Matching and stabilization of constrained systems

Guido Blankenstein
Département de Mathématiques
École Polytechnique Fédérale de Lausanne
MA-Ecublens, 1015 Lausanne
Switzerland

Abstract

In this paper we discuss the stabilization by means of structure preserving feedback laws (i.e., *matching*) of constrained systems described as implicit port-controlled Hamiltonian systems. The theory is applied to underactuated mechanical systems with kinematic constraints.

1 Introduction

In recent literature new methods have been described for the stabilization of underactuated mechanical systems. The key idea of these methods is to look for stabilizing feedback controllers which preserve the mathematical structure of the system. The *controlled Lagrangians method* [5, 6, 10, 7, 1] considers underactuated mechanical systems described as Euler-Lagrange systems and looks for controllers which preserve the Euler-Lagrange format in closed-loop. The conditions under which such a controller exists are called matching conditions. The method has been extended to general Euler-Lagrange systems in [11]. On the Hamiltonian side an analogous method has been described for mechanical systems in [14]. This method originates from the more general *interconnection and damping assignment passivity based control (IDA-PBC) method* as described in [15, 16]. The latter is applicable to the general class of port-controlled Hamiltonian systems, including mechanical systems but also electrical or electromechanical systems (e.g. power converters, synchronous motors, see [15, 16] for examples). We refer to [3, 4, 8] for a discussion on the connections between the two methods.

In this paper we present the interconnection and damping assignment passivity based control method for constrained systems. We consider the class of *implicit* port-controlled Hamiltonian systems and derive the matching conditions for these systems. The result is applied to the class of underactuated mechanical systems with (possibly nonholonomic) kinematic constraints. Our exposition will be closely related to the work of [12] on the controlled Lagrangians method for constrained mechanical systems, see also [20] for some work in this direction.

2 Constrained systems

In this paper we consider constrained systems described as implicit port-controlled Hamiltonian systems of the following form

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) + g(x)\lambda + b(x)u, \tag{2.1}$$

$$0 = g^T(x)\frac{\partial H}{\partial x}(x), \tag{2.2}$$

$$y = b^T(x)\frac{\partial H}{\partial x}(x), \tag{2.3}$$

where $x \in \mathcal{X}$ (the state-space manifold), $J : T^*\mathcal{X} \to T\mathcal{X}$ is a skew-symmetric vector bundle map, $g(x) : \mathbb{R}^l \to T_x\mathcal{X}$ the (independent) constrained vector fields (i.e., $g(x)\lambda$ ($\lambda \in \mathbb{R}^l$) are the reaction forces), $u \in \mathcal{F}$ (the input-space, assumed to be a vector space), $b(x) : \mathcal{F} \to T_x\mathcal{X}$ represents the (independent) input vector fields, $y \in \mathcal{F}^*$ are the ouputs. $H(x) \in C^\infty(\mathcal{X})$ is the Hamiltonian, or energy, function of the system.

Remark 2.1. The system (2.1)–(2.3) can be written in a coordinate free way as an implicit port-controlled Hamiltonian system $(\mathcal{X}, \mathcal{F}, \mathcal{D}, H)$, where the geometric structure corresponding to J, g, b is described by a so-called *Dirac structure* $\mathcal{D}$. We refer to [17, 2] and references therein for more details.

The constraints are given by (2.2) and define the constraint manifold

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid g^T(x)\frac{\partial H}{\partial x}(x) = 0\}. \tag{2.4}$$

We assume that the constraints satisfy the following assumption:

Assumption 2.1. *Let $g(x) = [g_1(x), \ldots, g_l(x)]$, with $g_1(x), \ldots, g_l(x)$ independent vector fields over $\mathcal{X}$. Assume that the $l \times l$ matrix $[L_{g_i} L_{g_j} H(x)]_{i,j=1,\ldots,l}$ is invertible for all $x \in \mathcal{X}_c$.*

Define the (full rank) distributions $G_0(x) = \mathrm{Im}\ g(x) \subset T_x\mathcal{X}$ and $B(x) = \mathrm{Im}\ b(x) \subset T_x\mathcal{X}$.

Assumption 2.2. *We assume that $(G_0 \cap B) \mid_{\mathcal{X}_c} = \{0\}$, that is, the control directions do not coincide with the directions of the constraint forces.*

As shown in [19, 2] under Assumption 2.1 the constraints can be eliminated and the system (2.1)-(2.3) can be written as an *explicit* generalized Hamiltonian system on $\mathcal{X}_c$ in the following way:

In [19, 2] it is shown that Assumption 2.1 implies that $T_{x_c}\mathcal{X}_c \cap G_0(x_c) = \{0\}, \forall x_c \in \mathcal{X}_c$. Therefore, the tangent bundle to $\mathcal{X}$ restricted to $\mathcal{X}_c$ can be written as a direct sum

$$T_{x_c}\mathcal{X} = T_{x_c}\mathcal{X}_c \oplus G_0(x_c), \quad \forall x_c \in \mathcal{X}_c. \tag{2.5}$$

Define the projection map $\pi(x_c) : T_{x_c}\mathcal{X} \to T_{x_c}\mathcal{X}_c$ according to the decomposition in (2.5). Dually to (2.5) we have

$$T^*_{x_c}\mathcal{X} = T^*_{x_c}\mathcal{X}_c \oplus (G_0(x_c))^*, \quad \forall x_c \in \mathcal{X}_c, \tag{2.6}$$

which defines a projection $\Pi(x_c) : T^*_{x_c}\mathcal{X} \to T^*_{x_c}\mathcal{X}_c$. Notice that the projection π on the tangent bundle defines the inclusion $\pi^* : T^*\mathcal{X}_c \hookrightarrow T^*\mathcal{X} \mid_{\mathcal{X}_c}$ on the cotangent bundle. In fact,

$$\pi^* \circ \Pi \mid_{(\mathrm{ann}\ G_0|_{\mathcal{X}_c})} = \text{ identity } = \Pi \circ \pi^*. \tag{2.7}$$

Define the skew-symmetric vector bundle map

$$J_c = \pi \circ J \mid_{\mathcal{X}_c} \circ \pi^* : T^*\mathcal{X}_c \to T\mathcal{X}_c. \tag{2.8}$$

By Assumption 2.2 the distribution B projects to a nonzero full rank distribution $B_c = \pi B$ on $\mathcal{X}_c$.

