on $v \in \mathcal{V}$. Often we call $\mathcal{V}$ the space of *flows* f, and $\mathcal{V}^*$ the space of *efforts* e, with the power of a signal $(f, e) \in \mathcal{V} \times \mathcal{V}^*$ denoted as $< e \mid f >$.

Note that, contrary to other treatments, we have used 'Occam's razor' by *not* necessarily endowing $\mathcal{V}$ with an inner product structure $<,>$. Of course, in this latter case $\mathcal{V}^*$ can be naturally identified with $\mathcal{V}$ in such a way that $< e \mid f >=< e, f >$, $f, e \in \mathcal{V} \simeq \mathcal{V}^*$.

Closely related to the definition of power there exists a canonically defined *bilinear form*

$\ll, \gg$ on the space of power variables $\mathcal{V} \times \mathcal{V}^*$, defined as

$$\ll (f^a, e^a),(f^b, e^b) \gg := \\ < e^a \mid f^b > + < e^b \mid f^a >, \quad (f^a, e^a),(f^b, e^b) \in \mathcal{V} \times \mathcal{V}^*. \tag{2.2}$$

Definition 2.1. *[11, 12] A (constant) Dirac structure on is a subspace*

$$\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$$

such that $\mathcal{D} = \mathcal{D}^\perp$, where $\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$.

If $\mathcal{V}$ is a *finite-dimensional* linear space then it is easily seen that necessarily $\dim \mathcal{D} = \dim \mathcal{V}$ for any Dirac structure $\mathcal{D}$. Moreover, in this case a Dirac structure can be alternatively characterized as a subspace $\mathcal{D}$ of $\mathcal{V} \times \mathcal{V}^*$ such that

(i) $< e \mid f >= 0$, for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{V}$.

Note that condition (i) expresses power conservation. Condition (ii) is perhaps more open to discussion. Although this condition holds for all “normal” interconnection structures such as Kirchhoff’s laws, transformers, gyrators, Newton’s third law, kinetic pairs, kinematic constraints, etc., one could imagine power-conserving elements such as the “nullator” in the electric domain (setting *both* voltage and current to be equal to zero) which violate this condition. The discussion boils down to the validity (or usefulness!) of the usually expressed statement that a physical element can*not* determine at the same time both its voltage and its current, or both its force and velocity.

Remark 2.2. *The property $\mathcal{D} = \mathcal{D}^\perp$ can be regarded as a generalization of Tellegen’s theorem, since it describes a constraint between two different realizations of the power variables (in contrast to condition (i)).*

Remark 2.3. *For many systems, especially those with mechanical components, the interconnection structure is actually modulated by energy or geometric variables. This leads to the notion of non-constant Dirac structures on manifolds, see e.g. [11, 12, 6, 7, 13]. Because of space limitations and for reasons of clarity of exposition we focus in the current paper on the constant case, although everything can be extended to the case of Dirac structures on manifolds.*

Constant Dirac structures admit different *matrix representations*. Here we just list a number of them. Let $\mathcal{D} \subset \mathcal{V} \times \mathcal{V}^*$, with $\dim \mathcal{V} = n$, be a constant Dirac structure.

1. *(Kernel and Image representation)* Every Dirac structure $\mathcal{D}$ can be represented in *kernel representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid Ff + Ee = 0\} \tag{2.3}$$

for $n \times n$ matrices F and E satisfying

$$\begin{array}{ll} (i) & EF^T + FE^T = 0, \\ & \\ (ii) & \text{rank } [F \vdots E] = n. \end{array} \tag{2.4}$$

It follows that $\mathcal{D}$ can be also written in *image representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = E^T \lambda, e = F^T \lambda, \lambda \in \mathbb{R}^n\} \tag{2.5}$$

2. (*Constrained input-output representation*) $\mathcal{D} = \{(f, e) \in \mathcal{V} \times \mathcal{V}^* \mid f = Je + G\lambda, G^T e = 0\}$ for a skew-symmetric matrix J and a matrix G such that Im $G = \{f \mid (f, 0) \in \mathcal{D}\}$. Furthermore, Ker $J = \{e \mid (0, e) \in \mathcal{D}\}$.

