Symmetric nonsquare factorization of selfadjoint rational matrix functions and algebraic Riccati inequalities

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Abstract

In this paper we shall present a parametrization of all symmetric, possibly nonsquare minimal factorizations of a positive semidefinite rational matrix function. It turns out that a pole-pair of such a nonsquare factor is the same as a pole pair for a specific square factor. The location of the zeros is then determined by a solution to a certain algebraic Riccati inequality.

We shall also consider the case where the function we wish to factorize in a symmetric way has only constant signature. A connection with Bezoutians is given as well.

1 Introduction

Consider an $m \times m$ rational matrix function, $\Phi(\lambda)$, that has positive semidefinite values on the imaginary axis, $i\mathbf{R}$ and is regular. Note that, in this case, it is possible that Φ may have poles or zeros on $i\mathbf{R}$. Furthermore, we shall mostly assume that $\Phi(\infty) = I_m$. The McMillan degree of Φ is always even as is well known and is denoted by $2n$. We say that an $m \times p$ rational matrix function $W(\lambda)$ is a *minimal spectral factor* of $\Phi(\lambda)$ if

$$\Phi(\lambda) = W(\lambda)W(-\overline{\lambda})^* \tag{1.1}$$

is a minimal factorization. In other words, the McMillan degree of Φ is twice that of W. Here we denote the McMillan degree of W by $\delta(W)$.

Square spectral factors (i.e., with $p = m$) have been studied from many points of view in the past (see [1, 5, 6, 7, 8, 9, 10, 11, 14, 24, 26, 27, 28, 29, 33, 35, 36, 37, 39].) Our present interest lies in giving a simple parametrization of all, possibly nonsquare, spectral factors. Such parametrizations were obtained for the stable spectral factors in [12, 13, 30]. In the present paper we summarize the main results of [31] and [32] which deal with the general case.

We shall also consider the case where $\Phi(\lambda)$ has constant signature on $i\mathbf{R}$, and where $\Phi(\infty) = J$. Here, J is a selfadjoint invertible matrix. In that case we may expect a factorization of the form

$$\Phi(\lambda) = W(\lambda)\tilde{J}W(-\overline{\lambda})^* \tag{1.2}$$

for some selfadjoint $\tilde{J}$ and $m \times p$ rational matrix function $W(\lambda)$. In contrast to the positive semidefinite case such factorization with a square W may fail to exist, as is well-known. We shall assume existence of one square factorization of this type and give a parametrization of all possibly non-square factorizations of this type for which the poles of the non-square factor are the same as the ones of the given square factor. Such J-symmetric factorizations were studied from several points of view in earlier papers, see, e.g., [15, 17, 19, 20, 38, 40].

2 Preliminaries

If $W(\lambda)$ is a rational matrix function with $W(\infty) = D$, a *realization* of $W(\lambda)$ is a representation of W in the form

$$W(\lambda) = D + C(\lambda I_n - A)^{-1}B.$$

As is well known, this always exists. It is called a *minimal realization* if the number n is as small as possible. In that case n is called the *McMillan degree* of W, which we denote by $\delta(W)$.

If D is invertible, then

$$W(\lambda)^{-1} = D^{-1} - D^{-1}C(\lambda - A^{\times})^{-1}BD^{-1},$$

where $A^{\times} = A - BD^{-1}C$.

If the realization is minimal then the eigenvalues of A are the poles of $W(\lambda)$ and eigenvalues of $A^{\times}$ are the zeros of $W(\lambda)$.

In general if $W(\lambda) = W_1(\lambda)W_2(\lambda)$ then $\delta(W) \leq \delta(W_1) + \delta(W_2)$. In case equality holds we say that the factorization is *minimal*. Let $W(\lambda) = D + C(\lambda I_n - A)^{-1}B$ be a minimal realization. Assume D is invertible. Let $\mathcal{M}$ be A-invariant, $\mathcal{M}^{\times}$ $A^{\times}$ invariant and $\mathcal{M} \oplus \mathcal{M}^{\times} = \mathbb{C}^n$. Let Π be the projection onto $\mathcal{M}$ along $\mathcal{M}^{\times}$ and let $D = D_1D_2$. Put

$$W_1(\lambda) = D_1 + C|_{\mathcal{M}}(\lambda - A|_{\mathcal{M}})^{-1}(I - \Pi)BD_2^{-1}$$

$$W_2(\lambda) = D_2 + D_1^{-1}C\Pi(\lambda - \Pi A \Pi)^{-1}B$$

Then $W = W_1W_2$ is a minimal factorization with square factors and all minimal factorizations with square factors are obtained this way (see [3, 4]).