Proposition 2.1. *The system (2.1)-(2.3) when restricted to the constraint manifold can be written as the following explicit generalized Hamiltonian system*

$$\dot{x}_c = J_c(x_c)\frac{\partial H_c}{\partial x_c}(x_c) + b_c(x_c)u, \tag{2.9}$$

$$y = b_c^T(x_c)\frac{\partial H_c}{\partial x_c}(x_c), \tag{2.10}$$

$x_c \in \mathcal{X}_c$*, where* $H_c = H \mid_{\mathcal{X}_c}$*,* b_c *is defined by* $b_c = \pi b$ *so that* $B_c = Im\, b_c$*.*

Proof. Consider (2.1), $X_H - J\, dH - bu \in G_0$, where $\dot{x} = X_H \in T\mathcal{X} \mid_{\mathcal{X}_c}$. Under the projection by π this results in $\pi X_H - \pi J \mid_{\mathcal{X}_c} dH - \pi bu = 0$. The constraint manifold (2.4) can be equivalently defined as

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid dH(x) \in \text{ann}\, G_0(x)\}. \tag{2.11}$$

Let $H_c = H \mid_{\mathcal{X}_c}$ be the restriction of the Hamiltonian to the constrained manifold. Then from (2.11) it follows that $\Pi dH = d(H \mid_{\mathcal{X}_c}) = dH_c$, which by (2.7) implies $dH \mid_{\mathcal{X}_c} = \pi^* dH_c$. Finally, let $X_{H_c} = \pi X_H$, then it follows that $\dot{x}_c = X_{H_c} = J_c dH_c + b_c u$, which equals (2.9). Equation (2.10) follows from $y = \mathbf{i}_b dH = \mathbf{i}_b \pi^* dH_c = \mathbf{i}_{\pi b} dH_c = \mathbf{i}_{b_c} dH_c$. □

Remark 2.2. The system (2.9, 2.10) can be written as a port-controlled Hamiltonian system $(\mathcal{X}_c, \mathcal{F}, \mathcal{D}_c, H_c)$ in a coordinate free way as the restriction of $(\mathcal{X}, \mathcal{F}, \mathcal{D}, H)$ to the constraint manifold X_c, see [2].

3 Matching and stabilization

As stated in the introduction, the principal idea of the controlled Lagrangians and the IDA-PBC method is to consider stabilizing feedback control laws which preserve the mathematical structure of the system. Therefore we assume that the closed loop system is described by

$$\dot{x} = \tilde{J}(x)\frac{\partial \tilde{H}}{\partial x}(x) + \tilde{g}(x)\mu + b(x)v, \tag{3.1}$$

$$0 = \tilde{g}^T(x)\frac{\partial \tilde{H}}{\partial x}(x), \tag{3.2}$$

$$\tilde{y} = b^T(x)\frac{\partial \tilde{H}}{\partial x}(x). \tag{3.3}$$

(For simplicity we restrict our attention to affine static state feedback controllers of the form $u(x, v) = \alpha(x) + v$.) Following [12] we define the notion of *constraint related* systems:

Definition 3.1. *The systems (2.1)–(2.3) and (3.1)–(3.3) are said to be* constraint related *if i) the constraint manifolds coincide, i.e. , (2.2) is equivalent to (3.2), and ii) the distributions* $G_0 = Im\, g$ *and* $\tilde{G}_0 = Im\, \tilde{g}$ *coincide.*

In particular, when the systems are constraint related then the projections π and $\tilde{\pi}$, as well as the projections Π and $\tilde{\Pi}$, are the same, which corresponds to the definition given in [12].

Assume that the first condtion in Definition 3.1 is satisfied, that is, the constraint manifold of (3.1)–(3.3) is given by $\mathcal{X}_c$. Then, under Assumption 2.1, the system (3.1)–(3.3) can be restricted to an explicit generalized Hamiltonian system on $\mathcal{X}_c$ by Proposition 2.1

$$\dot{x}_c = \tilde{J}_c(x_c)\frac{\partial \tilde{H}_c}{\partial x_c}(x_c) + b_c(x_c)v, \tag{3.4}$$

$$\tilde{y} = b_c^T(x_c)\frac{\partial \tilde{H}_c}{\partial x_c}(x_c). \tag{3.5}$$

Definition 3.2. *The systems (2.1)-(2.3) and (3.1)-(3.3) are called* matching *if the restricted systems (2.9, 2.10) and (3.4, 3.5) are matching; that is, there exists a feedback law* $u(x_c, v) = \alpha(x_c) + v$ *such that (2.9, 2.10) becomes the closed loop system (3.4, 3.5).*

Consider the matching conditions for the restricted systems (2.9, 2.10) and (3.4, 3.5). From the above definition it is clear that these systems are matching if and only if there exists an $\alpha(x_c) \in \mathcal{F}$ such that

$$J_c(x_c)\frac{\partial H_c}{\partial x_c}(x_c) + b_c(x_c)\alpha(x_c) = \tilde{J}_c(x_c)\frac{\partial \tilde{H}_c}{\partial x_c}(x_c), \quad \forall x_c \in \mathcal{X}_c. \tag{3.6}$$

Let $b_c^\perp \subset T^*\mathcal{X}_c$ denote a full rank left annihilator of b_c, i.e., $b_c^\perp b_c = 0$, then (3.6) is equivalent to

$$b_c^\perp(x_c)\left(J_c(x_c)\frac{\partial H_c}{\partial x_c}(x_c) - \tilde{J}_c(x_c)\frac{\partial \tilde{H}_c}{\partial x_c}(x_c)\right) = 0. \tag{3.7}$$

These are the matching conditions for the restricted systems (2.9, 2.10) and (3.4, 3.5). They correspond to the matching conditions for explicit port-controlled Hamiltonian systems as described in the IDA-PBC method, see [15].

For constraint related systems we can translate the matching conditions (3.7) into conditions for the original *implicit* port-controlled Hamiltonian systems as follows: Using the results of the previous section (3.7) can be written as

$$(\pi^* b_c^\perp)\left(J\mid_{\mathcal{X}_c} dH\mid_{\mathcal{X}_c} - \tilde{J}\mid_{\mathcal{X}_c} d\tilde{H}\mid_{\mathcal{X}_c}\right) = 0. \tag{3.8}$$

Notice that $\pi^* b_c^\perp \subset T^*\mathcal{X}\mid_{\mathcal{X}_c}$ is a left annihilator of $b\mid_{\mathcal{X}_c}$, i.e., $(\pi^* b_c^\perp)(b\mid_{\mathcal{X}_c}) = b_c^\perp(\pi b) = b_c^\perp b_c = 0$. By construction it also annihilates G_0. Therefore $\pi^* b_c^\perp \subset \left(b^\perp \cap \operatorname{ann} G_0\right)\mid_{\mathcal{X}_c}$, where $b^\perp$ is a full rank left annihilator of b. In fact, a dimension argument learns that equality holds, that is, $\pi^* b_c^\perp = \left(b^\perp \cap \operatorname{ann} G_0\right)\mid_{\mathcal{X}_c}$. Together with (3.8) this yields

$$\left[\left(b^\perp \cap \operatorname{ann} G_0\right)\left(J\frac{\partial H}{\partial x} - \tilde{J}\frac{\partial \tilde{H}}{\partial x}\right)\right]\mid_{\mathcal{X}_c} = 0, \tag{3.9}$$

which are the matching conditions for the implicit systems (2.1)-(2.3) and (3.1)-(3.3). In conclusion:

Proposition 3.1. *The constraint related systems (2.1)-(2.3) and (3.1)-(3.3) are matching if and only if the matching conditions (3.9) hold.*

Remark 3.1. Applied to the class of underactuated mechanical systems with kinematic constraints, Proposition 3.1 yields Theorem 9 in [12]. See also the next section.

