Investigating Duality on Stability Conditions*

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Abstract

This paper is devoted to investigate the role played by duality in stability analysis of linear time-invariant systems. We seek for a dual statement of a recently developed method for generating stability conditions, which combines Lyapunov stability theory with Finsler's Lemma. This method, developed in the time domain, is able to generate a set of (primal) equivalent stability tests involving extra multipliers. The resulting tests have very attractive properties. Stability is characterized via Linear Matrix Inequalities and we use optimization theory to obtain the duals. The dual problems are given a frequency domain interpretation.

Notation

$\mathbb{R}^n$ ($\mathbb{C}^n$) denotes the space of real (complex) vectors of dimension n. $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) denotes the space of real (complex) matrices of dimension $m \times n$. $\mathbb{S}^n$ ($\mathbb{H}^n$) denotes the space of real symmetric (complex Hermitian) matrices of dimension $n \times n$. For real or complex scalars or matrices $(\cdot)^T$, $(\cdot)^*$, and $(\cdot)^H$, indicate, respectively, transposition, complex conjugate, and complex conjugate plus transposition. $X \succ 0$ is used to denote that $X \in \mathbb{S}^n$ ($\mathbb{H}^n$) is *positive definite*. Analogous definitions follow for $X \prec 0$, $X \succeq 0$, $X \preceq 0$, which indicate, respectively that $X \in \mathbb{S}^n$ ($\mathbb{H}^n$) is, respectively, *negative definite*, *positive semidefinite*, and *negative semidefinite*.

*The author is supported by a grant from *FAPESP*, São Paulo, Brazil.

1 Motivation

Consider the continuous-time linear time-invariant system described by the equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) : [0, \infty) \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.

Using Lyapunov stability theory, an stability test for system (1) can be obtained as follows. Define the quadratic form $V : \mathbb{R}^n \to \mathbb{R}$ as

$$V(x) := x^T P x, \tag{2}$$

where $P \in \mathbb{S}^n$. Compute the quadratic form $\dot{V} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\dot{V}(x(t), \dot{x}(t)) := x(t)^T P \dot{x}(t) + \dot{x}(t)^T P x(t). \tag{3}$$

That is, the time derivative of the quadratic form (2) expressed as a function of $x(t)$ and $\dot{x}(t)$. According to Lyapunov stability theory, system (1) is asymptotically stable if for any $x(0) = x_0 \neq 0$ there exists $V(x(t)) > 0$, $\forall x(t) \neq 0$ such that[1]

$$\dot{V}(x(t), \dot{x}(t)) < 0, \quad \forall \dot{x}(t) = Ax(t), \quad (x(t), \dot{x}(t)) \neq 0, \quad t \in [0, \infty). \tag{4}$$

That is, if the quadratic form (3) is negative *along all trajectories* of system (1). The standard approach to verify the stability condition (4) is to explicitly substitute for $\dot{x}(t)$ into $\dot{V}(\cdot)$ using the system equation (1). This provides the equivalent condition

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA\right) x(t) < 0, \quad \forall x(t) \neq 0, \quad t \in [0, \infty), \tag{5}$$

and the well know stability test.

Lemma 1 (Lyapunov) *The time-invariant linear system is asymptotically stable if, and only if,* $\exists\, P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0.$

Although the Lyapunov method provides only a sufficient condition for stability, one can prove necessity of Lemma 1 by showing that if (1) is stable then there is always a positive definite matrix P that makes the condition given in Lemma 1 feasible.

Some authors have develop alternative stability tests from a different starting point, on the frequency domain (see [2, 3]). In this context, after taking the Laplace transform, asymptotic stability of the linear system (1) can be formulated as the existence of no zeros (nontrivial solutions) of the algebraic equation

$$(sI - A)\, q = 0, \quad q \neq 0, \tag{6}$$

[1]The *classic* statement of Lyapunov stability requires (4) to be verified for all $x(t) \neq 0$. It has been shown in [1] that testing for all $(x(t), \dot{x}(t)) \neq 0$ can be done without loss of generality.

where $s \in \mathbb{C}$, has positive real part. That is, asymptotic stability can be characterized as

$$\nexists\, s \in \mathbb{C}, q \in \mathbb{C}^n : s + s^* \geq 0, \quad (sI - A)\, q = 0, \quad q \neq 0. \tag{7}$$

Condition (7) is of little use unless the zeros of (6) are explicitly computed. In order to derive an indirect stability test, that does not require computing the zeros of a rational matrix, we use a lemma given in [2].

