# Hamiltonian Attitude Dynamics for a Spacecraft with a Point Mass Oscillator

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**Abstract**

Two noncanonical Hamiltonian models are presented for the dynamics of a rigid body with an constrained moving point mass. One of these models is used to analyze stability of steady principal axis rotation of a rigid body with a spring-mass oscillator. The analysis gives necessary and sufficient conditions for stability of steady major axis rotation, as well as sufficient conditions for instability of intermediate and minor axis rotation.

## 1 Introduction

A rigid body coupled to a moving point mass is a dynamic model which often appears in vehicle dynamics and control literature. If the coupling is a spring-damper mechanism, then this system provides a relatively low-dimensional model for analyzing the effect of vibrational modes on rigid body motion. Damped point mass oscillators have been used to study effects such as fuel slosh in aircraft dynamics [1] and flexible modes in gyrostats [3]. Alternatively, by treating the force of interaction between the rigid body and the point mass as an input, one may consider certain vehicle control problems. In [11], for example, "principal axis misalignments" were intentionally introduced using two independently controlled point masses in order to asymptotically stabilize steady rotation of a prolate, axisymmetric spacecraft. Servo-actuated internal masses have also been proposed to control maneuverable re-entry vehicles [9, 10]. A number of underwater gliders, which are winged underwater vehicles propelled by gravity and buoyancy, also use internal moving masses for attitude control; see [4] and references contained there for examples.

Analysis and control design for systems such as the ones above has largely been limited to linear or numerical methods. Perhaps this is because analysis of moderate-dimensional nonlinear systems can be tedious. Exploiting symmetry, however, can make nonlinear control design and analysis tractable and can possibly lead to improved results. For example, the method of controlled Lagrangians [2], or the equivalent technique of interconnection and damping assignment, passivity-based control [8], can provide nonlinear control laws for a class of mechanical systems which includes vehicles of interest to engineers. Both techniques involve shaping the energy, damping, and dynamic structure of a given system. Stability analysis techniques for systems with symmetry, such as the energy-Casimir or energy-momentum methods, can be used to construct Lyapunov functions which prove stability and provide region of attraction estimates [7].

Drawing on results presented in [12], this paper presents two reduced-dimensional Hamiltonian models for a rigid spacecraft with a single degree-of-freedom point mass. A "potential shaping" control law, which can be realized as a simple spring, is applied to the point mass. The energy-Casimir method provides sufficient conditions for nonlinear stability of steady rotation about the principal axis of greatest inertia. Spectral analysis shows that this condition is necessary, as well as sufficient. If the spring which couples the point mass to the rigid body is sufficiently stiff, then the equilibrium is stable; otherwise, the equilibrium is unstable. Spectral analysis also provides conditions under which steady rotation about the principal axes of intermediate or least inertia is unstable. These results pose interesting implications for the demise of Explorer I, which is widely blamed on energy dissipation due to internal damping [6].

## 2 Two Hamiltonian Models

Consider a rigid body with a coordinate frame fixed in the principal axes and an internal point mass constrained to move in a slot parallel to the body 1-axis. We assume that the system's center of mass passes through the body coordinate origin as the point mass passes through the body 2-3 plane. For simplicity, we assume that the system consists of a non-axisymmetric ellipsoid with uniformly distributed mass and inertia matrix $\tilde{\boldsymbol{J}} = \mathrm{diag}(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$, a moving point mass $\bar{m}$ located at $\boldsymbol{r}_{\mathrm{m}} = [r_{\mathrm{m}_1}, 0, \Delta]^T$, and a fixed point mass $\bar{m}$ located at $\boldsymbol{r}_{\mathrm{f}} = [0, 0, -\Delta]^T$. (See Figure 1.) The mass of the complete system is $m$.

