Stability of Hybrid Control Systems Based on Time-State Control Forms

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Abstract

Time-State Control Form was proposed as one of the control method for nonholonomic systems. But this method needs input switching, so there is a drawback that the switching conditions may spoil the stability of the system. From the above backgrounds, this research considers the property of the control method based on Time-State Control Form from the viewpoint of hybrid systems. Then, we derive the switching conditions which stabilize the systems by using Lyapunov functions for each mode. Furthermore, the controlled object is limited to chained system, and the conditions to stabilize the system are shown by introducing the Lyapunov functions which are invariant to input switching.

1 Introduction

The systems such as wheeled mobile robots or space robots are nonholonomic systems which have particular mechanical constraint. Since the systems cannot be stabilized by continuous static state feedback controller, it is well known that the systems are very hard to control theoretically [1]. Recently, the various control methods for stabilizing the state of nonholonomic systems are proposed [2].

As one in them, there is a control method which changes the differential equation of the system into Time-State Control Form, and designs a stabilization controller [3]. This method can apply general control theories easily according to various control specifications. Therefore, the method has the advantage of being rich in the extendibility of a control system design. But it is necessary to perform input switching in order to stabilize the state of the system. So there is a problem that the switching conditions may spoil stability of the system.

On the other hand, since the control method using Time-State Control Form needs input switching inevitably, the method is a kind of the hybrid controller which combined the continuous time input and the discrete switching input. Many researches which systematize the analysis / control scheme of a hybrid system are performed briskly in recent years [4][5]. However, since the class of the hybrid systems is very large, the present condition is having not resulted in construction of the scheme. Therefore, it is effective in construction of the control scheme of hybrid systems to consider the stability conditions of the systems including input switching.

From the above backgrounds, in this paper, the control method based on Time-State Control Form is treated from the framework as hybrid control systems, and the property of the system is considered. Especially, it takes into consideration about the influence which the input switching has on stability, and two kinds of conditions for guaranteeing the stability of the systems are derived. In one of them, general nonholonomic systems are considered as the controlled objects and input switching condition which guarantees the stability of the systems are derived by using Lyapunov functions for each mode. In another, the controlled object is limited to chained system, and the conditions to stabilize the systems are shown by introducing the Lyapunov functions which are invariant to input switching.

2 Time-State Control Form

Consider the drift-less nonholonomic system described by the equation

$$\frac{d\boldsymbol{x}}{dt} = \sum_{i=1}^{m} g_i(\boldsymbol{x})u_i, \tag{2.1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state variable, $\boldsymbol{u} = [u_1, u_2, \dots, u_m]^T \in \mathbb{R}^m$ is the control input, $g_i(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ are smooth vector functions and assume $m < n$. It is known that if the system (2.1) satisfies some conditions, it can be changed into Time-State Control Form described by the following equations

$$\frac{d\boldsymbol{\xi}}{d\tau} = f_0(\boldsymbol{\xi}) + \sum_{i=1}^{m-1} f_i(\boldsymbol{\xi})\mu_i \tag{2.2a}$$

$$\frac{d\tau}{dt} = \mu_m \tag{2.2b}$$

which are obtained by using some suitable coordinate and input transformations as follows:

$$\begin{bmatrix} \boldsymbol{\xi} \\ \tau \end{bmatrix} = T(\boldsymbol{x}) \qquad (T(0) = 0) \tag{2.3}$$

$$\mu_i = V_i(\boldsymbol{x}, \boldsymbol{u}) \qquad (i = 1, 2, \dots, m). \tag{2.4}$$

where $T(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ and $V_i(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$.

Time-State Control Form (2.2) consists of two subsystems. The equation (2.2a) is called state control part which has state variable $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$. Note that the time scale of state control part is not actual time t, but one of the state $\tau \in \mathbb{R}$ obtained by the coordinate transformation (2.3). The another equation (2.2b) is called time control part. The state of this part is τ, which is the time scale of state control part. Using one of the new control input μ_m, we can control the time scale of state control part arbitrary.