If the matching conditions are satisfied then the corresponding feedback law can be obtained from (3.6) as

$$\begin{aligned}\alpha &= (b_c^T b_c)^{-1} b_c^T \left(\tilde{J}_c d\tilde{H}_c - J_c dH_c\right) \\ &= (b_c^T b_c)^{-1} b_c^T \, \pi \left(\tilde{J} d\tilde{H} - J dH\right) |_{\mathcal{X}_c} \, . \end{aligned} \tag{3.10}$$

The control law is then given by $u(x_c, v) = \alpha(x_c) + v$. Notice that the control law is only defined on the constraint manifold $\mathcal{X}_c$, where the actual motion takes place.

Now, let $x^* \in \mathcal{X}_c$ be a stationary point of the original *constrained* Hamiltonian H_c, i.e., $\frac{\partial H_c}{\partial x}(x^*) = 0$. Then x^* is an equilibrium point of (2.9, 2.10), or equivalenlty (2.1)-(2.3), with $u = 0$, which possibly is unstable. If we can find a matching control law such that x^* is a *strict* local minimum for the new constrained Hamiltonian $\tilde{H}_c$, then x^* is a Lyapunov stable equilibrium point of the closed loop dynamics (3.1)-(3.3), with $v = 0$. Indeed, the Lyapunov function is given by $\tilde{H}_c$ and satisfies $\frac{d}{dt}\tilde{H}_c(x(t)) = \frac{d}{dt}\tilde{H}(x(t)) = 0$ along solutions of (3.1)-(3.3) with $v = 0$. Furthermore, if we apply the negative output feedback $v = -\tilde{y} = -b^T \frac{\partial \tilde{H}}{\partial x}$, then we get the energy balance

$$\frac{d}{dt}\tilde{H}_c(x(t)) = \frac{d}{dt}\tilde{H}(x(t)) = -\frac{\partial \tilde{H}}{\partial x}^T (x(t)) b(x(t)) b^T(x(t)) \frac{\partial \tilde{H}}{\partial x}(x(t)) \le 0, \tag{3.11}$$

which shows that the energy of the system is monotonically decreasing along the trajectories of the system. By LaSalle's theorem any trajectory starting close enough to x^* will converge to the largest invariant set Ω_{inv} (with respect to (3.1)-(3.3) with $v = 0$) contained in

$$\Omega = \{x \in \mathcal{X}_c \mid b^T(x) \frac{\partial \tilde{H}}{\partial x}(x) = 0\}. \tag{3.12}$$

If Ω_{inv} turns out to be exactly x^*, then x^* is an asymptotically stable equilibrium point of the closed loop system.

In conclusion, we have obtained the following result, which can be regarded as an extension of the *interconnection and damping assignment passivity based control (IDA-PBC) method* [15, 16] to the class of constrained systems.

Theorem 3.1. *Consider the constrained system (2.1)-(2.3), and let $x^* \in \mathcal{X}_c$ be a stationary point of the Hamiltonian $H_c = H |_{\mathcal{X}_c}$. Suppose we can find a function $\tilde{H}$ and a matrix $\tilde{J}$, such that (3.2) with $\tilde{g}(x) \triangleq g(x)$ is equivalent to (2.2) and such that the matching conditions (3.9) hold. Furthermore, suppose that we can choose $\tilde{H}$ in such a way that x^* is a strict local minimum of $\tilde{H}_c = \tilde{H} |_{\mathcal{X}_c}$. Then the state feedback law*

$$u(x_c) = \alpha(x_c) - b^T(x_c) \frac{\partial \tilde{H}}{\partial x}(x_c), \quad x_c \in \mathcal{X}_c, \tag{3.13}$$

with $\alpha(x_c)$ defined by (3.10), stabilizes the equilibrium point x^*. In other words, x^* is a Lyapunov stable equilibrium point of the closed loop system. Furthermore, all trajectories starting close enough to x^* will converge to the largest invariant subset Ω_{inv} in Ω (3.12).

Remark 3.2. Applying negative output feedback is equivalent to the introduction of damping in the system. Indeed, (3.1)-(3.3) with $v = -\tilde{y} = -b^T \frac{\partial \tilde{H}}{\partial x}$ can be equivalently written as

$$\dot{x} = (\tilde{J}(x) - R(x)) \frac{\partial \tilde{H}}{\partial x}(x) + \tilde{g}(x)\mu, \tag{3.14}$$

$$0 = \tilde{g}^T(x) \frac{\partial \tilde{H}}{\partial x}(x), \tag{3.15}$$

(leaving out the outputs (3.3)), where $R(x) = b(x)b^T(x) \geq 0$. The introduction of the dissipation (or, damping) matrix R is called *damping assignment.*

4 Constrained mechanical systems

In this section we will apply the theory to underactuated mechanical systems with kinematic constraints. The Hamiltonian, or energy, function is given by the sum of kinetic and potential energy

$$H(q,p) = \frac{1}{2} p^T M^{-1}(q) p + V(q), \tag{4.1}$$

where $q \in \mathbb{R}^n$ are the configuration coordinates and $p = M(q)\dot{q} \in \mathbb{R}^n$ are the generalized momenta, and $M(q) = M^T(q)$ describes the generalized mass matrix of the system, assumed to be positive definite. The system is underactuated in the sense that the externally supplied forces F_{external} are assumed to lie in the image of a full rank matrix $B(q) : \mathbb{R}^l \to \mathbb{R}^n$ describing the admissible force fields. Suppose that the system has to satisfy the kinematic constraints described by $A^T(q)\dot{q} = 0$, where $A(q)$ is a full rank $k \times n$ matrix ($k < n$). Depending on the matrix $A(q)$ the constraints can be holonomic or nonholonomic. If we assume that the constraints are ideal, i.e., produce no work, then the constraints generate constraint forces $F_{\text{constraint}} \in \text{Im } A(q)$. The system can be described as an implicit port-controlled Hamiltonian system in the following way