A pair of matrices (C, A) is called a *pole pair* for the rational matrix function $W(\lambda)$ if there is a matrix B such that

$$W(\lambda) - C(\lambda I - A)^{-1}B$$

is analytic over the whole complex plane, and (A, B, C) is minimal. For the theory of pole and zero pairs, and more generally, spectral triples, see [2].

3 The positive semidefinite case

The parametrization we have in mind starts from a minimal realization

$$\Phi(\lambda) = I_m + C(\lambda I - A)^{-1}B.$$

It is well-known that the minimality implies the existence of an invertible skew-hermitian matrix H such that $HA = -A^*H$ and $HB = C^*$. The first of these relations can be rephrased as saying that iA is selfadjoint in the indefinite inner product given by iH. This allows us to use the results and techniques of the theory of indefinite inner product spaces. One element of this theory is the following. A subspace $\mathcal{M}$ is called H-Lagrangian if $H\mathcal{M} = \mathcal{M}^{\perp}$. It is well-known that in the parametrization of all square spectral factors invariant Lagrangian subspaces play a crucial role. In fact, introduce also $A^{\times} = A - BC$, then it is easily seen that also $HA^{\times} = -(A^{\times})^*H$. Then we have the following result [36].

Theorem 3.1. *There is a one-one correspondence between all square spectral factors $W(\lambda)$ with $W(\infty) = I_m$ and all pairs of subspaces $\mathcal{M}, \mathcal{M}^{\times}$, where $\mathcal{M}$ is A-invariant, $\mathcal{M}^{\times}$ is $A^{\times}$-invariant, and both these subspaces are H-Lagrangian.*

For given $\mathcal{M}$ and $\mathcal{M}^{\times}$ of this type, let Π be the projection onto $\mathcal{M}$ along $\mathcal{M}^{\times}$. Then the corresponding factor W is given by $W(\lambda) = I + C(\lambda I - A)^{-1}\Pi B$.

In many cases Φ arises as a product

$$\Phi(\lambda) = W_1(\lambda)W_1(-\bar{\lambda})^*,$$

where this is minimal, and we have a minimal realization for $W_1(\lambda)$:

$$W_1(\lambda) = I_m + C(\lambda I_n - A)^{-1}B.$$

We are then looking for all $W(\lambda)$ such that $\Phi(\lambda) = W(\lambda)W(-\bar{\lambda})^*$ minimally, and for which (C, A) is a pole pair for $W(\lambda)$. This problem was considered in [33], from which we summarize the following.

From the realization for W_1 we build a minimal realization for Φ:

$$\Phi(\lambda) = I_m + \begin{pmatrix} C & -B^* \end{pmatrix} \left(\lambda I_{2n} - \begin{pmatrix} A & -BB^* \\ 0 & -A^* \end{pmatrix}\right)^{-1} \begin{pmatrix} B \\ C^* \end{pmatrix}.$$

Then $\mathcal{M} = \operatorname{Im} \begin{pmatrix} I \\ 0 \end{pmatrix}$, $\mathcal{A}^{\times} = \begin{pmatrix} A - BC & 0 \\ -CC^* & -A^* + C^*B^* \end{pmatrix}$, $H = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. An H-Lagrangian $\mathcal{A}^{\times}$-invariant subspace is of the form $\mathcal{M}^{\times} = \operatorname{Im} \begin{pmatrix} X \\ I \end{pmatrix}$, where $X = X^*$ satisfies the algebraic Riccati equation

$$XCC^*X + X(A - BC) + (A^* - C^*B^*)X = 0.$$

The converse also holds. If X is a selfadjoint solution to the algebraic Riccati equation, then the corresponding $W(\lambda)$ is given by:

$$W(\lambda) = I_m + C(\lambda I_n - A)^{-1}(B - XC^*).$$

Now consider

$$\frac{W_1(\mu)W_1(-\bar{\lambda})^* - W(\mu)W(-\bar{\lambda})^*}{\mu - \lambda}$$