Let $\boldsymbol{\xi} = 0$ be an equilibrium point for state control part, i.e. $f_0(0) = 0$, and linearized system in a neighborhood of the origin is controllable. Then, it is known that original nonlinear system (2.2a) is locally controllable in a neighborhood of the origin, and there exists a continuous static state feedback controller which stabilizes the system asymptotically [6]. Therefore, we consider two state feedback controller for the system (2.2a) according to an increase/decrease of time scale τ:

- A controller $\mu_i = \alpha_i(\boldsymbol{\xi}) \quad (i = 1, \dots, m-1)$, which stabilizes state control part (2.2a) locally *when time scale* τ *increases.*
- A controller $\mu_i = \beta_i(\boldsymbol{\xi}) \quad (i = 1, \dots, m-1)$, which stabilizes state control part (2.2a) locally *when time scale* τ *decreases.*

Since we can control the change of time scale τ by using the control input μ_m, we can choose these feedback controllers $\alpha_i(\boldsymbol{\xi})$ and $\beta_i(\boldsymbol{\xi})$ suitably. Repeating the change of time scale τ and choice of the feedback controllers, the state $\boldsymbol{\xi}$ can be converged asymptotically.

We now treat Time-State Control Form from as the framework of hybrid control systems by using BBM model which is one of the general description of hybrid systems. Details of BBM model are in the reference [7]. Suppose that the inputs μ_m for time control part (2.2b) are given according to the time increase mode and the time decrease mode respectively as follows:

$$\left\{ \begin{array}{lll} \mu_m^+ & \in & \{\mu_m | \mu_m \in \mathbb{R}, \mu_m > 0\} \\ \mu_m^- & \in & \{\mu_m | \mu_m \in \mathbb{R}, \mu_m < 0\} \end{array} \right. . \tag{2.5}$$

Moreover, assume that the state feedback controllers which stabilize state control part (2.2a) are already designed by some suitable method as follows:

$$\left\{ \begin{array}{lll} \mu_i^+ & = & \alpha_i(\boldsymbol{\xi}) \\ \mu_i^- & = & \beta_i(\boldsymbol{\xi}) \end{array} \right. \quad (i = 1, 2, \dots, m-1). \tag{2.6}$$

By the way, combining the two subsystems (2.2a) and (2.2b), Time-State Control Form (2.2) is given by:

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \tau \end{bmatrix} = \begin{bmatrix} f_0(\boldsymbol{\xi}) \\ 1 \end{bmatrix} \mu_m + \sum_{i=1}^{m-1} \begin{bmatrix} f_i(\boldsymbol{\xi}) \\ 0 \end{bmatrix} \mu_i \mu_m.$$

Moreover, let $\kappa(t) \in \{0, 1\}$ be a discrete state variable which describes the mode of the system. We define that $\kappa(t) = 0$ denotes the time scale increase mode and $\kappa(t) = 1$ denotes the time scale decrease mode. Using this state variable, closed loop systems of each mode can be combined as follows:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \tau \end{bmatrix} &= \begin{bmatrix} f_0 \\ 1 \end{bmatrix} \{(1-\kappa)\mu_m^+ + \kappa\mu_m^-\} \\ &\quad + \sum_{i=1}^{m-1} \begin{bmatrix} f_i \\ 0 \end{bmatrix} \{(1-\kappa)\alpha_i\mu_m^+ + \kappa\beta_i\mu_m^-\}. \end{aligned}$$

From the above discussion, Time-State Control Form (2.2) described by BBM model is given by

$$\frac{d}{dt}\begin{bmatrix} \boldsymbol{\xi} \\ \tau \\ \kappa \end{bmatrix} = \begin{bmatrix} f_0 \\ 1 \\ 0 \end{bmatrix} \{(1-\kappa)\mu_m^+ + \kappa\mu_m^-\}$$

$$+ \sum_{i=1}^{m-1} \begin{bmatrix} f_i \\ 0 \\ 0 \end{bmatrix} \{(1-\kappa)\alpha_i\mu_m^+ + \kappa\beta_i\mu_m^-\} \tag{2.7a}$$

$$\begin{bmatrix} \boldsymbol{\xi} \\ \tau \\ \kappa \end{bmatrix}^+ = \begin{bmatrix} \boldsymbol{\xi} \\ \tau \\ g(\boldsymbol{\xi}, \tau, \kappa) \end{bmatrix} \tag{2.7b}$$

where the equation (2.7a) is a ordinary continuous time system which describes the continuous change of state, and the equation (2.7b) describes the discrete change of state. The mapping $g : \mathbb{R}^{n-1} \times \mathbb{R} \times \{0, 1\} \mapsto \{0, 1\}$ is a function of mode swiching condition.