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \partial_q H \\ \partial_p H \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} u + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda, \tag{4.2}$$

$$0 = [0 \quad A^T(q)] \begin{bmatrix} \partial_q H \\ \partial_p H \end{bmatrix}, \tag{4.3}$$

$$y = [0 \quad B^T(q)] \begin{bmatrix} \partial_q H \\ \partial_p H \end{bmatrix}. \tag{4.4}$$

The constraint manifold is defined by

$$X_c = \{(q,p) \in \mathbb{R}^n \times \mathbb{R}^n \mid A^T(q) M^{-1}(q) p = 0\}. \tag{4.5}$$

Since in this case the matrix $[L_{g_i} L_{g_j} H(x)]_{i,j=1,\ldots,l}$ is given by $A^T(q)M^{-1}(q)A(q)$, which has full rank, Assumption 2.1 is satisfied and we can restrict the system (4.2)–(4.4) to an explicit generalized Hamiltonian system of the form (2.9, 2.10). Define the new coordinates $(q, \hat{p}_1, \hat{p}_2) = (q, S^T(q)p, A^T(q)p) \in \mathbb{R}^n \times \mathbb{R}^{n-k} \times \mathbb{R}^k$, where $S(q)$ is any full rank $n \times (n-k)$ matrix such that $A^T(q)S(q) = 0$. Then the constraints are given by $\frac{\partial \hat{H}}{\partial \hat{p}_2}(q, \hat{p}_1, \hat{p}_2) = 0$, where $\hat{H}(q, \hat{p}_1, \hat{p}_2) = H(q,p)$.

By Assumption 2.1 and the Implicit Function Theorem, on the constraint manifold $\hat{p}_2$ can be written as a function of $q, \hat{p}_1$. It follows that $(q, \hat{p}_1)$ are local coordinates for $\mathcal{X}_c$. Because of (4.1) it follows that the constrained Hamiltonian $H_c = H \mid_{\mathcal{X}_c}$ has the form

$$H_c(q, \hat{p}_1) = \frac{1}{2}\hat{p}_1^T M_c^{-1}(q)\hat{p}_1 + V(q), \quad M_c(q) > 0. \tag{4.6}$$

In [18, 13] it is shown that the restricted skew-symmetric matrix J_c and the restricted input vector fields b_c have the form

$$J_c(q, \hat{p}_1) = \begin{bmatrix} 0 & S(q) \\ -S^T(q) & (-p^T[S_i, S_j](q))_{i,j} \end{bmatrix}, \quad b_c(q) = \begin{bmatrix} 0 \\ S^T(q)B(q) \end{bmatrix}, \tag{4.7}$$

where p is expressed as a function of $q, \hat{p}_1$, and $[S_i, S_j](q)$ denotes the Lie bracket between the i-th and j-th column of the matrix $S(q)$. Furthermore, in [18] it has been shown that the Poisson bracket corresponding to the structure matrix J_c satisfies the Jacobi identities (i.e., the integrability conditions) if and only if the kinematic constraints are holonomic.

4.1 Matching of constrained mechanical systems

According to the principal idea of the IDA-PBC method, consider a closed loop kinematically constrained underactuated mechanical system with new energy function given by

$$\tilde{H}(q, p) = \frac{1}{2}p^T\tilde{M}^{-1}(q)p + \tilde{V}(q), \tag{4.8}$$

where $\tilde{M}(q) = \tilde{M}^T(q)$ is the new generalized mass matrix, assumed to be positive definite, and $\tilde{V}(q)$ the new potential energy function. The closed loop system should also satisfy the kinematic constraints $A^T(q)\dot{q} = 0$ which can be written as $A^T(q)M^{-1}(q)\tilde{M}(q)\partial_p\tilde{H}(q, p) = 0$. Since in (4.2) q is a nonactuated coordinate, the relation $\dot{q} = M^{-1}(q)p$ should also hold in closed loop. It follows that the closed loop system can be written in the form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & M^{-1}\tilde{M} \\ -\tilde{M}M^{-1} & J_2(q, p) \end{bmatrix} \begin{bmatrix} \partial_q\tilde{H} \\ \partial_p\tilde{H} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} v + \begin{bmatrix} 0 \\ \tilde{M}M^{-1}A \end{bmatrix} \mu, \tag{4.9}$$

$$0 = [0 \quad A^T M^{-1}\tilde{M}] \begin{bmatrix} \partial_q\tilde{H} \\ \partial_p\tilde{H} \end{bmatrix}, \tag{4.10}$$

$$\tilde{y} = [0 \quad B^T] \begin{bmatrix} \partial_q\tilde{H} \\ \partial_p\tilde{H} \end{bmatrix}, \tag{4.11}$$

where $J_2(q, p)$ is an arbitrary skew-symmetric matrix. Since Assumption 2.1 holds, the system (4.9)–(4.11) can be written as an explicit generalized Hamiltonian system of the form (2.9, 2.10) on the constraint manifold (4.5). The expression for $\tilde{J}_c$, analogously to (4.7) (and written in the coordinates $q, \hat{p}_1$!), is quite involved and will not be given here. The restricted systems are matching if the matching conditions (3.7) hold.

However, if the systems (4.2)–(4.4) and (4.9)–(4.11) are constraint related, then we can use Proposition 3.1 to obtain a much simpler form of the matching conditions. Since by construction the constraint manifolds are equal, i.e., (4.3) is equivalent to (4.10), the systems are constraint related if and only if $G_0 = \tilde{G}_0$, i.e., if and only if

$$\text{Im } A(q) = \text{Im } \tilde{M}(q)M^{-1}(q)A(q). \tag{4.12}$$

Notice that (4.12) gives quite a strong condition on $\tilde{M}$, see also Example 4.2 at the end of this paper of a nonholonomically constrained particle stabilized by a control law corresponding to a closed loop constraint related mechanical system.