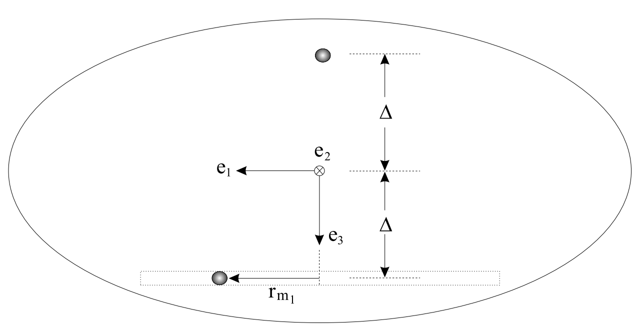


Figure 1: Rigid body and a moving point mass

There are at least two ways to describe the dynamics of this system in Hamiltonian form. The first uses the point mass velocity relative to the vehicle as a coordinate velocity. The second uses the absolute point mass velocity as a coordinate velocity.

### 2.1 Case 1

Define the body angular velocity $\boldsymbol{\Omega}$ and the body translational velocity $\boldsymbol{v}$. The inertial velocity of the moving point mass, written in body coordinates, is

$$\boldsymbol{v}_{\mathrm{m}} = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r}_{\mathrm{m}} + \dot{\boldsymbol{r}}_{\mathrm{m}}.$$

Note that, because the other point mass is rigidly constrained, $\boldsymbol{v}_{\mathrm{f}}$ is defined by the body angular and translational velocity.

Define the operator $\hat{\cdot}$ which converts a vector into a $3\times3$ skew-symmetric matrix satisfying $\hat{\boldsymbol{x}}\boldsymbol{y} = \boldsymbol{x} \times \boldsymbol{y}$ for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. The rotational inertia of the system with the point mass locked in place is

$$\boldsymbol{J}(r_{\mathrm{m}_1}) = \tilde{\boldsymbol{J}} - \bar{m}\widehat{\boldsymbol{r}_{\mathrm{f}}}^2 - \bar{m}\widehat{\boldsymbol{r}_{\mathrm{m}}}^2 = \begin{pmatrix} J_1 & 0 & -\bar{m}\Delta r_{\mathrm{m}_1} \\ 0 & J_2 + \bar{m}r_{\mathrm{m}_1}^2 & 0 \\ -\bar{m}\Delta r_{\mathrm{m}_1} & 0 & J_3 + \bar{m}r_{\mathrm{m}_1}^2 \end{pmatrix},$$

where

$$\begin{aligned} J_1 &= \tilde{J}_1 + 2\bar{m}\Delta^2 \\ J_2 &= \tilde{J}_2 + 2\bar{m}\Delta^2 \\ J_3 &= \tilde{J}_3. \end{aligned}$$

Define $\boldsymbol{\eta}_1 = [\boldsymbol{\Omega}^T, \boldsymbol{v}^T, \dot{r}_{\mathrm{m}_1}]^T$. The system kinetic energy is

$$T_1 = \frac{1}{2}\boldsymbol{\eta}_1 \cdot \mathbb{M}_1\boldsymbol{\eta}_1, \qquad \text{where} \qquad \mathbb{M}_1 = \begin{pmatrix} \boldsymbol{J}(r_{\mathrm{m}_1}) & \bar{m}\boldsymbol{r}_{\mathrm{m}}\hat{\boldsymbol{e}}_1 & \bar{m}\Delta\boldsymbol{e}_2 \\ -\bar{m}\boldsymbol{r}_{\mathrm{m}}\hat{\boldsymbol{e}}_1 & m\mathbb{I} & \bar{m}\boldsymbol{e}_1 \\ \bar{m}\Delta\boldsymbol{e}_2^T & \bar{m}\boldsymbol{e}_1^T & \bar{m} \end{pmatrix}, \tag{1}$$

$\mathbb{I}$ represents the $3\times3$ identity matrix, and we have defined

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The momentum conjugate to $\boldsymbol{\eta}_1$ is

$$\boldsymbol{\nu}_1 = \frac{\partial T_1}{\partial \boldsymbol{\eta}_1} = \begin{pmatrix} \boldsymbol{\Pi} \\ \boldsymbol{P} \\ P_{\mathrm{m}_1} \end{pmatrix} = \mathbb{M}_1\boldsymbol{\eta}_1.$$