A straightforward computation gives that this can be written as

$$C(\mu - A)^{-1}\mathbf{T}(\lambda + A^*)^{-1}C^*$$

for some hermitian $\mathbf{T}$ which is called the *Bezoutian*. It turns out that in the present case we have $\mathbf{T} = X$. Moreover, $\operatorname{Ker}\mathbf{X}$ describes the common right zero structure of W and W_1. For the definition and properties of the Bezoutian, see, e.g., [16, 18, 25]. For the application to symmetric factorization see, e.g., [21, 22, 23, 24].

4 Minimal realizations for nonsquare minimal symmetric factors

Consider again positive semidefinite $\Phi(\lambda)$, with minimal realization

$$\Phi(\lambda) = I_m + C(\lambda I_{2n} - A)^{-1}B$$

and the corresponding $H = -H^*$ for which $HA = -A^*H$ and $HB = C^*$. Now we are looking for all possibly nonsquare minimal symmetric factorizations:

$$\Phi(\lambda) = W(\lambda)W(-\bar{\lambda})^*.$$

This problem was studied in [31], from which we present the main results. See also [12, 13, 30, 32] for other approaches and additional results. The first step is the following key observation:

Proposition 4.1. *If W is a minimal (possibly nonsquare) symmetric factor, then there is an A-invariant H-Lagrangian subspace $\mathcal{M}$ such that $(C|_{\mathcal{M}}, A|_{\mathcal{M}})$ is a pole pair for W.*

If W is a minimal symmetric factor of Φ then $W(\infty) = V$ is a co-isometry, i.e., $VV^* = I_m$. With respect to appropriate bases we can take $V = \begin{pmatrix} I_m & 0 \end{pmatrix}$. We describe all minimal symmetric factors for which $W(\infty) = \begin{pmatrix} I_m & 0 \end{pmatrix}$, thereby describing all minimal symmetric factors up to choice of bases.

We now state the main result

Theorem 4.1. *There is a one-to-one correspondence between the set of minimal symmetric factors $W(\lambda)$ of $\Phi(\lambda)$ such that $W(\infty) = \begin{pmatrix} I_m & 0 \end{pmatrix}$ and the set of triples $\{\mathcal{M}, X, \widehat{B}_1\}$ described below.*

- *$\mathcal{M}$ is an A-invariant H-Lagrangian subspace.*

To describe X and $\widehat{B}_1$, let A_1 and C_1 be given by $A_1 = A|_{\mathcal{M}}$ and $C_1 = C|_{\mathcal{M}}$. Furthermore, suppose that $\mathcal{M}^\times$ is the $A^\times = (A - BC)$-invariant, H-Lagrangian subspace such that $\sigma(A^\times|_{\mathcal{M}^\times}) \subset \overline{\mathbb{C}}_-$.

Let π be the projection onto $\mathcal{M}$ along $\mathcal{M}^\times$ and denote a matrix representation for πB by $\widetilde{B}_1$.

- *Then X solves the Riccati inequality*

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X \leq 0$$

- *$\widehat{B}_1$ satisfies*

$$XC_1^*C_1X - X(A_1 - \widetilde{B}_1C_1)^* - (A_1 - \widetilde{B}_1C_1)X = -\widehat{B}_1\widehat{B}_1^*.$$

This correspondence is given by

$$W(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} + C_1(\lambda I - A_1)^{-1} \begin{pmatrix} XC_1^* + \widetilde{B}_1 & \widehat{B}_1 \end{pmatrix}.$$

Some remarks are in order.

1. The proof uses that we have a minimal square symmetric factor

$$W_1(\lambda) = I + C_1(\lambda I - A_1)^{-1}\tilde{B}_1.$$

   We know how to obtain this from the results of the previous section.