Next, let us work out explicitly the matching conditions (3.9). Let $\Lambda(q)$ be a full rank matrix such that $\text{Im } \Lambda(q) = \text{Im } A(q) \oplus \text{Im } B(q)$, and let $\Lambda^{\perp}(q)$ denote a full rank (matrix) left annihilator of $\Lambda(q)$. Then $b^{\perp} \cap \text{ann } G_0 = \text{ann } (\text{Im } b \oplus G_0) = \begin{bmatrix} I_n & 0 \\ 0 & \Lambda^{\perp}(q) \end{bmatrix}$. The matching conditions (3.9) become

$$\begin{bmatrix} I_n & 0 \\ 0 & \Lambda^{\perp}(q) \end{bmatrix} \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \partial_q H \\ \partial_p H \end{bmatrix} - \begin{bmatrix} 0 & M^{-1}\tilde{M} \\ -\tilde{M}M^{-1} & J_2(q,p) \end{bmatrix} \begin{bmatrix} \partial_q \tilde{H} \\ \partial_p \tilde{H} \end{bmatrix} \right) = 0, \quad \forall (q,p) \in \mathcal{X}_c. \tag{4.13}$$

Since the first line of (4.13) is trivially satisfied, the matching conditions take the form

$$\Lambda^{\perp} \left(\partial_q H - \tilde{M}M^{-1}\partial_q \tilde{H} + J_2 \tilde{M}^{-1} p \right) = 0, \quad \forall (q,p) \in \mathcal{X}_c. \tag{4.14}$$

The matching conditions (4.14) are very much alike the matching conditions for *un*constrained mechanical systems as described in [14]. The conditions (4.14) represent the Hamiltonian analogue of the matching conditions given in Theorem 9 [12]. Now notice that for each $q \in \mathbb{R}^n$ the state $(q, p = 0) \in \mathcal{X}_c$. Then as in the case of unconstrained mechanical systems, using (4.1) and (4.8) the matching conditions (4.14) can be equivalenlty written as a set of the following two equations (representing the p-dependent, respectively the p-independent, part of (4.14))

$$\begin{aligned} \Lambda^{\perp}(q) \Big(\partial_q (\tfrac{1}{2} p^T M^{-1}(q) p) - \tilde{M}(q) M^{-1}(q) \partial_q (\tfrac{1}{2} p^T \tilde{M}^{-1}(q) p) \\ + J_2(q,p) \tilde{M}^{-1}(q) p \Big) = 0, \quad \forall (q,p) \in \mathcal{X}_c, \end{aligned} \tag{4.15}$$

and

$$\Lambda^{\perp}(q) \left(\partial_q V(q) - \tilde{M}(q) M^{-1}(q) \partial_q \tilde{V}(q) \right) = 0, \quad \forall q \in \mathbb{R}^n. \tag{4.16}$$

In conclusion:

Proposition 4.1. *Assume that the systems (4.1),(4.2)–(4.4) and (4.8),(4.9)–(4.11) are constraint related, i.e., condition (4.12) is satisfied. Then the systems are matching if and only if the matching conditions (4.15) and (4.16) hold.*

Remark 4.1. With respect to the matching conditions as described in the controlled Lagrangians framework by [12], the Hamiltonian analogue described above includes an extra degree of freedom of design represented by the parameter J_2. In [3] it is shown that the "unconstrained" closed loop structure matrix

$$\tilde{J}(q,p) = \begin{bmatrix} 0 & M^{-1}(q)\tilde{M}(q) \\ -\tilde{M}(q)M^{-1}(q) & J_2(q,p) \end{bmatrix} \tag{4.17}$$

satisfies the integrability conditions if and only if

$$\begin{aligned} J_2 =& \tilde{M}M^{-1}\left([\partial_q(M\tilde{M}^{-1}p)]^T - \partial_q(M\tilde{M}^{-1}p)\right)M^{-1}\tilde{M} \\ &+ \tilde{M}M^{-1}\left([\partial_q Q]^T - \partial_q Q\right)M^{-1}\tilde{M}, \end{aligned} \tag{4.18}$$

for some smooth $\mathbb{R}^n$-valued function $Q(q)$. The function $Q(q)$ respresents the introduction of gyroscopic terms in the closed loop system. We remark that it is possible to include the extra degree of freedom represented by J_2 in the constrained controlled Lagrangians framework, by taking into account ("uncontrollable") external forces. This has been done in the *un*constrained case in [8].

Secondly, the form (4.15, 4.16) of the matching conditions gives the possibility to extend the so-called λ-method of [1] to constrained mechanical systems, see section 4.2 of [3].

Let $(q^*, \hat{p}_1 = 0)$ be a stationary point of the constrained Hamiltonian (4.6), i.e., $\frac{\partial V}{\partial q}(q^*) = 0$. Equivalenlty, $(q^*, p = 0)$ is a stationary point of the Hamiltonian (4.1). Indeed, $\hat{p}_1 = S^T p = 0$ implies $p = A\nu$ for some $\nu \in \mathbb{R}^k$. However, since $(q^*, \hat{p}_1 = 0) \in \mathcal{X}_c$ it follows that $A^T M^{-1} p = A^T M^{-1} A\nu = 0$ which implies $\nu = 0$ since $A^T M^{-1} A$ is invertible, and therefore $p = 0$. If $V(q)$ has a strict local minimum at q^*, then the point $(q^*, 0)$ is Lyapunov stable (since M_c is positive definite). If not, then one can try to shape $V(q)$ in such a way that the new, shaped, potential energy function $\tilde{V}$ has a strict local minimum at q^*. However, it is well know that even for unconstrained *underactuated* systems it is generally not possible to stabilize the system by only shaping the potential energy. In general, one also needs to change the kinetic energy of the system (see e.g. the well know example of a cart and pendulum, where one tries to stabilize the upright position of the pendulum by applying a controlled force to the cart). In fact, we can immediately translate Theorem 3.1 to the class of mechanical systems.

Theorem 4.1. *Consider the constrained mechanical system (4.1),(4.2)–(4.4), and let $(q^*, 0)$ be a stationary point of the total energy function H, i.e., $\frac{\partial V}{\partial q}(q^*) = 0$. Suppose we can find a positive definite matrix $\tilde{M}$,[1] a function $\tilde{V}$ and a skew-symmetric matrix J_2, such that (4.12) and the matching conditions (4.15,4.16) hold. Furthermore, suppose that we can choose $\tilde{V}$ in such a way that q^* is a strict local minimum of $\tilde{V}$. Then the state feedback*

$$\begin{aligned} u(q,p) =& (B^T S S^T B)^{-1} B^T S S^T \left(\partial_q H - \tilde{M}M^{-1}\partial_q \tilde{H} + J_2 \tilde{M}^{-1} p\right) - B^T \tilde{M}^{-1} p, \\ & (q,p) \in \mathcal{X}_c, \end{aligned} \tag{4.19}$$

(where we left out the argument q for clarity) stabilizes the equilibrium point $(q^*, 0)$. In other words, $(q^*, 0)$ is a Lyapunov stable equilibrium point of the closed loop system. Furthermore, all trajectories starting close enough to $(q^*, 0)$ will converge to the largest invariant (with respect to (4.8),(4.9)–(4.11) with $v = 0$) subset Ω_{inv} in

$$\Omega = \{(q,p) \in \mathbb{R}^{2n} \mid A^T(q)M^{-1}(q)p = 0 \textit{ and } B^T(q)\tilde{M}^{-1}(q)p = 0\}. \tag{4.20}$$