3 Switching Condition for a Guaranty of Stability

In this section, we discuss switching conditions which guarantee stability of the system (2.7). First of all, for the system (2.7), we define mode switching time.

Definition 3.1 (Mode switching time). *Consider the system (2.7). The time T is called mode switching time (simply called switching time) if it satisfies:*

$$\kappa(\lim_{\epsilon \to -0}(T + \epsilon)) \neq \kappa(\lim_{\epsilon \to +0}(T + \epsilon)).$$

Moreover, T_0 denotes the initial time and T_n denote the nth switching time.

Since the inputs μ_i are given by (2.6), asymptotic stability of the state $\boldsymbol{\xi}$ is guaranteed if any mode switching don't occur. Then, we can consider the Lyapunov function $V_\beta(\boldsymbol{\xi})$ which corresponds to the time scale decrease mode.

But, monotone decreasing of $V_\beta(\boldsymbol{\xi})$ is not guaranteed when time scale τ increase. However, since state control part (2.2a) is asymptotically stabilized by the input (2.6), the state $\boldsymbol{\xi}$ can converge to zero if time scale τ enough increase. The convergence of the state $\boldsymbol{\xi}$ makes $V_\beta(\boldsymbol{\xi})$ decrease (note that it isn't always monotonous). So the following lemma is realized.

Lemma 3.1. *Let $U \subset \mathbb{R}^{n-1}$ be some neighborhood containing the origin $\boldsymbol{\xi} = 0$ and assume that the time scale τ increase monotonously. Then for all initial state $\boldsymbol{\xi}_0 = \boldsymbol{\xi}(\tau_0) \in U$ and $\gamma(0 < \gamma < 1)$, there exists $\bar{\tau} > 0$ such that:*

$$V_\beta(\boldsymbol{\xi}(\tau_0 + \tau)) \leq \gamma V_\beta(\boldsymbol{\xi}_0), \qquad \forall \tau > \bar{\tau}.$$

The previous lemma guarantees $V_\beta(\boldsymbol{\xi})$ to be less than initial value $V_\beta(\boldsymbol{\xi}_0)$ if time scale τ increase enough (more than $\bar{\tau}$). From this lemma, a sufficient condition of the switching function g for stabilization of system (2.7) is given as follows.

Theorem 3.1. *Let $U \subset \mathbb{R}^{n-1}$ be some neighborhood containing the origin $\boldsymbol{\xi} = 0$. For all initial state $\boldsymbol{\xi}(T_0) \in U$ and $\tau(T_0)$, the system (2.7) is locally stable if there exists switching function g such that the following two conditions are satisfied.*

$$\begin{aligned} &\text{i)} \quad \tau(T_n)\tau(T_{n+1}) \leq 0 \qquad \forall n \geq 0 \\ &\text{ii)} \quad V_\beta(\boldsymbol{\xi}(T_{n+1})) \leq \gamma V_\beta(\boldsymbol{\xi}(T_n)) \\ &\qquad\qquad \forall n \in \{n \geq 0 | \kappa(t) = 0, T_n < t < T_{n+1}\}, \end{aligned}$$

where γ is a scalar such that $0 < \gamma < 1$.

The condition ii) is more important out of two conditions in the previous theorem. Lemma 1 guarantees the existence of switching time T_{n+1} satisfying the condition ii). This condition implies that the Lyapunov function $V_\beta(\boldsymbol{\xi})$ decrease at the each switching times. We can avoid the probability to spoil the stability of the system which is caused by mode switching.

The condition i) makes the time scale τ to pass the origin $\tau = 0$ for each modes. This condition guarantees that the time scale τ can converge to zero after the state $\boldsymbol{\xi}$ converge to the origin.