3. *(Hybrid input-output representation, cf. [14])* Let $\mathcal{D}$ be given by square matrices E and F as in 1. Suppose rank $F = m (\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \vdots F_2]$, and correspondingly $E = [E_1 \vdots E_2], f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. Then the matrix $[F_1 \vdots E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \left[\begin{array}{c} f_1 \\ f_2 \end{array} \right], \left[\begin{array}{c} e_1 \\ e_2 \end{array} \right] \middle| \left[\begin{array}{c} f_1 \\ e_2 \end{array} \right] = J \left[\begin{array}{c} e_1 \\ f_2 \end{array} \right] \right\} \tag{2.6}$$

with $J := -[F_1 \vdots E_2]^{-1} [F_2 \vdots E_1]$ skew-symmetric.

4. *(Canonical coordinate representation), cf. [11].* There exist linear coordinates (q, p, r, s) for $\mathcal{V}$ such that $(f, e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s) \in \mathcal{D}$ iff

$$\left\{ \begin{array}{rclrcl} f_q & = & e_p, & f_p & = & -e_q \\ & & & & & \\ f_r & = & 0, & e_s & = & 0 \end{array} \right. \tag{2.7}$$

2.2 Port-Hamiltonian systems

Now let us consider a lumped-parameter physical system given by a power-conserving interconnection defined by a constant Dirac structure $\mathcal{D}$, and k energy-storing elements with energy-variables x_i. For simplicity we assume that the energy-variables are living in *linear* spaces $\mathcal{X}_i$, although everything can be generalized to the case of manifolds. The constitutive relations of the energy-storing elements are specified by their stored energy functions $H_i(x_i),\ i = 1, \cdots, k$. Define the total linear state space $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$, and the total energy $H(x_1, \cdots, x_k) := H_1(x_1) + \cdots + H_k(x_k)$.

The space of flow variables for the Dirac structure $\mathcal{D}$ is split as $\mathcal{X} \times \mathcal{F}$, with $f_x \in \mathcal{X}$ the flows corresponding to the energy-storing elements, and $f \in \mathcal{F}$ denoting the remaining flows (corresponding to dissipative elements and ports/sources). Correspondingly, the space of effort variables is split as $\mathcal{X}^* \times \mathcal{F}^*$, with $e_x \in \mathcal{X}^*$ the efforts corresponding to the energy-storing elements and $e \in \mathcal{F}^*$ the remaining efforts. The bilinear form $\ll, \gg$ then takes the form:

$$\begin{aligned} &\ll (f_x^a, e_x^a, f^a, e^a), (f_x^b, e_x^b, f^b, e^b) \gg := \\ &< e_x^a \mid f_x^b > + < e_x^b \mid f_x^a > + < e^a \mid f^b > + < e^b \mid f^a > \end{aligned} \tag{2.8}$$

with $f_x^a, f_x^b \in \mathcal{X}, f^a, f^b \in \mathcal{F}, e_x^a, e_x^b \in \mathcal{X}^*, e^a, e^b \in \mathcal{F}^*$. It follows that $\mathcal{D}$ can be represented in kernel representation as

$$\mathcal{D} = \{(f_x, e_x, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^* \mid F_x f_x + E_x e_x + Ff + Ee = 0\}, \tag{2.9}$$

with

$$\begin{aligned} &\text{(i)} \quad E_x F_x^T + F_x E_x^T + EF^T + FE^T = 0, \\ &\text{(ii)} \quad \text{rank } [F_x \vdots E_x \vdots F \vdots E] = \dim(\mathcal{X} \times \mathcal{F}). \end{aligned} \tag{2.10}$$

Now the *flows* of the energy-storing elements are given by $\dot{x_i}$, and these are equated to $-f_{xi}, i = 1, \cdots, k$ (the minus sign is again included to have a consistent energy flow diraction). Furthermore, the efforts e_{xi} corresponding to the energy-storing elements are given as $e_{xi} = \frac{\partial H}{\partial x_i}, i = 1, \cdots, k$. Substitution in (2.9) leads to the description of the physical system by the set of DAE's

$$F_x \dot{x}(t) = E_x \frac{\partial H}{\partial x}(x(t)) + Ff(t) + Ee(t), \tag{2.11}$$

with f, e the *port* power variables (some of which are terminated by dissipative elements). The system of equations (2.11) is called a *port-Hamiltonian system.*

The definition of a *port-Hamiltonian system* is not dependent on the particular representation of the Dirac structure (a kernel representation in (2.11)). In fact, the port-Hamiltonian system is defined by the Dirac structure $\mathcal{D}$ (a geometric object), together with the Hamiltonian H and the specification of the ports. Thus in the case of no energy-dissipating ports we denote the port-Hamiltonian system by $(\mathcal{X}, \mathcal{F}, \mathcal{D}, H)$.