That is,

$$
\begin{aligned}
\boldsymbol{\Pi} &= \boldsymbol{J}(r_{\mathrm{m}_1})\boldsymbol{\Omega} + \bar{m}\boldsymbol{r}_{\mathrm{m}}\hat{\boldsymbol{e}}_1\boldsymbol{v} + \bar{m}\Delta\dot{r}_{\mathrm{m}_1}\boldsymbol{e}_2 \\
\boldsymbol{P} &= -\bar{m}\boldsymbol{r}_{\mathrm{m}}\hat{\boldsymbol{e}}_1\boldsymbol{\Omega} + m\mathbb{I}\boldsymbol{v} + \bar{m}\dot{r}_{\mathrm{m}_1}\boldsymbol{e}_1 \\
P_{\mathrm{m}_1} &= \bar{m}\left(\Delta\boldsymbol{e}_2\cdot\boldsymbol{\Omega} + \boldsymbol{e}_1\cdot\boldsymbol{v} + \dot{r}_{\mathrm{m}_1}\right).
\end{aligned}
$$

The vector $\boldsymbol{\Pi}$ represents the total angular momentum of the system about the body coordinate origin, written in body coordinates. Similarly, $\boldsymbol{P}$ represents the total translational momentum in body coordinates. The term $P_{\mathrm{m}_1}$ represents the momentum of the moving point mass in the 1-axis direction, written in body coordinates.

Specializing the results of [12] to the case of a one degree of freedom point mass, the system dynamics are

$$
\begin{pmatrix} \dot{\boldsymbol{\Pi}} \\ \dot{\boldsymbol{P}} \\ \dot{r}_{\mathrm{m}_1} \\ \dot{P}_{\mathrm{m}_1} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\Pi}} & \hat{\boldsymbol{P}} & \mathbf{0} & \mathbf{0} \\ \hat{\boldsymbol{P}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & 1 \\ \mathbf{0} & \mathbf{0} & -1 & 0 \end{pmatrix} \nabla H_1 + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 1 \end{pmatrix} u, \tag{2}
$$

where $u$ is the internal force applied to the moving mass in the positive 1-axis direction, and where

$$
H_1(\boldsymbol{\Pi}, \boldsymbol{P}, r_{\mathrm{m}_1}, P_{\mathrm{m}_1}) = \frac{1}{2}\boldsymbol{\nu}_1 \cdot \mathbb{M}_1^{-1}\boldsymbol{\nu}_1.
$$

Here, we have assumed that no external forces or moments act on the system.

Note that when $u = 0$, equations (2) form a generalized (noncanonical) Hamiltonian system. The $8 \times 8$ skew-symmetric matrix is the "Poisson tensor" which generalizes the symplectic matrix from classical mechanics. Also note in this case that, in addition to the Hamiltonian $H_1$, the quantities

$$
C_1 = \frac{1}{2}\boldsymbol{P}\cdot\boldsymbol{P} \quad \text{and} \quad C_2 = \boldsymbol{\Pi}\cdot\boldsymbol{P}
$$

are conserved. These conserved quantities, called Casimir functions, can be useful in stability analysis. In fact, one may check that the Casimir functions are conserved regardless of the choice of $u$, reflecting conservation of inertial angular and translational momentum for the internally actuated system.

**Remark 2.1** *If one applies an internal force*

$$
u = -\frac{d\phi}{dr_{\mathrm{m}_1}}
$$

*to the point mass for some "artificial potential" function* $\phi(r_{\mathrm{m}_1})$*, then the system is Hamiltonian with respect to* $H = \frac{1}{2}\boldsymbol{\nu}_1 \cdot \mathbb{M}_1^{-1}\boldsymbol{\nu}_1 + \phi(r_{\mathrm{m}_1})$.