Remark 3.1. *We can also treat the above discussion by using another Lyapunov function $V_\alpha(\boldsymbol{\xi})$ which corresponds to the time scale increase mode.*

Now we show the simulation result. Consider the 3-dimensional 2-input nonholonomic system described by Time-State Control Form as follows:

$$\begin{aligned} \frac{d\xi}{d\tau} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mu_1 \\ \frac{d\tau}{dt} &= \mu_2. \end{aligned}$$

The following control inputs μ_1 and μ_2 for time increase mode and time decrease mode are designed respectively:

$$\begin{cases} \mu_1^+ = [-1.00 \quad -1.73]\boldsymbol{\xi} \quad , \qquad \mu_2^+ = 1 \\ \mu_1^- = [-1.00 \quad 1.73]\boldsymbol{\xi} \qquad , \qquad \mu_2^- = -1 \end{cases} .$$

The inputs μ_1 which stabilize state control part are designed by LQ optimal control theory. Lyapunov function $V_\beta(\boldsymbol{\xi})$ which corresponds to the time scale decrease mode is given by $V_\beta(\boldsymbol{\xi}) = \boldsymbol{\xi}^T P \boldsymbol{\xi}$, where P is a positive definite solution of Riccati equation. The γ shown in the condition ii) is $\gamma = 0.9$ and initial value of the states $\boldsymbol{\xi}$ and τ are -1 respectively.