Because of (2.10) we immediately obtain the *power balance*

$$\frac{dH}{dt} = \left(\frac{\partial H}{\partial x}(x) \right)^T \dot{x} = e^T f. \tag{2.12}$$

expressing that the increase of of internal energy of the port-Hamiltonian system is equal to the externally supplied power minus the power dissipated in the energy-dissipating elements.

Remark 2.4. *In case of a Dirac structure modulated by the energy variables x and state space $\mathcal{X}$ being an arbitrary manifold, the flows $f_x = -\dot{x}$ are elements of the tangent space $T_x\mathcal{X}$ at the state $x \in \mathcal{X}$, and the efforts e_x are elements of the co-tangent space $T_x^*\mathcal{X}$. We still obtain the kernel representation (2.11) for the resulting port-Hamiltonian system, but now the matrices F_x, E_x, F, E depend on x. See for an extensive treatment [6, 7, 13].*

3 Composition of Dirac structures

In this section we investigate the *compositionality* properties of Dirac structures. Physically it seems clear that the composition of a number of power-conserving interconnections with partially shared variables should yield again a power-conserving interconnection. We show how this can be formalized within the framework of Dirac structures.

First we consider the composition of *two* Dirac structures with partially shared variables. That is, we consider a Dirac structure $\mathcal{D}_{12}$ on a product space $\mathcal{V}_1 \times \mathcal{V}_2$ of two linear spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, and another Dirac structure $\mathcal{D}_{23}$ on a product space $\mathcal{V}_2 \times \mathcal{V}_3$, with also $\mathcal{V}_3$ being a linear space. The linear space $\mathcal{V}_2$ is the space of shared flow variables.

In order to compose $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$ a problem arises of *sign* convention for the power flow corresponding to the power variables $(f_2, e_2) \in \mathcal{V}_2 \times \mathcal{V}_2^*$. Indeed, if $< e \mid f >$ denotes *incoming* power (see the previous section), then for

$$\left(f_1, e_1, f_2^{12}, e_2^{12}\right) \in \mathcal{D}_{12}$$

the term $< e_2^{12} \mid f_2^{12} >$ denotes the incoming power *in* $\mathcal{D}_{12}$ due to the power variables $(f_2^{12}, e_2^{12}) \in \mathcal{V}_2 \times \mathcal{V}_2^*$, while for

$$\left(f_2^{23}, e_2^{23}, f_3, e_3\right) \in \mathcal{D}_{23}$$

the term $< e_2^{23} \mid f_2^{23} >$ denotes the incoming power *in* $\mathcal{D}_{23}$ due to the power variables $(f_2^{23}, e_2^{23}) \in \mathcal{V}_2 \times \mathcal{V}_2^*$. Since physically, the *incoming* power in $\mathcal{D}_{12}$ due to the power variables in $\mathcal{V}_2 \times \mathcal{V}_2^*$ should equal the *outgoing* power from $\mathcal{D}_{23}$ due to the power variables in $\mathcal{V}_2 \times \mathcal{V}_2^*$ there is a sign conflict if we would simply equate $f_2^{12} = f_2^{23}, e_2^{12} = e_2^{23}$. A simple way to resolve this sign problem is to set instead

$$\begin{aligned} f_2^{12} &= -f_2^{23} \\ e_2^{12} &= e_2^{23} \end{aligned} \tag{3.13}$$

Let us therefore define the *composition* $\mathcal{D}_{12} \parallel \mathcal{D}_{23}$ of the Dirac structures $\mathcal{D}_{12}$ and $\mathcal{D}_{23}$ as

$$\begin{aligned} \mathcal{D}_{12} \parallel \mathcal{D}_{23} \;:=\; & \{(f_1, e_1, f_3, e_3) \in \mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_3 \times \mathcal{V}_3^* \mid \exists (f_2, e_2) \in \mathcal{V}_2 \times \mathcal{V}_2^* \text{ s.t.} \\ & (f_1, e_1, f_2, e_2) \in \mathcal{D}_{12} \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_{23}\} \end{aligned} \tag{3.14}$$

The following theorem has been shown in [1] (with a preliminary version given in [2]).