Remark 4.2. The IDA-PBC method for unconstrained mechanical systems, see [14], allowing the shaping of the *total* energy of the system, is an extension of the classical passivity based control method by means of potential energy shaping. Naturally, also the method presented above includes

[1] In fact, it is sufficient for $\tilde{M}(q)$ to be positive definite locally around q^*.

the method of potential energy shaping, e.g. [13], as a special case. Indeed, taking $\tilde{M}(q)$ equal to $M(q)$ and $J_2 = 0$ automatically satisfies (4.12, 4.15) and the matching conditions (4.16) become $\Lambda^\perp(\partial_q V - \partial_q \tilde{V}) = 0$, or equivalently, $\partial_q V - \partial_q \tilde{V} \in \text{Im } A \oplus \text{Im } B$. These conditions are equivalent to the condition

$$S^T(q)\frac{\partial(V - \tilde{V})}{\partial q}(q) \in \text{Im } S^T(q)B(q), \tag{4.21}$$

which are the conditions as given in [13].

4.2 Fully actuated mechanical systems

It is instructive to specialize the results obtained above to the case of *fully actuated* constrained mechanical systems, that is, when $\text{Im } A(q) \oplus \text{Im } B(q) = \mathbb{R}^n$ (equivalently, $S^T(q)B(q)$ is invertible). In this case $\Lambda^\perp(q) = 0$ and it follows that the matching conditions (4.15, 4.16) are trivialy satisfied. In particular this means that we can *always* shape the potential energy function $V(q)$ to a new potential energy function $\tilde{V}(q)$ having a strict minimum at the stationary configuration point q^* (in fact, for *any* configuration point $q^* \in \mathbb{R}^n$ we can find such a $\tilde{V}$). By taking the new kinetic energy matrix $\tilde{M}(q)$ to be equal to $M(q)$, condition (4.12) is automatically satisfied and Theorem 4.1 implies that there exists a state feedback control law which stabilizes the equilibrium point $(q^*, 0)$.

Furthermore, in this case the largest invariant subset Ω_{inv} in (4.20) can be calculated to be

$$\Omega_{\text{inv}} = \{(q, 0) \in \mathbb{R}^{2n} \mid S^T(q)\frac{\partial \tilde{V}}{\partial q}(q) = 0\}, \tag{4.22}$$

see e.g. [13]. Indeed, $(q, p) \in \Omega$ implies $p = 0$, which can be seen as follows: $A^T M^{-1} p = 0$ implies that $M^{-1}p = S\nu$, for some $\nu \in \mathbb{R}^{n-k}$. Then $B^T \tilde{M}^{-1} p = B^T \tilde{M}^{-1} M S \nu = 0$. Now notice that (4.12) is equivalent to $\text{Im } \tilde{M}^{-1} M S = \text{Im } S$. This implies that $B^T \tilde{M}^{-1} M S \nu = B^T S L \nu = 0$ for some $(n - k) \times (n - k)$ invertible matrix L. Since $B^T S$ is invertible it follows that $\nu = 0$, i.e., $p = 0$. So $\Omega = \{(q, p) \in \mathbb{R}^{2n} \mid p = 0\}$. Now consider a point $(q, p = 0) \in \Omega$, then $\dot{q} = M^{-1}p = 0$ and $\dot{p} = -\frac{\partial \tilde{V}}{\partial q} + A\lambda$, which yields $S^T \dot{p} = -S^T \frac{\partial \tilde{V}}{\partial q}$. Notice that $(q, p = 0) \in \Omega_{\text{inv}}$ if and only if $\dot{p} = 0$. However, $\dot{p} = 0$ is equivalent to $S^T \dot{p} = 0$. For if $S^T \dot{p} = 0$, then $\dot{p} = Az$, for some $z \in \mathbb{R}^k$. Differentiate $A^T M^{-1} p = 0$ with respect to time and use $p = 0$ to obtain $A^T M^{-1} A z = 0$. Since $A^T M^{-1} A$ is invertible it follows that $z = 0$, i.e., $\dot{p} = 0$. In conclusion: Ω_{inv} is exactly given by (4.22).

To illustrate the above results, let us recall the example of a *knife edge* as treated in [13].

Example 4.1. Consider a knife edge moving in point contact on a plane surface. Setting all parameters equal to one, the equations are given by

$$\dot{x} = p_x, \quad \dot{p}_x = u_1 \cos\phi + \lambda \sin\phi, \tag{4.23}$$

$$\dot{y} = p_y, \quad \dot{p}_y = u_1 \sin\phi - \lambda \cos\phi, \tag{4.24}$$

$$\dot{\phi} = p_\phi, \quad \dot{p}_\phi = u_2, \tag{4.25}$$

$$p_x \sin\phi - p_y \cos\phi = 0, \tag{4.26}$$

where (x, y) denote the coordinates of the point of contact of the knife edge on the plane, ϕ denotes the heading angle on the plane, and p_x, p_y, p_ϕ denote the corresponding momenta. The nonholonomic kinematic constraints are given by the condition that the knife edge is only allowed to move in a direction on the plane tangential to the blade of the knife, and are described by (4.26). The dynamic equations can be written in the form (4.2)–(4.4), with $H(x, y, \phi, p_x, p_y, p_\phi) = \frac{1}{2}(p_x^2 + p_y^2 + p_\phi^2)$,

$$A(q) = \begin{bmatrix} \sin\phi \\ -\cos\phi \\ 0 \end{bmatrix}, \quad B(q) = \begin{bmatrix} \cos\phi & 0 \\ \sin\phi & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.27}$$

When $u_1 = u_2 = 0$ the origin is an unstable equilibrium point of the system. Indeed, a solution starting at $(x^0, y^0, \phi^0, p_x^0, p_y^0, p_\phi^0) = (0, 0, 0, 0, 0, \epsilon)$, $\epsilon > 0$, will satisfy $\dot{\phi}(t) = \epsilon$ and therefore runs of to infinity, i.e., $\phi(t) \to \infty$.