## 2.2 Case 2

Now define $\boldsymbol{\eta}_2 = [\boldsymbol{\Omega}^T, \boldsymbol{v}^T, v_{\mathrm{m}_1}]^T$. Note that the relative velocity $\dot{r}_{\mathrm{m}_1}$ of the moving point mass appearing in $\boldsymbol{\eta}_1$ has been replaced by its inertial velocity $v_{\mathrm{m}_1}$. Let

$$\mathbb{I}_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and define the inertia matrix

$$\bar{\boldsymbol{J}}(r_{\mathrm{m}_1}) = \tilde{\boldsymbol{J}} - \bar{m}\widehat{\boldsymbol{r}}_{\mathrm{f}}^{\,2} - \bar{m}\widehat{\boldsymbol{r}_{\mathrm{m}}}\mathbb{I}_{23}\widehat{\boldsymbol{r}_{\mathrm{m}}} = \begin{pmatrix} \bar{J}_1 & 0 & -\bar{m}\Delta r_{\mathrm{m}_1} \\ 0 & \bar{J}_2 + \bar{m}r_{\mathrm{m}_1}^2 & 0 \\ -\bar{m}\Delta r_{\mathrm{m}_1} & 0 & \bar{J}_3 + \bar{m}r_{\mathrm{m}_1}^2 \end{pmatrix}$$

where

$$\begin{aligned} \bar{J}_1 &= \tilde{J}_1 + 2\bar{m}\Delta^2 \\ \bar{J}_2 &= \tilde{J}_2 + \bar{m}\Delta^2 \\ \bar{J}_3 &= \tilde{J}_3. \end{aligned}$$

The system kinetic energy may be rewritten as

$$T_2 = \frac{1}{2}\boldsymbol{\eta}_2 \cdot \mathbb{M}_2\boldsymbol{\eta}_2, \qquad \text{where} \qquad \mathbb{M}_2 = \begin{pmatrix} \bar{\boldsymbol{J}}(r_{\mathrm{m}_1}) & \bar{m}\left(\widehat{\boldsymbol{r}}_{\mathrm{f}} + \widehat{\boldsymbol{r}}_{\mathrm{m}}\mathbb{I}_{23}\right) & \mathbf{0} \\ -\bar{m}\left(\widehat{\boldsymbol{r}}_{\mathrm{f}} + \mathbb{I}_{23}\widehat{\boldsymbol{r}_{\mathrm{m}}}\right) & m\mathbb{I} - \bar{m}\boldsymbol{e}_1\boldsymbol{e}_1^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{m} \end{pmatrix}. \tag{3}$$

The momentum conjugate to $\boldsymbol{\eta}_2$ is

$$\boldsymbol{\nu}_2 = \frac{\partial T_2}{\partial \boldsymbol{\eta}_2} = \begin{pmatrix} \bar{\boldsymbol{\Pi}} \\ \bar{\boldsymbol{P}} \\ \bar{P}_{\mathrm{m}_1} \end{pmatrix} = \mathbb{M}_2\boldsymbol{\eta}_2.$$

It can easily be checked that

$$\begin{aligned} \bar{\boldsymbol{\Pi}} &= \boldsymbol{\Pi} - \boldsymbol{r}_{\mathrm{m}} \times P_{\mathrm{m}_1}\boldsymbol{e}_1 \\ \bar{\boldsymbol{P}} &= \boldsymbol{P} - P_{\mathrm{m}_1}\boldsymbol{e}_1 \\ \bar{P}_{\mathrm{m}_1} &= P_{\mathrm{m}_1}. \end{aligned}$$

The vector $\bar{\boldsymbol{\Pi}}$ represents the total angular momentum of the system about the body coordinate origin, less that component due to the 1-axis momentum of the moving point mass. Similarly, $\bar{\boldsymbol{P}}$ represents the total translational momentum less the component due to the 1-axis momentum of the moving point mass. Finally, $\bar{P}_{\mathrm{m}_1}$ represents the momentum of the moving point mass in the 1-axis direction, written in body coordinates.