Theorem 3.1. *Let $\mathcal{D}_{12}, \mathcal{D}_{23}$ be Dirac structures as above. Then $\mathcal{D}_{12} \parallel \mathcal{D}_{23}$ defined in (3.14) is a Dirac structure.*

The compositionality of *multiple* Dirac structures follows easily from Theorem 3.1. In general, consider k port-Hamiltonian systems $(\mathcal{X}_i, \mathcal{F}_i, \mathcal{D}_i, H_i),\ i = 1, \cdots, k$, interconnected by a Dirac structure $\mathcal{D}_I \subset \mathcal{F}_1 \times \cdots \times \mathcal{F}_k \times \mathcal{F} \times \mathcal{F}_1^* \times \cdots \times \mathcal{F}_k^* \times \mathcal{F}^*$, with $\mathcal{F}$ a linear space of flow port variables, cf. Figure 1.

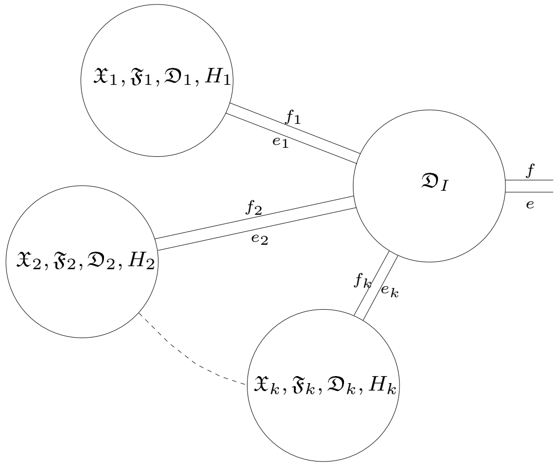

Figure 1: Interconnection of port-Hamiltonian systems

This can be easily seen to define a port-Hamiltonian system $(\mathcal{X}, \mathcal{F}, \mathcal{D}, H)$, where $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$, $H := H_1 + \cdots + H_k$, and where the Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^*$ is determined by $\mathcal{D}_1, \cdots, \mathcal{D}_k$ and $\mathcal{D}_I$. Indeed, consider the *product* of the Dirac structures $\mathcal{D}_1, \cdots, \mathcal{D}_k$, and compose this, as in Theorem 3.1, with the Dirac structure $\mathcal{D}_I$. This yields the Dirac structure $\mathcal{D}$.

A typical example of a power-conserving interconnection is the standard *feedback interconnection*

$$u_P = -y_C, \quad u_C = y_P \tag{3.15}$$

with u_P, y_P and u_C, y_C denoting the inputs and outputs of the plant, respectively, controller system. Identifying the inputs u_P, u_C with *flows*, and the outputs y_P, y_C with *efforts*, (3.15) defines a Dirac structure $\mathcal{D}_I$.

4 Achievable Dirac structures

In this section several questions about the composition of Dirac structures are addressed. The main idea is to investigate which closed-loop port-Hamiltonian systems can be achieved by interconnecting a given plant port-Hamiltonian system P with a controller port-Hamiltonian system C.

In the framework of the current paper this is restricted to the investigation of the *achievable Dirac structures* of the closed-loop system. That is, given the Dirac structure $\mathcal{D}_P$ of the plant system P and the to-be-designed Dirac structure $\mathcal{D}_C$ of the controller system C, what are the achievable Dirac structures $\mathcal{D}_P \parallel \mathcal{D}_C$, where $\parallel$ denotes the composition of $\mathcal{D}_P$ and $\mathcal{D}_C$ by interconnecting P and C as defined in the previous section.

Consider the general configuration given in Figure 2.

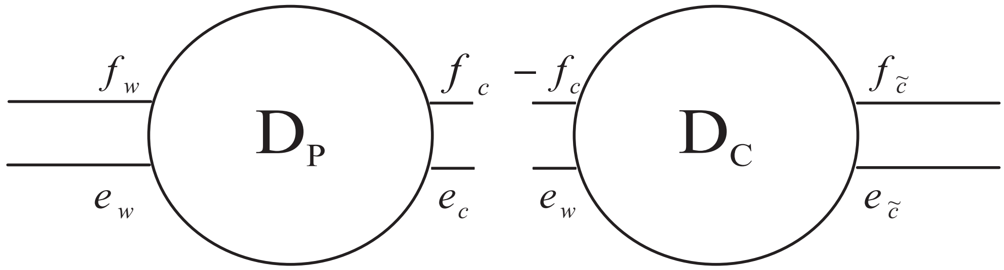

Figure 2: $\mathcal{D}_P \parallel \mathcal{D}_C$

Theorem 4.1. *Given any plant Dirac structure $\mathcal{D}_P$, a certain composed $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$ can be achieved by a proper choice of the controller Dirac structure $\mathcal{D}_C$ if and only if the following two conditions are satisfied*