2. Observe that we can write

$$W(\lambda) = \begin{pmatrix} W_1(\lambda) & 0 \end{pmatrix} + C_1(\lambda - A_1)^{-1} \begin{pmatrix} XC_1^* & B_{12} \end{pmatrix}.$$

3. Thus the co-isometry

$$U(\lambda) = W_1(\lambda)^{-1}W(\lambda)$$

   is given by

$$U(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} + C_1(\lambda - (A_1 - \widetilde{B}_1C_1))^{-1} \begin{pmatrix} XC_1^* & B_{12} \end{pmatrix}.$$

5 J-symmetric factorization

Consider an $m \times m$ rational matrix function $\Phi(\lambda)$ with only selfadjoint (not nonnegative) values on $i\mathbb{R}$. Given is also an invertible hermitian $m \times m$ matrix J. A square rational matrix function $W(\lambda)$ is called a *J-symmetric factor* of $\Phi(\lambda)$ if

$$\Phi(\lambda) = W(\lambda)JW(-\bar{\lambda})^*.$$

Obviously necessary is that the number of positive and negative eigenvalues of the matrix $\Phi(\lambda)$ does not depend on $\lambda \in i\mathbb{R}$ (at least outside of poles and zeros), i.e., $\Phi(\lambda)$ has *constant signature*.

Without loss of generality we can take $W(\infty) = J$. Let $\Phi(\lambda) = J + C(\lambda I_n - A)^{-1}B$ be a minimal realization. Recall that there is an invertible matrix H with

$$H = -H^*, \quad HA = -A^*H, \quad HB = C^*.$$

Also we have

$$A^\times = A - BJ^{-1}C.$$

Necessary conditions for existence of *minimal* J-symmetric factorization are the following (see [38])

1. $\Phi(\lambda)$ has constant signature
2. there exists an A-invariant H-Lagrangian $\mathcal{M}$
3. there exists an $A^\times$-invariant H-Lagrangian $\mathcal{M}^\times$

However, we no longer have that automatically $\mathcal{M} \oplus \mathcal{M}^\times = \mathbb{C}$. So, even if these necessary conditions are satisfied, a minimal J-symmetric factorization may fail to exist. As an example consider

$$\Phi(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & \lambda^{-2} \end{pmatrix}$$

A minimal realization is

$$\Phi(\lambda) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \left(\lambda - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case

$$H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A^\times = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

There is a unique invariant H-Lagrangian subspace $\mathcal{M} = \text{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. All necessary conditions are satisfied, yet no minimal factorization exists.

We now turn to the case where we do have automatic matching of $\mathcal{M}$ and $\mathcal{M}^\times$.

$\Phi(\lambda)$ is said to have a *complete set of minimal J-symmetric factorizations* if for any A-invariant H-Lagrangian subspace $\mathcal{M}$ and any $A^\times$-invariant H-Lagrangian subspace $\mathcal{M}^\times$ we have

$$\mathcal{M} \oplus \mathcal{M}^\times = \mathbb{C}^n.$$

The next result was proved in [19].

Theorem 5.1. *The following are equivalent:*

1. *there is a complete set of J-symmetric factorizations*

2. *for every A-invariant H-Lagrangian subspace $\mathcal{M}$, and for every nonzero vector $x \in \mathcal{M}$ we have*

$$\langle H(\lambda - A^{\times})^{-1}x, x\rangle \not\equiv 0, \quad \lambda \in \rho(A^{\times}),$$

3. *for every $A^{\times}$-invariant H-Lagrangian subspace $\mathcal{M}^{\times}$, and for every nonzero vector $x \in \mathcal{M}^{\times}$ we have*

$$\langle H(\lambda - A)^{-1}x, x\rangle \not\equiv 0, \quad \lambda \in \rho(A).$$

6 Minimal nonsquare J-symmetric factorization

Let Φ be as in the previous section. We assume the existence of a square minimal factor

$$\Phi(\lambda) = W_1(\lambda)JW_1(-\bar{\lambda})^*.$$

We are then looking for all $W(\lambda)$ such that

1. $\Phi(\lambda) = W(\lambda)\begin{pmatrix} J & 0 \\ 0 & J_{22}\end{pmatrix}W(-\bar{\lambda})^*,$
   for some $J_{22} = J_{22}^*$, and this is a minimal factorization,

2. W has the same pole pair as W_1.

Concerning this problem we have the following results (see [34])

Our first observation is that without loss of generality we may take $\widetilde{J}$ in the $\widetilde{J}$-spectral factorization

$$\Phi(\lambda) = W(\lambda)\widetilde{J}W(-\overline{\lambda})^*. \tag{6.3}$$

to be of the form

$$\widetilde{J} = \begin{pmatrix} J & 0 \\ 0 & J_{22}\end{pmatrix}, \tag{6.4}$$

and at the same time we may assume that $W(\infty) = \begin{pmatrix} I & 0\end{pmatrix}$.