Specializing the results of [12], the system dynamics in the absence of dissipation are

$$\begin{pmatrix} \dot{\bar{\mathbf{\Pi}}} \\ \dot{\bar{\boldsymbol{P}}} \\ \dot{r}_{\mathrm{m}_1} \\ \dot{\bar{P}}_{\mathrm{m}_1} \end{pmatrix} = \begin{pmatrix} \hat{\bar{\mathbf{\Pi}}} + \widehat{\boldsymbol{r}_{\mathrm{m}} \times \bar{P}_{\mathrm{m}_1} \boldsymbol{e}_1} & \hat{\boldsymbol{P}} + \bar{P}_{\mathrm{m}_1} \hat{\boldsymbol{e}}_1 & \Delta \boldsymbol{e}_2 & \mathbf{0} \\ \hat{\boldsymbol{P}} + \bar{P}_{\mathrm{m}_1} \hat{\boldsymbol{e}}_1 & \mathbf{0} & \boldsymbol{e}_1 & \mathbf{0} \\ -\Delta \boldsymbol{e}_2^T & -\boldsymbol{e}_1^T & 0 & 1 \\ \mathbf{0} & \mathbf{0} & -1 & 0 \end{pmatrix} \nabla H_2 + \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 1 \end{pmatrix} u,$$

where

$$H_2(\bar{\mathbf{\Pi}}, \bar{\boldsymbol{P}}, r_{\mathrm{m}_1}, \bar{P}_{\mathrm{m}_1}) = \frac{1}{2} \boldsymbol{\nu}_2 \cdot \mathbb{M}_2^{-1} \boldsymbol{\nu}_2.$$

In addition to the Hamiltonian $H_2$, the Casimir functions

$$\bar{C}_1 = \frac{1}{2} (\bar{\boldsymbol{P}} + \bar{P}_{\mathrm{m}_1} \boldsymbol{e}_1) \cdot (\bar{\boldsymbol{P}} + \bar{P}_{\mathrm{m}_1} \boldsymbol{e}_1) \quad \text{and} \quad \bar{C}_2 = (\bar{\mathbf{\Pi}} + \boldsymbol{r}_{\mathrm{m}} \times \bar{P}_{\mathrm{m}_1} \boldsymbol{e}_1) \cdot (\bar{\boldsymbol{P}} + \bar{P}_{\mathrm{m}_1} \boldsymbol{e}_1)$$

are conserved. These functions correspond to the Casimirs $C_1$ and $C_2$ in Case 1.

# 3 Stability of Steady Rotation

One application for the models described in Section 2 is to investigate the effect of unmodeled dynamics on spacecraft motion. For example, suppose we let

$$u = -k r_{\mathrm{m}_1} \tag{4}$$

where $k > 0$ is a spring stiffness corresponding, for example, to the lowest natural frequency of a flexible appendage. Note that this control law can be written as the negative gradient of a scalar potential,

$$u = -\frac{d\phi}{dr_{\mathrm{m}_1}}$$

where $\phi = \frac{1}{2} k r_{\mathrm{m}_1}^2$. In the setting of Section 2.1, the "closed-loop" dynamics are Hamiltonian with respect to $H = \frac{1}{2} \boldsymbol{\nu}_1 \cdot \mathbb{M}_1^{-1} \boldsymbol{\nu}_1 + \phi$.

We are interested in determining conditions under which the equilibrium

$$\mathbf{\Pi}|_e = \Pi_1^0 \boldsymbol{e}_1, \qquad \boldsymbol{P}|_e = \mathbf{0}, \qquad r_{\mathrm{m}_1}|_e = 0, \qquad P_{\mathrm{m}_1}|_e = 0, \tag{5}$$

which corresponds to steady rotation about the 1-axis, is stable. This is a "nongeneric" equilibrium. When $\boldsymbol{P} = \mathbf{0}$, the rank of the Poisson tensor drops from six to five. Correspondingly, the dynamics evolve on a lower-dimensional invariant surface than when $\boldsymbol{P} \neq \mathbf{0}$. (Recall that $\|\boldsymbol{P}\|$ is conserved, so if $\boldsymbol{P} = \mathbf{0}$ initially, it remains so for all time.) In this case, there exists a special conserved quantity, known as a "subcasimir"

$$C_3 = \frac{1}{2} \mathbf{\Pi} \cdot \mathbf{\Pi}.$$

One may study nonlinear stability of (5) using the energy-Casimir method. However, the stability results may only be valid when the dynamics are restricted to the lower-dimensional "invariant leaf" corresponding to the special case $\boldsymbol{P} = \mathbf{0}$. In this case, one says that the equilibrium is "leafwise stable." One may apply the results of [5] to extend conclusions about stability to the entire phase space.