$$\mathcal{D}_P^0 \subset D^0 \tag{4.16}$$

$$D^\pi \subset \mathcal{D}_P^\pi \tag{4.17}$$

$$\textit{where} \begin{cases} \mathcal{D}_P^0 := \{(f_w, e_w) \mid (f_w, e_w, 0, 0) \in \mathcal{D}_P\} \\ \mathcal{D}_P^\pi := \{(f_w, e_w) \mid \exists (f_c, e_c) : (f_w, e_w, f_c, e_c) \in \mathcal{D}_P\} \\ \mathcal{D}^0 := \{(f_w, e_w) \mid (f_w, e_w, 0, 0) \in \mathcal{D}\} \\ \mathcal{D}^\pi := \{(f_w, e_w) \mid \exists (f_{\tilde{c}}, e_{\tilde{c}}) : (f_w, e_w, f_{\tilde{c}}, e_{\tilde{c}}) \in \mathcal{D}\} \end{cases} \tag{4.18}$$

Remark 4.2. *A restricted version of this theorem for the case $f_{\tilde{c}} = 0, e_{\tilde{c}} = 0$ was given in [1].*

The simple proof of this theorem (compare with the proof given in [1]!) is based on the following 'inverse' $\mathcal{D}_P^*$ of the plant Dirac structure $\mathcal{D}_P$:

$$\mathcal{D}_P^* := \{(f_w, e_w, f_c, e_c) \mid (-f_w, e_w, -f_c, e_c) \in \mathcal{D}_P\} \tag{4.19}$$

It is easily seen that $\mathcal{D}_P^*$ is a Dirac structure if and only if $\mathcal{D}_P$ is a Dirac structure.

Proof of Theorem 4.1

Necessity of (4.16, 4.17) is obvious. Sufficiency is shown using the controller Dirac structure

$$\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D}$$

(see Figure 3).

To check that $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$, consider $(w', c) \in \mathcal{D}$. Because $w' \in \mathcal{D}^\pi$, applying (4.17) yields that $\exists d$ such that $(w', d) \in \mathcal{D}_P$. Define the partial sign reversal operator

$$Rd := (-f_d, e_d), \text{for} \quad d = (f_d, e_d)$$

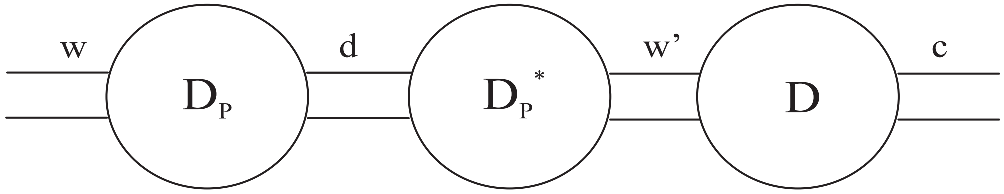

Figure 3: $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_P^* \parallel \mathcal{D}$.

It follows that $(Rd, Rw') \in \mathcal{D}_P^*$ [3], and thus $(w', c) \in \mathcal{D}_P \parallel \mathcal{D}_C$ (take $w = w'$ in Figure 3). Therefore, $\mathcal{D} \subset \mathcal{D}_P \parallel \mathcal{D}_C$.

To check that $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$, consider $(w, c) \in \mathcal{D}_P \parallel \mathcal{D}_C$. Then there exist d, w' such that

$$(w, d) \in \mathcal{D}_P \tag{4.20}$$

$$(w', d) \in \mathcal{D}_P \tag{4.21}$$

$$(w', c) \in \mathcal{D} \tag{4.22}$$

Subtracting (4.21) from (4.20) we get

$$(w - w', 0) \in \mathcal{D}_P \Longleftrightarrow w - w' \in \mathcal{D}_P^0 \tag{4.23}$$

Using (4.23) and (4.16) we get

$$(w - w', 0) \in \mathcal{D} \tag{4.24}$$

Finally, adding (4.22) and (4.24), we get $(w, c) \in \mathcal{D}$, and so $\mathcal{D}_P \parallel \mathcal{D}_C \subset \mathcal{D}$.