Then the main result is the following theorem.

Theorem 6.1. *Suppose that the rational matrix function Φ with constant signature and with $\Phi(\infty) = J$ has a minimal square J-spectral factor W_1 given by the minimal realization*

$$W_1(\lambda) = I_m + C_1(\lambda I - A_1)^{-1}\widetilde{B}_1. \tag{6.5}$$

Put $Z = A_1 - \widetilde{B}_1C_1$. For any $X = X^*$ form $XZ^* + ZX - XC_1^*JC_1X$ and let X_2 and J_{22} be any matrices such that

$$XZ^* + ZX - XC_1^*JC_1X = X_2J_{22}X_2^*. \tag{6.6}$$

Then for any such X, X_2 and J_{22} the function

$$W(\lambda) = \begin{pmatrix} I_m & 0 \end{pmatrix} + C_1(\lambda I - A_1)^{-1} \begin{pmatrix} XC_1^* J + \widetilde{B}_1 & X_2 \end{pmatrix} \tag{6.7}$$

is a $\widetilde{J}$-spectral factor of Φ, where $\widetilde{J}$ is given by (6.4).

Conversely, given $\widetilde{J}$ as in (6.4) *all $\widetilde{J}$-spectral factors of Φ are given by* (6.7) *where X and X_{22} satisfy* (6.6).

For special choices of $\widetilde{J}$, i.e., special choices of J_{22}, we obtain the following corollary.

Corollary 6.1. *Let $\widetilde{J}$ be given by* (6.4). *Under the assumptions of Theorem* 6.1 *the following hold.*

(a) *Let $\Pi_+(J) = \Pi_+(\widetilde{J})$, where $\Pi_+(J)$ (resp., $\Pi_+(\widetilde{J})$) denotes the number of positive eigenvalues of J (resp., $\widetilde{J}$). There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair (C_1, A_1) and with value $\begin{pmatrix} I & 0 \end{pmatrix}$ at infinity, and pairs of matrices (X, X_2) satisfying*

$$XZ^* + ZX - XC_1^* J C_1 X \leq 0$$

and

$$XZ^* + ZX - XC_1^* J C_1 X = -X_2 J_{22} X_2^*.$$

This one-to-one correspondence is given by (6.7).

(b) *Let $\Pi_-(J) = \Pi_-(\widetilde{J})$, where $\Pi_-(J)$ (resp., $\Pi_-(\widetilde{J})$) denotes the number of negative eigenvalues of J (resp., $\widetilde{J}$). There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair (C_1, A_1) and with value $\begin{pmatrix} I & 0 \end{pmatrix}$ at infinity, and pairs of matrices (X, X_2) satisfying*

$$XZ^* + ZX - XC_1^* J C_1 X \geq 0$$

and

$$XZ^* + ZX - XC_1^* J C_1 X = X_2 J_{22} X_2^*.$$

This one-to-one correspondence is given by (6.7).

(c) *Let $\Pi_+(J) = \Pi_+(\widetilde{J})$ and $\Pi_-(J) = \Pi_-(\widetilde{J})$. There is a one-to-one correspondence between $\widetilde{J}$-spectral factors of Φ with pole pair (C_1, A_1) and with value $\begin{pmatrix} I & 0 \end{pmatrix}$ at infinity, and matrices X satisfying*

$$XZ^* + ZX - XC_1^* J C_1 X = 0$$

This one-to-one correspondence is given by (6.7).

Part (c) of the above corollary corresponds to the square case which, for instance, is discussed in [19].

One can also show, like in the positive semidefinite case, that the solutions of the particular algebraic Riccati equation arising in the parametrization of all J-nonsquare spectral factors (see Theorem 6.1) can be interpreted as generalized Bezoutians in the sense of (see [16] and [18]).

Also, in case $J_{22} > 0$, the kernel of the Bezoutian again describes the common right zero structure of W and W_1.

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