Lemma 2 *There exists a vector $p = sq \neq 0$ for some $s+s^* \geq 0$ if, and only if, $pq^H + qp^H \succeq 0$.*

Hence, introducing the vector $p \in \mathbb{C}^n$, $p = sq$, condition (7) can be rewritten as

$$\nexists\, p \in \mathbb{C}^n, q \in \mathbb{C}^n : pq^H + qp^H \succeq 0, \quad p - Aq = 0, \quad (p, q) \neq 0. \tag{8}$$

Substituting $p = Aq$ into the first inequality in (8) we obtain

$$\nexists\, q \in \mathbb{C}^n : Aqq^H + qq^H A^T \succeq 0, \quad q \neq 0. \tag{9}$$

From this point on, we proceed by defining the rank-one Hermitian matrix $Q \in \mathbb{H}^n$

$$Q = qq^H \succeq 0,$$

which provides the equivalent stability condition

$$\nexists\, Q \in \mathbb{H}^n : Q \succeq 0, \quad AQ + QA^T \succeq 0, \quad \operatorname{rank}(Q) = 1. \tag{10}$$

The above is summarized as the following stability test [2].

Lemma 3 (Ben-Tal) *The time-invariant linear system is asymptotically stable if, and only if, $\nexists\, Q \in \mathbb{S}^n : Q \succeq 0, \quad AQ + QA^T \succeq 0$.*

Again, with the steps shown above, we can only conclude that the condition in Lemma 2 is sufficient for stability. Some more steps are required to show that, for a system in the form (1), the rank constraint can be relaxed and that Q can be taken to be symmetric (instead of Hermitian) with no loss of generality. See [2, 3] for details or wait until Section 3 for a rigorous proof.

Comparing Lemmas 1 and 3, we notice that besides the qualitative *time domain* versus *frequency domain* duality, the given tests are also mathematical duals. In fact, by defining the Lagrangian function

$$\mathcal{L}(P, Q) = \operatorname{trace}\left[\left(A^T P + PA\right) Q\right] = \operatorname{trace}\left[\left(AQ + QA^T\right) P\right],$$

it follows from duality theory [4] that

$$\exists P \in \mathbb{S}^n : P \succ 0, \quad A^T P + PA \prec 0 \quad \Leftrightarrow \quad \nexists\, Q \in \mathbb{S}^n : Q \succeq 0, \quad AQ + QA^T \succeq 0$$

which shows that the stability conditions in Lemmas 1 and 3 are indeed duals. This fact is the main motivation of this paper. In the next sections we depart from the recently developed (time domain) Lyapunov stability analysis tool given in [1] aiming at the derivation of its (frequency domain) dual. We use optimization theory to derive the duals and show that one of the obtained dual stability conditions is closely related to the method developed in the recent paper [3].

2 Stability Analysis in the Time Domain

In the work [1] the authors have developed a method for performing stability analysis that combine the method of Lyapunov with a result known as Finsler's Lemma [5]. In the context of stability analysis of system (1), the main idea explored in [1] is to avoid explicitly substituting for $\dot{x}(t)$ into $\dot{V}(\cdot)$ using the system equation (1). Instead, the authors work directly with the stability conditions (4) rather than (5). This method will be briefly reviewed here.

As a first step, define the following vector and matrices

$$p := \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}, \qquad \mathcal{Q}(P) := \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix}, \qquad \mathcal{B} := \begin{bmatrix} A & -I \end{bmatrix}. \tag{11}$$

Using these definitions the stability condition (4) can be rewritten as

$$\dot{V}(p) = p^T \mathcal{Q}(P) p < 0, \quad \forall \mathcal{B}p = 0, \quad p \neq 0, \tag{12}$$

which is in a form suitable for application of the following result.

Lemma 4 (Finsler) *Let $p \in \mathbb{C}^n$, $\mathcal{Q} \in \mathbb{H}^n$ and $\mathcal{B} \in \mathbb{C}^{m \times n}$ such that* $\operatorname{rank}(\mathcal{B}) < n$. *The following statements are equivalent:*

i) $p^T \mathcal{Q} p < 0, \quad \forall \mathcal{B}p = 0, \quad p \neq 0.$

ii) ${\mathcal{B}^{\perp}}^H \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$

iii) $\exists\, \mu \in \mathbb{R} : \mathcal{Q} - \mu \mathcal{B}^H \mathcal{B} \prec 0.$

iv) $\exists\, X \in \mathbb{C}^{n \times m} : \mathcal{Q} + X\mathcal{B} + \mathcal{B}^H X^H \prec 0.$

Proof: See [1]. □

Recalling that the requirement $V(x(t)) > 0$, $\forall x(t) \neq 0$ can be simply stated as $P \succ 0$, the next theorem follows from the application of Lemma 4 on (12).