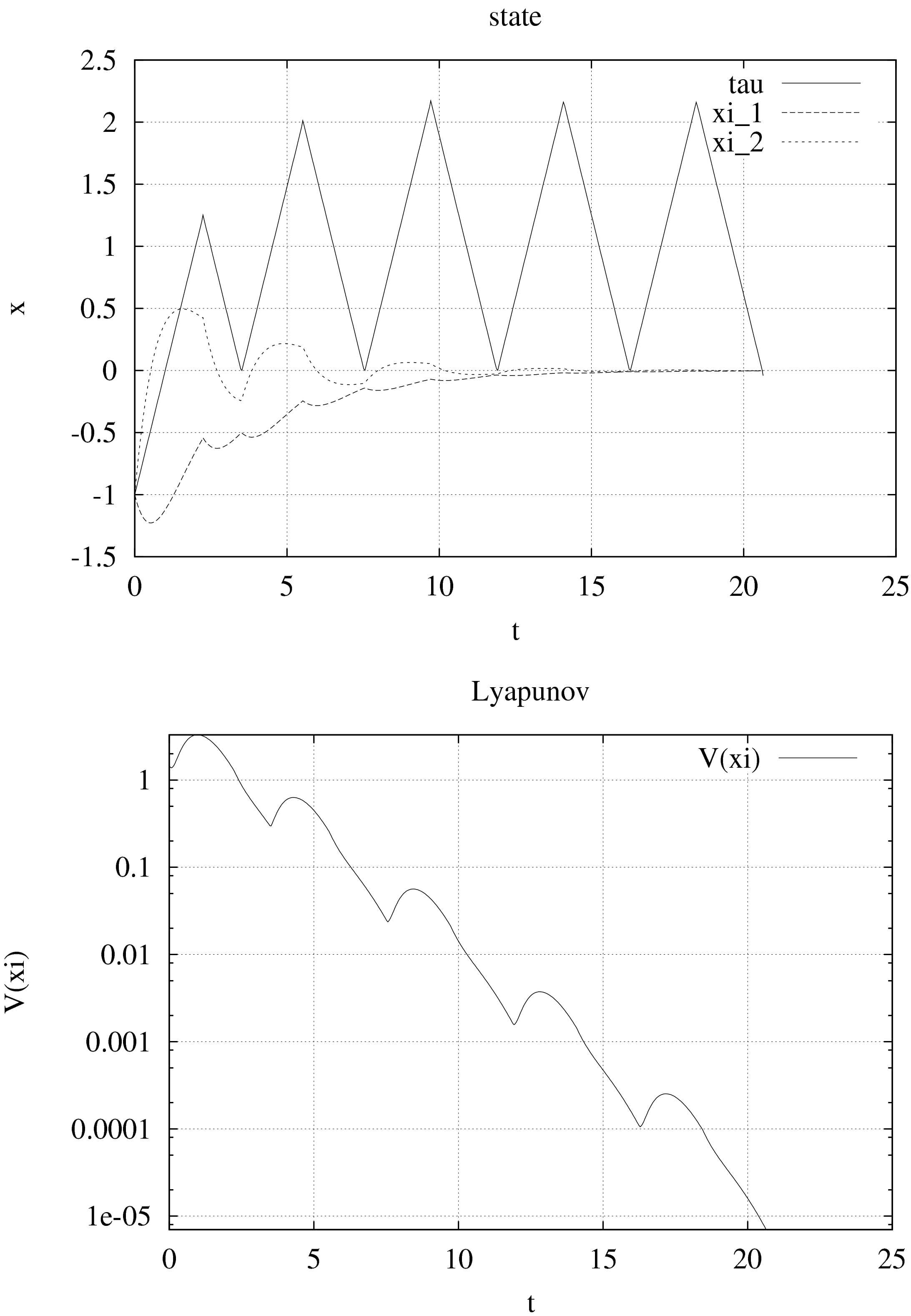

Figure 1: Simulation result(1)

Fig.1 shows the simulation result controlled by above controller. The above is the change of the state $\boldsymbol{\xi}$ and time scale τ. We can show that the change of τ is like a triangular wave because of mode switching and $\boldsymbol{\xi}$ converges to zero.

The bottom is the change of the Lyapunov function $V_\beta(\boldsymbol{\xi})$. We know that $V_\beta(\boldsymbol{\xi})$ decrease monotonously when time scale τ decrease because of its property. When time scale τ increase, $V_\beta(\boldsymbol{\xi})$ decrease but not monotonously. However, since the switching condition ii), $V_\beta(\boldsymbol{\xi})$ is at least less than last switching time.

4 Stability Condition for Chained System

In this section, the controled object is limited to chained system, which is one of the canonical forms of nonholonomic systems, and the conditions to stabilize the system are shown. Chained system is the n-dimensional ($n \geq 3$) 2-input nonholonomic system described by the following equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2. \tag{4.8}$$

For this system, we apply the following coordinate and input transformations.

$$\begin{bmatrix} \boldsymbol{\xi} \\ \hline \tau \end{bmatrix} = \begin{bmatrix} x_n \\ \vdots \\ x_2 \\ \hline x_1 \end{bmatrix}, \quad \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} u_2/u_1 \\ u_1 \end{bmatrix}$$

Then we obtain the following Time-State Control Form:

$$\frac{d\boldsymbol{\xi}}{d\tau} = A\boldsymbol{\xi} + B\mu_1 \tag{4.9a}$$

$$\frac{d\tau}{dt} = \mu_2, \tag{4.9b}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let μ_1^+ be a following feedback controller which stabilizes state control part (4.9a) when time scale τ increase.

$$\mu_1^+ = K_+ \boldsymbol{\xi} = \left[\begin{array}{ccc} -k_1 & \cdots & -k_{n-1} \end{array} \right] \boldsymbol{\xi} \tag{4.10}$$

It is easy to check that if the controller for time increase mode is given by (4.10), the closed loop system controlled by the following controller μ_1^- in time decrease mode has poles as same as those of closed loop system controlled by (4.10) in time increase mode.

$$\mu_1^- = K_- \boldsymbol{\xi} = \left[\begin{array}{ccc} (-1)^{n-2}k_1 & \cdots & (-1)^0 k_{n-1} \end{array} \right] \boldsymbol{\xi} \tag{4.11}$$

The closed loop systems of state control part (4.9a) for each mode are given by the following equations:

$$\left\{ \begin{array}{rcl} \dfrac{d\boldsymbol{\xi}}{d\tau} & = & [A + BK_+]\boldsymbol{\xi} \\ \dfrac{d\boldsymbol{\xi}}{d\tau'} & = & [-A - BK_-]\boldsymbol{\xi} \end{array} \right. , \tag{4.12}$$

where, in time decrease mode, $\tau' := -\tau$ is new time scale. From the definition of τ', time scale τ' increase monotonously.

By the way, since the feedback controllers for each mode (4.10) and (4.11) are given to have same poles of closed loop respectively, we obtain:

$$-A - BK_- = E_{n-1}[A + BK_+]E_{n-1},$$

where

$$E_n = E_n^{-1} = \mathrm{diag} \left[\begin{array}{cccc} 1 & -1 & \cdots & (-1)^{n-1} \end{array} \right].$$

The above equation implies that the system (4.12) can be described by:

$$\left\{ \begin{array}{rcl} \dfrac{d\boldsymbol{\xi}}{d\tau} & = & [A + BK_+]\boldsymbol{\xi} \\ \dfrac{d\bar{\boldsymbol{\xi}}}{d\tau'} & = & [A + BK_+]\bar{\boldsymbol{\xi}} \end{array} \right. \tag{4.13}$$

with the following coordinate transformation.

$$\bar{\boldsymbol{\xi}} = E_{n-1}\boldsymbol{\xi} \tag{4.14}$$

Therefore, we can identify the mode change occurred by input switching with the state jump which is defined by the coordinate transformation (4.14).