The system is fully actuated and therefore we can arbitrarily shape the potential energy of the system. We choose $\tilde{V}(x, y, \phi) = \frac{1}{2}(x^2 + y^2 + \phi^2)$ which has a strict minimum at the origin. We leave the kinetic energy unchanged and choose $J_2 = 0$. The corresponding feedback law (4.19) is given by, see also [13],

$$u_1 = -(x\cos\phi + y\sin\phi + p_x\cos\phi + p_y\sin\phi), \tag{4.28}$$

$$u_2 = -(\phi + p_\phi). \tag{4.29}$$

The origin is a Lyapunov stable equilibrium of the closed loop system. All trajectories will converge to Ω_{inv} (4.22), which in this case is given by (note that $S(q) = B(q)$)

$$\Omega_{\text{inv}} = \{(x, y, z, p_x, p_y, p_z) \mid x = 0,\ \phi = 0,\ p_x = 0,\ p_y = 0,\ p_\phi = 0\}, \tag{4.30}$$

i.e., the y-axis. Notice that by Brockett's necessary condition we cannot find a smooth state feedback control law which asymptotically stabilizes the origin of the system.

The next example is a nonholonomically constrained particle as treated in [12]. The system is underactuated and can be stabilized by *total energy* shaping.

Example 4.2. Consider a particle in $\mathbb{R}^3$ with mass 1. The vector $q = (x, y, z) \in \mathbb{R}^3$ denotes the coordinates of the particle, and $p = (p_x, p_y, p_z) \in \mathbb{R}^3$ are the corresponding momenta. The total energy of the particle is defined by

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{1}{2}xz^2 + yz. \tag{4.31}$$

Assume the particle has to satisfy the nonholonomic kinematic constraints $\dot{x} + z\dot{y} = 0$, or equivalenlty, $p_x + zp_y = 0$. The particle is actuated by a force in the y-direction. Then the dynamics can be written in the form (4.2)–(4.4), with

$$A(q) = \begin{bmatrix} 1 \\ z \\ 0 \end{bmatrix}, \quad B(q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{4.32}$$

When $u = 0$, the origin is an unstable equilibrium point of the system, as can be seen by calculating the constraint multipliers λ and checking that the linearization of the system around the origin has an unstable eigenvalue $+1$.

Notice that $\Lambda^{\perp}(q) = [0 \quad 0 \quad 1]$. Stabilization of the origin by potential energy shaping only does not work, since the matching conditions $\Lambda^{\perp}(\partial_q V - \partial_q \tilde{V}) = 0$ imply that $\tilde{V}(q) = -\frac{1}{2}xz^2 + yz + \Phi(x,y)$ for an arbitrary smooth function $\Phi(x,y)$. Since $\tilde{V}(0,0,0) = \tilde{V}(0,0,z)$ for all z, $\tilde{V}(q)$ can never have a strict local minimum in the origin.[2] Therefore, in order to stabilize the system we need to shape the total energy of the system.

Any symmetric matrix $\tilde{M}(q)$ satisfying the condition (4.12) has the form

$$\tilde{M}(q) = \begin{bmatrix} a + c(1-z^2) & cz & -bz \\ cz & a & b \\ -bz & b & d \end{bmatrix}, \tag{4.33}$$

where a, b, c, d are arbitrary smooth functions of q, see [12]. As in [12] we choose $\tilde{M}$ to be

$$\tilde{M}(q) = \begin{bmatrix} 4 & 0 & -z \\ 0 & 4 & 1 \\ -z & 1 & 1 \end{bmatrix}, \tag{4.34}$$

which is positive definite as long as $z \in]-\sqrt{3}, \sqrt{3}[$. If we choose

$$\tilde{V}(q) = \frac{1}{2}(x^2 + y^2 + (z-y)^2), \tag{4.35}$$

then the closed loop Hamiltonian $\tilde{H}(q,p)$ (4.8) has a strict local minimum in the origin. Notice that this choice of $\tilde{V}(q)$ is slightly different than the one in [12]. The matching conditions (4.15, 4.16) are satisfied with

$$J_2(q,p) = \frac{1}{z^2-3} \begin{bmatrix} 0 & 0 & zp_x - p_y + 4p_3 \\ 0 & 0 & 0 \\ -(zp_x - p_y + 4p_3) & 0 & 0 \end{bmatrix}. \tag{4.36}$$

The matrix (4.36) is different from the "integrable" choice (4.18), which in fact does not satisfy the matching conditions for any $Q(q)$. The necessity of matching by non-integrable closed loop structure matrices (i.e., not corresponding to classical Euler-Lagrange systems) is reflected in Hambergs concept of *generalized (Euler-Lagrange) matching* [11, 12]. Notice however that the possibility of "non-integrable" matching comes natural in the IDA-PBC method by using the more general framework of implicit port-controlled Hamiltonian systems. Furthermore, it is interesting to remark that (4.36) is not the only choice for J_2 for which the matching conditions hold. In fact, there is a whole family of (non-integrable) skew-symmetric matrices J_2 which solve the matching conditions (4.15). This freedom might be used for additional goals other than stabilization.

[2] Strictly speaking this does not imply that stabilization by potential energy shaping is not possible at all, however, it is not possible by the method described in Theorem 4.1. For example $\Phi(x,y) = x$ puts all the eigenvalues of the linearized system in the origin, and more advanced methods are needed to decide on stability of the nonlinear system.

The stabilizing feedback controller (4.19) is on the constraint manifold $\mathcal{X}_c = \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6 \mid p_x = -zp_y\}$ given by

$$u(q,p) = -\frac{1}{2}z^3 + yz^2 + (4x+4)z - 7y \\ + \frac{1}{z^2-3}\left(p_y - p_z + zp_z((z^2+1)p_y - 4p_z)\right), \tag{4.37}$$

(i.e., $S^T(q) = \begin{bmatrix} z & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$), which is defined as long as $z \in]-\sqrt{3}, \sqrt{3}[$.

All trajectories of the closed loop system starting close enough to the origin (such that $z(t) \in]-\sqrt{3}, \sqrt{3}[,\ \forall t$) will converge to the largest invariant subset Ω_{inv} contained in

$$\Omega = \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6 \mid p_x + zp_y = 0,\ p_y - p_z = 0\}. \tag{4.38}$$

We can calculate Ω_{inv} as follows: The trajectories of any invariant subset in Ω should satisfy $\frac{d}{dt}(p_x + zp_y) = 0 = \frac{d}{dt}(p_y - p_z)$, that is

$$\dot{p}_x + z\dot{p}_y + p_z p_y = 0, \quad \dot{p}_y - \dot{p}_z = 0. \tag{4.39}$$

The first of these equations defines the constraint multiplier μ. Using this, together with the fact that $p_x + zp_y = 0,\ p_y - p_z = 0$, the second equation yields that $2y - (x+1)z = 0$. Setting the time-derivative of this last equation to zero, etc., finally implies that $p_z = 0$ and therefore also $p_x = p_y = 0$. This means that any invariant subset contained in Ω necessarily is contained in the set $\{(q,p) \in \mathbb{R}^{2n} \mid p = 0\}$. Analogously to section 4.2 we can then show (using (4.12), which implies that $\text{Im } S = \text{Im } M^{-1}\tilde{M}S$) that the largest invariant subset Ω_{inv} contained in Ω is actually defined by (4.22), which in this case is given by

$$\Omega_{\text{inv}} = \{(x, y, z, 0, 0, 0) \in \mathbb{R}^6 \mid (x-1)y = 0 \text{ and } y = z\}, \tag{4.40}$$

i.e., the union of the line $\{x = 1, y = z\}$ with the $x - axis$. In particular this implies that any trajectory starting very close to the origin will converge to the x-axis.