Theorem 5 (Primal) *The following statements are equivalent:*

i) The linear time-invariant system (1) is asymptotically stable.

ii) $\exists\, P \in \mathbb{S}^n : P \succ 0, \quad {\mathcal{B}^{\perp}}^T \mathcal{Q}(P) \mathcal{B}^{\perp} \prec 0.$

iii) $\exists\, P \in \mathbb{S}^n, \mu \in \mathbb{R} : P \succ 0, \quad \mathcal{Q}(P) - \mu \mathcal{B}^T \mathcal{B} \prec 0.$

iv) $\exists\, P \in \mathbb{S}^n, X \in \mathbb{R}^{2n \times n} : P \succ 0, \quad \mathcal{Q}(P) + X\mathcal{B} + \mathcal{B}^T X^T \prec 0.$

In Theorem 5 the variables μ and X are extra variables (multipliers) that add degrees of freedom to the stability tests. These additional degrees of freedom can be used, for instance, to derive less conservative stability conditions for uncertain systems (see [1] for details).

By letting Lemma 4 handle (12) we completely ignore the functional relation between the entries of the vector p, that is, we are ignoring the fact that the entries of p are $x(t)$ and $\dot{x}(t)$. Surprisingly, in this case, this can be done with no loss of generality. We can see this by evaluating

$$\mathcal{B}^{\perp} = \begin{bmatrix} I \\ A \end{bmatrix}$$

which reveals that

$$ii)\ \exists\, P \in \mathbb{S}^n : P \succ 0, \quad {\mathcal{B}^{\perp}}^T \mathcal{Q}(P) \mathcal{B}^{\perp} = A^T P + PA \prec 0,$$

is the familiar Lyapunov stability conditions given in Lemma 1. Also notice that, as shown in [1], a multiplier $X \in \mathbb{R}^{n \times m}$ can be assumed without loss of generality whenever $\mathcal{B}$ is real, which leads to the simplified form of condition $iv)$ in Theorem 5.

The above "recipe" for stability analysis can cope with with a plethora of stability problems, including the analysis of linear systems of order higher then first order without necessarily building a state space realization, the analysis of robustness with parameter dependent Lyapunov functions when the uncertainty lies in a given polytope, analysis of robustness and performances based on IQC (Integral Quadratic Constraints) and many more. The interested reader is referred to [1] for more details and applications.

3 Duality Via Optimization Theory

Of the three (time domain) stability conditions developed in the previous section, the (frequency domain) dual of the first one has already been considered in the introduction and is given in Lemma 3. For the other two, we developed dual statements in this section using optimization theory. The obtained dual conditions will be given a frequency domain interpretation in the Section 4. The steps performed here can also be repeated to generate a rigorous proof of Lemma 3.

Consider the following optimization problems.

$$\left.\begin{aligned} \rho_1 := \inf \quad & \rho \\ \text{s.t.} \quad & \mathcal{Q}(P) - \mu \mathcal{B}^T \mathcal{B} \prec \rho I, \\ & \operatorname{trace}(P) = 1, \\ & P \in \mathbb{S}^n, \quad \mu \in \mathbb{R}, \quad \rho \in \mathbb{R}, \quad P \succ 0. \end{aligned}\right\}, \tag{13}$$

$$\left.\begin{aligned} \rho_2 := \inf \quad & \rho \\ \text{s.t.} \quad & \mathcal{Q}(P) + X\mathcal{B} + \mathcal{B}^T X^T \prec \rho I, \\ & \mathrm{trace}(P) = 1, \\ & P \in \mathbb{S}^n, \quad X \in \mathbb{R}^{2n\times n}, \quad \rho \in \mathbb{R}, \quad P \succ 0. \end{aligned}\right\} . \tag{14}$$

It is straightforward to verify that the stability conditions given in items *iii*) and *iv*) of Theorem 5 are satisfied if, and only if, the values of $\rho_i < 0$, for $i = \{1, 2\}$. Although the values of the optimal ρ_i, $i = \{1, 2\}$, do not necessarily coincide, they are all strictly negative whenever system (1) is asymptotically stable. The scaling condition $\mathrm{trace}(P) = 1$ has been introduced with the purpose of making the optimization problems well posed.