From the above discussion, if there exists a candidate of Lyapunov function which is invariant to coordinate transformation (4.14), we can use it as common Lyapunov function for each mode. The following lemma gives us the necessary and sufficient condition of positive definite matrix P which generate the common Lyapunov function.

Lemma 4.1. *Let $V(\boldsymbol{\xi}) = \boldsymbol{\xi}^T P \boldsymbol{\xi}$ $(P > 0)$ be a candidate of Lyapunov function for system (4.9a). Then, $V(\boldsymbol{\xi})$ is invariant to the coordinate transformation (4.14), i.e. $V(\boldsymbol{\xi}) = V(\bar{\boldsymbol{\xi}})$ is satisfied for all $\boldsymbol{\xi}$, if and only if P satisfies following equation:*

$$\begin{aligned} P_{ij} = 0, \quad \forall i, j \in \{i, j | i + j = 2l + 1, \\ 1 \le l \le n - 1,\ l \in \mathbb{N}\}, \end{aligned} \tag{4.15}$$

where P_{ij} is ij-th component of the matrix P.

The equation (4.15) implies that P must have zero on its ij-th component, where $i+j$ is odd number. Using the matrix P, we obtain the sufficient condition of state feedback controller which stabilizes state control part (4.9a). The condition is independent of switching function g.

Theorem 4.1. *If there exists symmetric matrix $P > 0$ and vector $K_+ \in \mathbb{R}^{n-1}$ such that they satisfy:*

$$(A + BK_+)^T P + P(A + BK_+) \le 0 \tag{4.16}$$

and P satisfies (4.15), then the feedback controller defined by (4.10) and (4.11) stabilize the system (4.9). The stability is independent of time scale switching function $g(\cdot)$.

It is difficult to solve the condition (4.16) because it is a nonlinear matrix inequality condition (NLMI) with respect to the matrices P and K_+. But by using following matrix transformation

$$X := P^{-1} \quad , \quad G := K_+ P^{-1},$$

we can reduce the condition (4.16) to the following linear matrix inequality (LMI).

$$AX + XA^T + BG + G^T B^T \ \le \ 0 \tag{4.17}$$

If the LMI condition (4.17) is feasible, we can obtain the matrices P and K_+ as follows.

$$P = X^{-1} \quad , \quad K_+ = GX^{-1}$$

Now we show the simulation result. Controlled object is same in the previous simulation, i.e. 3-dimensional chained system. The control inputs μ_1 and μ_2 are given by:

$$\begin{cases} \mu_1^+ = [-0.70 \quad -1.29]\boldsymbol{\xi} \quad , \qquad \mu_2^+ = 1 \\ \mu_1^- = [-0.70 \quad 1.29]\boldsymbol{\xi} \qquad , \qquad \mu_2^- = -1 \end{cases}$$

which is obtained by numerical solutions of the LMI condition (4.17) which is transformed from (4.16) where P satisfies the condition (4.15). Switching condition g is given by simple rule, i.e. the time scale τ goes and returns between $\tau = 0$ and $\tau = 2$. The initial states are $x_1(T_0) = 0 (= \tau)$, $x_2(T_0) = x_3(T_0) = -1$.