Proving nonlinear stability using the energy-Casimir method involves constructing a function $H_\Phi = H + \Phi(C_1, C_2, C_3)$ which has a minimum or a maximum at the equilibrium of interest. One first requires that the derivative of $H_\Phi$ vanishes at the equilibrium. Let

$$\Phi^i = \frac{\partial \Phi}{\partial C_i} \qquad \text{and} \qquad \Phi^{ij} = \frac{\partial^2 \Phi}{\partial C_i \partial C_j}.$$

The derivative of $H_\Phi(\mathbf{\Pi}, \boldsymbol{P}, P_{\mathrm{m}_1}, r_{\mathrm{m}_1})$ is

$$DH_\Phi = \begin{pmatrix} \mathbf{\Omega} + \Phi^2 \boldsymbol{P} + \Phi^3 \mathbf{\Pi} \\ \boldsymbol{v} + \Phi^1 \boldsymbol{P} + \Phi^2 \mathbf{\Pi} \\ \dot{r}_{\mathrm{m}_1} \\ \boldsymbol{e}_1 \cdot (-\boldsymbol{P}_{\mathrm{m}} \times \mathbf{\Omega}) + k r_{\mathrm{m}_1} \end{pmatrix},$$

which vanishes at the equilibrium (5) provided

$$\Phi^2_e = 0 \qquad \text{and} \qquad \Phi^3_e = -\frac{1}{J_1}. \tag{6}$$

(The order of $P_{\mathrm{m}_1}$ and $r_{\mathrm{m}_1}$ has been reversed in the argument list for $H_\Phi$ in order to simplify the analysis.)

One next requires that the second derivative of $H_\Phi$ be (positive or negative) definite when evaluated at the equilibrium. Evaluating the matrix of the second derivative of $H_\Phi$ at the equilibrium, and substituting from (6), one obtains

$$\begin{pmatrix}
-\frac{(\Pi_1^0)^2}{J_1} & 0 & 0 & \Phi_e^{23}(\Pi_1^0)^2 & 0 & 0 & 0 & 0 \\
0 & \frac{m-\bar{m}}{\alpha} - \frac{1}{J_1} & 0 & \frac{\bar{m}\Delta}{\alpha} & 0 & 0 & -\frac{\bar{m}\Delta}{\alpha} & 0 \\
0 & 0 & \frac{1}{J_3} - \frac{1}{J_1} & 0 & 0 & 0 & 0 & \frac{\bar{m}\Delta\Pi_1^0}{J_1 J_3} \\
\Phi_e^{23}(\Pi_1^0)^2 & \frac{\bar{m}\Delta}{\alpha} & 0 & a_{44} & 0 & 0 & -\frac{J_2}{\alpha} & 0 \\
0 & 0 & 0 & 0 & \frac{1}{m} + \Phi_e^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{m} + \Phi_e^1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{mJ_2}{\bar{m}\alpha} & 0 \\
0 & 0 & \frac{\bar{m}\Delta\Pi_1^0}{J_1 J_3} & 0 & 0 & 0 & 0 & a_{88}
\end{pmatrix}$$

where

$$\begin{aligned}
\alpha &= (m - \bar{m})J_2 - m\bar{m}\Delta^2 \\
a_{44} &= \frac{J_2 - \bar{m}\Delta^2}{\alpha} + \Phi_e^1 + \Phi_e^{22}(\Pi_1^0)^2, \qquad \text{and} \\
a_{88} &= \frac{(\bar{m}\Delta\Pi_1^0)^2}{J_3 J_1^2} + k.
\end{aligned}$$

Consider the case where the axis of rotation is the major axis; that is, $J_1 > J_2$ and $J_1 > J_3$. It is straightforward to check that choosing the function $\Phi$ to satisfy

$$\Phi_e^{23} = 0 \quad \text{and} \quad \Phi_e^1 + \Phi_e^{22}(\Pi_1^0)^2 > \frac{1}{\alpha}\left(-(J_2 - \bar{m}\Delta^2)^2 + \frac{J_1(\bar{m}\Delta)^2}{(m-\bar{m})J_1 - \alpha}\right)$$

makes the second derivative of $H_\Phi$ positive definite at the equilibrium provided

$$k > \frac{(\bar{m}\Delta\Pi_1^0)^2}{J_1^2(J_1 - J_3)}. \tag{7}$$

For such a choice of $\Phi$, the equilibrium is a minimum of $H_\Phi$. Thus, by Lyapunov's stability theorem, steady major axis rotation is stable provided the spring is sufficiently stiff.