In practice, problems $(13 - 14)$ can be solved to verify the feasibility of items *iii*) and *iv*) in Theorem 5. Here, the purpose of introducing these optimization problems is to be able to use optimization theory to search for a dual version of Theorem 5. Let us start by dualizing problem (13). The so called *dual problem* associated with (13) can be obtained by introducing the *dual variables* $H \in \mathbb{S}^{2n}$, $H \succeq 0$, $\lambda \in \mathbb{R}$, and the Lagrangian function

$$\begin{aligned} \mathcal{L}_1(P, \rho, \mu, H, \lambda) &= \rho + \mathrm{trace}\left[H\left(\mathcal{Q}(P) - \mu \mathcal{B}^T\mathcal{B} - \rho I\right)\right] + \lambda\left[1 - \mathrm{trace}(P)\right], \\ &= \lambda + \mathrm{trace}\left[P\left(\mathcal{S}(H) - \lambda I\right)\right] + \rho\left[1 - \mathrm{trace}(H)\right] - \mu\, \mathrm{trace}\left(\mathcal{B}H\mathcal{B}^T\right). \end{aligned}$$

On the above, $\mathcal{S}(H)$ is a *dual mapping*, i.e., a map of the dual matrix H computed such that

$$\mathrm{trace}\left(H\mathcal{Q}(P)\right) = \mathrm{trace}\left(\mathcal{S}(H)P\right). \tag{15}$$

The functional $\mathcal{Q}(P)$ and $\mathcal{B}$ are defined as in (11). Since $\mathcal{Q}(\cdot)$ is affine, the dual mapping $\mathcal{S}(H)$ can be explicitly computed as

$$\mathcal{S}(H) := \begin{bmatrix} I & 0 \end{bmatrix} H \begin{bmatrix} 0 \\ I \end{bmatrix} + \begin{bmatrix} 0 & I \end{bmatrix} H \begin{bmatrix} I \\ 0 \end{bmatrix}. \tag{16}$$

The existence of an explicit form for the dual mapping let us pose the dual problem to program (13)

$$\left.\begin{aligned} \lambda_1 := \max \quad & \lambda \\ \text{s.t.} \quad & \mathcal{S}(H) \succeq \lambda I, \\ & \mathrm{trace}\left(\mathcal{B}H\mathcal{B}^T\right) = 0, \\ & \mathrm{trace}(H) = 1, \\ & H \in \mathbb{S}^n, \quad \lambda \in \mathbb{R}, \quad H \succeq 0. \end{aligned}\right\} . \tag{17}$$

The above procedure can be repeated with respect to condition *iv*) and problem (14). Using the dual variables $H \in \mathbb{S}^{2n}$, $H \succeq 0$, $\lambda \in \mathbb{R}$, the Lagrangian function associated with program (14) is

$$\begin{aligned} \mathcal{L}_2(P, \rho, X, H, \lambda) &= \rho + \mathrm{trace}\left[H\left(\mathcal{Q}(P) + X\mathcal{B} + \mathcal{B}^T X^T - \rho I\right)\right] + \lambda\left[1 - \mathrm{trace}(P)\right], \\ &= \lambda + \mathrm{trace}\left[P\left(\mathcal{S}(H) - \lambda I\right)\right] + \rho\left[1 - \mathrm{trace}(H)\right] + 2\, \mathrm{trace}\left(X\mathcal{B}H\right). \end{aligned}$$

The dual mapping associated with the above Lagrangian is $\mathcal{S}(H)$, obtained in (16), so that the dual of the optimization program (14) is given by

$$\left.\begin{array}{rl} \lambda_2 := \max & \lambda \\ \text{s.t.} & \mathcal{S}(H) \succeq \lambda I, \\ & \mathcal{B}H = 0, \\ & \operatorname{trace}(H) = 1, \\ & H \in \mathbb{S}^n, \quad \lambda \in \mathbb{R}, \quad H \succeq 0. \end{array}\right\}. \tag{18}$$

Before using these problems to characterize stability it is necessary to verify to what extent the solution of the duals indeed provide useful information. It follows from duality theory (see, for instance, [4]) that

$$\lambda_i \leq \rho_i, \quad i = \{1, 2\}.$$

A straightforward conclusion is that

$$\lambda_i \geq 0, \quad \Rightarrow \quad \rho_i \geq 0, \quad i = \{1, 2\},$$

and, therefore, that the system (1) is *not* asymptotically stable whenever λ_i or ρ_i, $i = \{1, 2\}$ are positive. The complementary statement, that is,

$$\lambda_i < 0, \quad \Rightarrow \quad \rho_i < 0, \quad i = \{1, 2\},$$

which would enable us to conclude on asymptotic stability, can only be established in the presence of extra conditions. Fortunately, the strict character of the inequalities in the primal problems $(13-14)$ can be used as a *Slater constraint qualification*, which is able to ensure the existence of a nonempty set of solutions to the dual programs $(17-18)$ and no duality gap. That is, $\lambda_i = \rho_i$, $i = \{1, 2\}$. Conversely, the existence of a nonempty and bounded set of optimal solutions to the dual problems $(17-18)$ also guarantees no duality gap. These statements are based on Theorems 4.1.3 and 4.1.4 given in [4].