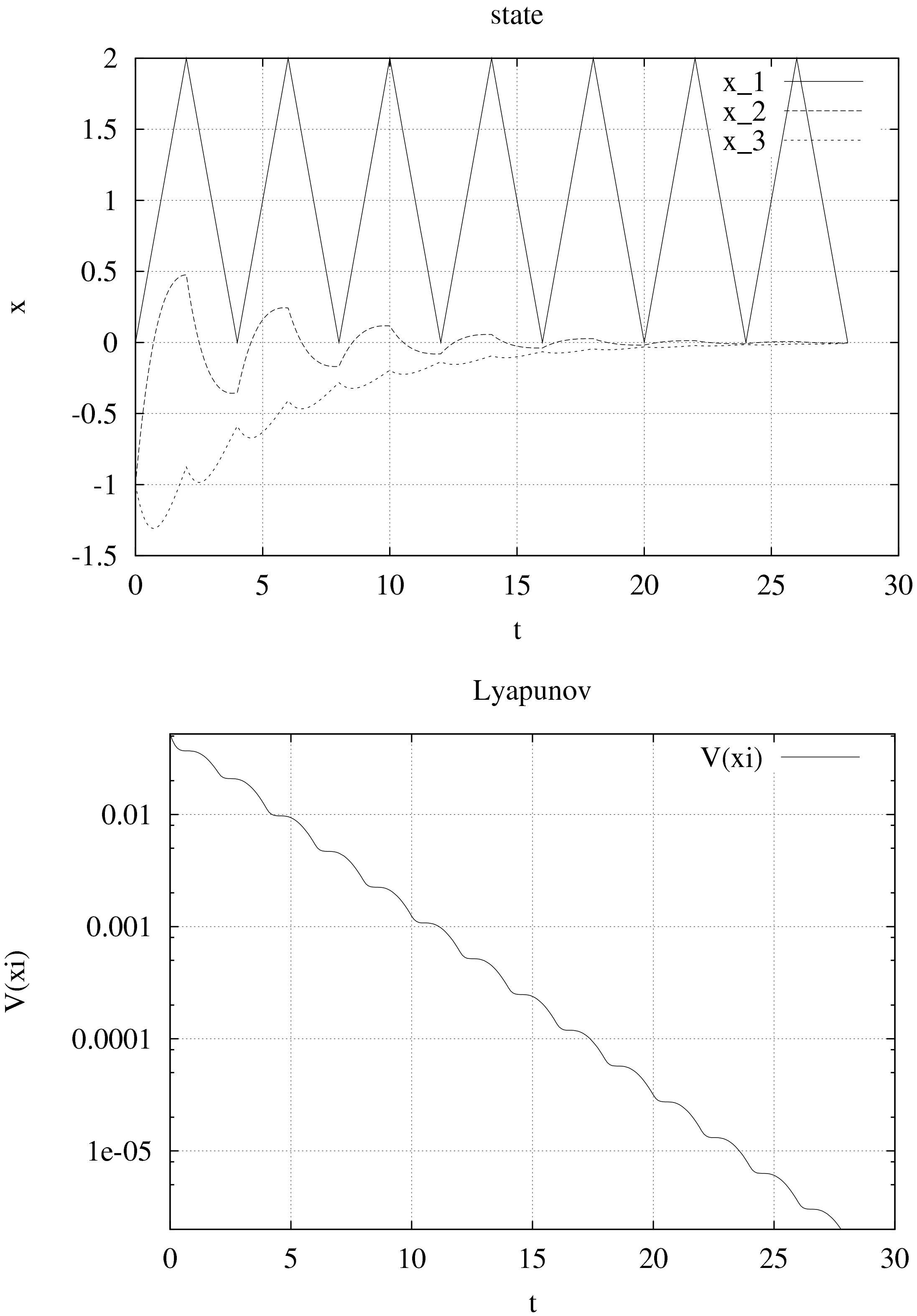

Figure 2: Simulation result(2)

Fig.2 shows the simulation result controlled by above controller. The aspect of Fig.2 is as same as Fig.1. As in the previous simulation, τ changes like a triangular wave and $\boldsymbol{\xi}$ converges to zero.

Unlike before, however, the Lyapunov function $V(\boldsymbol{\xi}) = \boldsymbol{\xi}^T P \boldsymbol{\xi}$ decrease monotonously for each mode. So $V(\boldsymbol{\xi})$ plays a role as the common Lyapunov function. This property is independent from the mode switching condition $g(\cdot)$.

5 Conclusion

In this paper, we treat the control method for nonholonomic systems based on Time-State Control Form with a viewpoint of hybrid control systems. This control method has a drawback that the switching conditions may spoil the stability of the system. Then, we introduce two conditions which guarantee the stability of the system. And the effectiveness of introduced condition are shown by some numerical simulations.

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