To sharpen the stability condition, one may consider the linearized dynamics. The characteristic polynomial of the system linearized about (5) is

$$\begin{aligned}(\bar{m}\alpha J_1^2 J_3)\,\lambda^4 + (\bar{m}(\Pi_1^0)^2(J_1 - J_2)((m-\bar{m})(J_1 - J_3) - m\bar{m}\Delta^2 J_3) + kmJ_1^2 J_2 J_3)\,\lambda^2 \\ + \tfrac{m(\Pi_1^0)^2}{J_1^2}(J_1 - J_2)\,(kJ_1^2(J_1 - J_3) - (\bar{m}\Delta\Pi_1^0)^2)\,. \end{aligned} \tag{8}$$

One can easily see from the last term, i.e., the coefficient of $\lambda^0$, that condition (7) is both necessary and sufficient, disregarding the case of equality.

**Theorem 3.1** *If $J_1 > J_2$ and $J_2 > J_3$, then the equilibrium (5) of the dynamics (2) with (4) is leafwise stable if and only if*

$$k > \frac{(\bar{m}\Delta\Pi_1^0)^2}{J_1^2(J_1 - J_3)}.$$

Referring to the characteristic polynomial (8), one may also show that the following conditions are sufficient for *instability* of the equilibrium (5) when $k > 0$:

*i)* $J_2 < J_1 < J_3$
*ii)* $(J_2 - J_1) < J_3 < J_1 < J_2$
*iii)* $J_1 < J_2$, $J_1 < J_3$, and
$m\bar{m}\Delta^2(J_1 + J_3 - J_2) > (m - \bar{m})(J_2 - J_1)(J_3 - J_1) + kmJ_1^2 J_2 J_3$.

These conditions follow from a rudimentary spectral stability analysis; a more thorough analysis may yield sharper conditions.

**Remark 3.2** *Condition (iii) above may have interesting implications for the demise of Explorer I, which is widely blamed on energy dissipation due to internal damping [6]. Explorer I was a prolate, axisymmetric spacecraft with four radial whip antennae. It was intended that the spacecraft rotate about its axis of symmetry, i.e., the minor axis, however the spacecraft quickly tumbled away from the desired equilibrium. Modeling this system as in Section 2.1, with three of the antennae rigidly fixed, condition (iii) above becomes*

$$\frac{\bar{m}\Delta^2}{J_1} > \frac{m - \bar{m}}{m}\left(\frac{J_2 - J_1}{J_1}\right)^2 + kJ_2^2.$$

*This condition is met when $\Delta$ is large, $J_1 \approx J_2$, and $k$ is small. Thus, minor axis rotation of a rigid body with a flexible mode can be unstable even in the absence of dissipation.*

# 4 Concluding Remarks

Two noncanonical Hamiltonian models have been presented for the dynamics of a rigid body with a single moving point mass. One of these models was used to obtain conditions for stability of steady principal axis rotation of the rigid body. The analysis gives necessary and sufficient conditions for stability of steady major axis rotation, as well as sufficient conditions for instability of steady intermediate and minor axis rotation. The latter result implies that minor axis rotation of a rigid body can be destabilized by a simple point mass oscillator, even without any damping in the system.

The models presented here can easily be extended to allow for multiple moving masses. In [11], a very similar system was presented in which both point masses are controlled in the 1-axis direction. A control law was developed, based on the linearized dynamics, which succeeded in (locally) asymptotically stabilizing steady minor axis rotation. This approach was dubbed "principal axis misalignment." Ongoing research may reveal a more general form of principal axis misalignment in terms of kinetic-shaping feedback.

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