□

The above proof also immediately provides us with a closed expression for a controller Dirac structure $\mathcal{D}_C$ such that $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$, for $\mathcal{D}$ satisfying the conditions of Theorem 4.1. We state this as a separate proposition.

Proposition 4.3. *Given a plant Dirac structure $\mathcal{D}_P$, and $\mathcal{D}$ satisfying the conditions of Theorem 4.1. Then $\mathcal{D}_C := \mathcal{D}_P^* \parallel \mathcal{D}$, with $\mathcal{D}_P^*$ defined as in (4.19), achieves $\mathcal{D} = \mathcal{D}_P \parallel \mathcal{D}_C$.*

5 Achievable Casimirs and constraints

An important application of Theorem 4.1 concerns the characterization of the *Casimir functions* which can be achieved for the closed-loop system by interconnecting a given plant port-Hamiltonian system with associated Dirac structure $\mathcal{D}_P$ with a controller port-Hamiltonian system with associated Dirac structure $\mathcal{D}_C$. This constitutes a cornerstone for passivity-based control of port-Hamiltonian systems as developed e.g. in [10, 15]. Dually, we may characterize the achievable *algebraic constraints* for the closed-loop system.

[3] Note that the definition of $\mathcal{D}_P^*$ compensates the sign change for interconnection of flows given in (3.13).

In order to explain these notions consider a port-Hamiltonian system without ports on a state space $\mathcal{X}$ with Dirac structure $\mathcal{D} \subset \mathcal{X} \times \mathcal{X}^*$. Then the following subspaces of $\mathcal{X}$, respectively $\mathcal{X}^*$, are of importance

$$\begin{aligned} G_1 &:= \{f_x \in \mathcal{X} \mid \exists e_x \in \mathcal{X}^* \text{ s.t. } (f_x, e_x) \in \mathcal{D}\} \\ P_1 &:= \{e_x \in \mathcal{X}^* \mid \exists f_x \in \mathcal{X} \text{ s.t. } (f_x, e_x) \in \mathcal{D}\} \end{aligned} \tag{5.25}$$

The subspace G_1 expresses the set of *admissible flows*, and P_1 the set of *admissible efforts*. It follows from the image representation (2.5) that

$$\begin{aligned} G_1 &= \text{ Im } E_x^T \\ P_1 &= \text{ Im } F_x^T \end{aligned} \tag{5.26}$$

A *Casimir function* $C : \mathcal{X} \to \mathbb{R}$ of the port-Hamiltonian system is defined to be a function which is constant along all trajectories of the port-Hamiltonian system, irrespectively of the Hamiltonian H. It follows from the above consideration of the admissible flows that the Casimirs are determined by the subspace G_1. Indeed, necessarily $f_x = -\dot{x}(t) \in G_1 = \text{ Im } E_x^T$, and thus

$$\dot{x}(t) \in \text{ Im } E_x^T, \quad t \in \mathbb{R}. \tag{5.27}$$

Therefore $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function if $\frac{dC}{dt}(x(t)) = \frac{\partial^T C}{\partial x}(x(t))\dot{x}(t) = 0$ for all $\dot{x}(t) \in$ Im E_x^T. Hence $C : \mathcal{X} \to \mathbb{R}$ is a Casimir of the port-Hamiltonian system if it satisfies the set of partial differential equations

$$\frac{\partial C}{\partial x}(x) \in \text{ Ker } E_x \tag{5.28}$$

Remark 5.1. *In the case of a non-constant Dirac structure the matrix E_x will depend on x, and Ker E_x will define a co-distribution on the manifold $\mathcal{X}$. Then the issue arises of integrability of this co-distribution, see [6].*

Dually, the algebraic constraints for the port-Hamiltonian system are determined by the space P_1, since necessarily $\frac{\partial^T H}{\partial x}(x) \in P_1$, which will induce constraints on the state variables x.