One final remark is the following: Notice that the feedback (4.37) drives every trajectory starting close enough to the origin to the invariant set (4.40). Its domain of attraction can be made arbitrary large by taking instead of (4.34) the matrix

$$\tilde{M}(q) = \begin{bmatrix} a & 0 & -z \\ 0 & a & 1 \\ -z & 1 & 1 \end{bmatrix}, \tag{4.41}$$

with a an arbitrary positive constant. This matrix is positive definite as long as $z \in]-\sqrt{a-1}, \sqrt{a-1}[$. The matching conditions are satisfied with

$$J_2(q,p) = \frac{1}{z^2 - a + 1}\begin{bmatrix} 0 & 0 & zp_x - p_y + ap_3 \\ 0 & 0 & 0 \\ -(zp_x - p_y + ap_3) & 0 & 0 \end{bmatrix}. \tag{4.42}$$

The stabilizing feedback controller (4.19) drives every trajectory starting close enough to the origin (such that $z(t) \in]-\sqrt{a-1}, \sqrt{a-1}[,\ \forall t$) to the invariant set (4.40).

5 Conclusions

In this paper we presented the extension of the *interconnection and damping assignment passivity based control (IDA-PBC) method* to the class of constrained systems. The constrained systems are modelled as implicit port-controlled Hamiltonian systems. The matching conditions for such systems are given and sufficient conditions for stabilizability of these systems are obtained. The stabilizing feedback law is explicitly calculated. The theory is applied to the class of underactuated mechanical systems with (nonholonomic) kinematic constraints. We remark that the method is depending on the solvability of a set of nonlinear PDEs (i.e., the matching conditions). However, it seems clear that various methods obtained in recent literature for solving these PDEs in the unconstrained case, such as the so-called simplified matching conditions [7], the λ-method [1, 3], or the transformation to ODEs [9], can be directly extended to the case of constrained systems.

References

[1] D. Auckly, L. Kapitanski, and W. White. Control of nonlinear underactuated systems. *Comm. Pure and Applied Math.*, 53:354–369, 2000.

[2] G. Blankenstein. *Implicit Hamiltonian systems: symmetry and interconnection.* PhD thesis, University of Twente, The Netherlands, november 2000.

[3] G. Blankenstein, R. Ortega, and A.J. van der Schaft. The matching conditions of controlled Lagrangians and IDA-passivity based control. *Int. Journal Control*, 2001. To appear.

[4] G. Blankenstein, R. Ortega, and A.J. van der Schaft. Matching of Euler-Lagrange and Hamiltonian systems. In *Proc. IFAC 2002*, Barcelona, Spain, 2002.

[5] A. Bloch, N. Leonard, and J.E. Marsden. Stabilization of mechanical systems using controlled Lagrangians. In *Proc. CDC*, volume 36, pages 2356–2361, San Diego, California, USA, 1997.

[6] A.M. Bloch, N.E. Leonard, and J.E. Marsden. Matching and stabilization by the method of controlled Lagrangians. In *Proc. CDC*, volume 37, pages 1446–1451, Tampa, Florida, USA, December 1998.

[7] A.M. Bloch, N.E. Leonard, and J.E. Marsden. Controlled Lagrangians and the stabilization of mechanical systems I: The first matching theorem. *IEEE Trans. Automatic Control*, 45(12):2253–2269, 2000.

[8] D.E. Chang, A.M. Bloch, N.E. Leonard, J.E. Marsden, and C.A. Woolsey. The equivalence of controlled Lagrangian and controlled Hamiltonian systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 2001. Submitted.

[9] F. Gómez-Estern, R. Ortega, F.R. Rubio, and J. Aracil. Stabilization of a class of underactuated mechanical systems via total energy shaping. In *Proc. CDC*, Orlando, Florida USA, December 2001.

[10] J. Hamberg. General matching conditions in the theory of controlled Lagrangians. In *Proc. CDC*, volume 38, pages 2519–2523, Phoenix, Arizona USA, December 1999.

[11] J. Hamberg. Controlled Lagrangians, symmetries and conditions for strong matching. In N.E. Leonard and R. Ortega, editors, *Preprints IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control*, pages 98–103, Princeton USA, March 2000.

[12] J. Hamberg. Simplified conditions for matching and for generalized matching in the theory of controlled Lagrangians. In *Proc. ACC*, Chicago, Illinois, USA, 2000.

[13] B.M. Maschke and A.J. van der Schaft. A Hamiltonian approach to the stabilization of nonholonomic mechanical systems. In *Proc. 33rd IEEE CDC*, pages 2950–2954, Orlando, FL, 1994.

[14] R. Ortega, M.W. Spong, F. Gómez-Estern, and G. Blankenstein. Stabilization of underactuated mechanical systems via interconnection and damping assignment. *IEEE Trans. Automatic Control*, 47(7), 2002.

[15] R. Ortega, A. van der Schaft, B. Maschke, and G. Escobar. Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems. *Automatica*, 38:585–596, 2002. See also *Stability and stabilization of nonlinear systems*, LNCIS Vol. 246, eds. D. Aeyels, F. Lamnabhi-Lagarrique and A.J. van der Schaft, Springer-Verlag, London, 1999.

[16] R. Ortega, A.J. van der Schaft, I. Mareels, and B. Maschke. Putting energy back in control. *IEEE Control Systems Magazine*, 21(2):18–33, 2001.

[17] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, London, second revised and enlarged edition, 2000.

[18] A.J. van der Schaft and B.M. Maschke. On the Hamiltonian formulation of nonholonomic mechanical systems. *Reports on Mathematical Physics*, 34(2):225–232, 1994.

[19] A.J. van der Schaft and B.M. Maschke. Mathematical modeling of constrained Hamiltonian systems. In *Preprints of the 3rd Nonlinear Control Systems Design Symposium, NOLCOS*, Tahoe City, CA, 1995.

[20] D.V. Zenkov, A.M. Bloch, N.E. Leonard, and J.E. Marsden. Matching and stabilization of low-dimensional nonholonomic systems. In *Proc. CDC*, volume 39, pages 1289–1295, 2000.