Let us now consider the question of characterizing the set of achievable Casimirs for the closed-loop system $\mathcal{D}_P \parallel \mathcal{D}_C$, where $\mathcal{D}_P$ is the given Dirac structure of the plant port-Hamiltonian system with Hamiltonian H, and $\mathcal{D}_C$ is the controller Dirac structure. In this case, the Casimirs will depend on the plant state x as well as on the controller state ξ. Since the controller Hamiltonian $H_C(\xi)$ is at our own disposal we will be primarily interested in the dependency of the Casimirs only on the plant state x. (Since we want to use the Casimirs for shaping the total Hamiltonian $H + H_C$ to a Lyapunov function, cf. [10, 15].)

Consider the notation given in Figure 2, and assume the ports in (f_w, e_w) are connected to the (given) energy storing elements of the plant port-Hamiltonian system (that is, $f_w = -\dot{x}, e_w = \frac{\partial^T H}{\partial x}$), while $(f_{\tilde{c}}, e_{\tilde{c}})$ are connected to the (to-be-designed) energy storing elements of a controller port-Hamiltonian system (that is, $f_{\tilde{c}} = -\dot{\xi}, e_{\tilde{c}} = \frac{\partial^T H_C}{\partial \xi}$). Note that the number of ports $(f_{\tilde{c}}, e_{\tilde{c}})$ can be freely chosen. In this situation the achievable Casimir functions are functions $C(x,\xi)$ such that $\frac{\partial^T C}{\partial x}(x,\xi)$ belongs to the space

$$P_{Cas} = \{e_w \mid \exists \mathcal{D}_C \text{ s.t. } \exists e_{\tilde{c}} : (0, e_w, 0, e_{\tilde{c}}) \in \mathcal{D}_P \parallel \mathcal{D}_C\} \tag{5.29}$$

Thus the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state x, is translated to finding a characterization of the space P_{Cas}. It is answered by the following theorem.

Theorem 5.2. *The space P_{Cas} defined in (5.29) is equal to the linear space*

$$\tilde{P} = \{e_w \mid \exists (f_c, e_c) : (0, e_w, f_c, e_c) \in \mathcal{D}_P\} \tag{5.30}$$

.

Proof $P_{Cas} \subset \tilde{P}$ trivially. By using the controller Dirac structure $\mathcal{D}_C = \mathcal{D}_P^*$, we immediately obtain $\tilde{P} \subset P_{Cas}$.

□

Remark 5.3. *For a non-constant Dirac structure on a manifold $\mathcal{X}$ P_{Cas} defines a co-distribution on $\mathcal{X}$.*

In a completely dual way we may consider the *achievable constraints* of the closed-loop system, characterized by the space

$$G_{Alg} = \{f_w \mid \exists \mathcal{D}_C \text{ s.t. } \exists f_{\tilde{c}} : (f_w, 0, f_{\tilde{c}}, 0) \in \mathcal{D}_P \parallel \mathcal{D}_C\} \tag{5.31}$$

Theorem 5.4. *The space G_{Alg} defined in (5.31) is equal to the linear space*

$$\{f_w \mid \exists (f_c, e_c) : (f_w, 0, f_c, e_c) \in \mathcal{D}_P\} \tag{5.32}$$

.

Remark 5.5. *For a non-constant Dirac structure G_{Alg} defines a distribution on the manifold $\mathcal{X}$.*

Example 5.6. *Consider the port-Hamiltonian plant system with inputs f_c and outputs e_c*

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x) f_c, \quad x \in \mathcal{X}, f_c \in \mathbb{R}^m \\ e_c &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad e_c \in \mathbb{R}^m \end{aligned} \tag{5.33}$$

where $J(x)$ is a skew-symmetric *$n \times n$ matrix. The corresponding Dirac structure is given by*

$$\begin{bmatrix} f_w \\ e_c \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_w \\ f_c \end{bmatrix} \tag{5.34}$$

It is easily seen that

$$P_{Cas} = \{e_w \mid \exists f_c \ s.t. \ 0 = J(x)e_w + g(x)f_c\}, \tag{5.35}$$

implying that the $x-$*dependency of the achievable Casimirs are the Hamiltonian functions corresponding to the input vector fields given by the columns of* $g(x)$*. Similarly, it is easily seen that*

$$G_{Alg} = \{f_w \mid \exists f_c \ s.t. \ f_w = -g(x)f_c\} = Im\ g(x), \tag{5.36}$$

and the achievable algebraic constraints are of the form $\frac{\partial^T H}{\partial x}(x)g(x) = k(\xi)$.

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