The analysis above establishes a complete symmetry between the primal programs $(13-14)$ and the dual programs $(17-18)$. In other words, asymptotic stability can be conclusively verified by solving either the primal or the dual programs. A dual version of Theorem 5 can be derived from Lemma 3 and $(17-18)$ as follows.

Theorem 6 (Dual) *The following statements are equivalent:*

i) The linear time-invariant system (1) is asymptotically stable.

ii) $\nexists\, Q \in \mathbb{S}^n : Q \succeq 0, \quad AQ + QA^T \succeq 0.$

iii) $\nexists\, H \in \mathbb{S}^{2n} : H \succeq 0, \quad \mathcal{S}(H) \succeq 0, \quad \operatorname{trace}\left(\mathcal{B}H\mathcal{B}^T\right) = 0.$

iv) $\nexists\, H \in \mathbb{S}^{2n} : H \succeq 0, \quad \mathcal{S}(H) \succeq 0, \quad \mathcal{B}H = 0.$

Notice that Theorem 6 is indeed a set of "instability condition" in the sense that asymptotically stability is concluded from the unfeasibility of given test sets. Also notice that conditions *iii*) and *iv*) are essentially the same, since

$$\operatorname{trace}\left(\mathcal{B}H\mathcal{B}^T\right) = 0 \quad \Leftrightarrow \quad \mathcal{B}H = 0$$

when $H \succeq 0$.

4 Stability Analysis in the Frequency Domain

In this section we provide the dual stability conditions *iii*) and *iv*) stated in Theorem 6 with a frequency domain interpretation. We depart from Lemma 2 and the associated stability condition (8). But instead of substituting $p = Aq$ into (8) we proceed as in [3], and define vector $h \in \mathbb{C}^{2n}$ and the rank-one Hermitian matrix $H \in \mathbb{H}^{2n}$

$$h := \begin{pmatrix} q \\ p \end{pmatrix}, \qquad\qquad H := hh^H \succeq 0. \tag{19}$$

Noticing that

$$pq^H + qp^H = \mathcal{S}(H), \qquad\qquad (p - Aq)\, h^H = \mathcal{B}H, \tag{20}$$

where $\mathcal{S}(H)$ is defined in (16) and $\mathcal{B}$ is defined in (11), the stability condition (8) can be equivalently written as

$$\nexists\, H \in \mathbb{H}^{2n} : \mathcal{S}(H) \succeq 0, \quad \mathcal{B}H = 0, \quad \operatorname{rank}(H) = 1. \tag{21}$$

The above condition should be compared with condition *iv*) in the dual Theorem 6. From the comparison it becomes clear that condition (21) can be relaxed in two points: first, H can be assumed to be real symmetric instead of Hermitian; second, and more importantly, the nonconvex rank constraint can be dropped. Since Theorem 6 provide necessary and sufficient conditions for stability, these relaxations can be performed without any loss of generality. The first relaxation is, at some extent, an expected consequence of the fact that the system (1) evolves on the field of real numbers. The second relaxation appears frequently in the context of combinatorial optimization and control [3, 6, 7].

Condition *iii*) of Theorem 6 can be generated via a similar relaxation procedure. Notice that using (19) and the equivalence

$$\mathcal{B}h = 0, \quad h \neq 0, \quad \Leftrightarrow \quad h^H \mathcal{B}^T \mathcal{B} h = \operatorname{trace}\left(\mathcal{B}H\mathcal{B}^T\right) = 0, \quad \operatorname{rank}(H) = 1,$$

provides

$$\nexists\, H \in \mathbb{H}^{2n} : H \succeq 0, \quad \mathcal{S}(H) \succeq 0, \quad \operatorname{trace}\left(\mathcal{B}H\mathcal{B}^T\right) = 0, \quad \operatorname{rank}(H) = 1, \tag{22}$$

whose relaxed version is item *iii*) of Theorem 6. This condition is also closely related with the optimization problem

$$\left.\begin{array}{rl} \nu := \min & \operatorname{trace}\left(\mathcal{B}H\mathcal{B}^T\right) \\ \text{s.t.} & \mathcal{S}(H) \succeq 0, \\ & \operatorname{trace}(H) = 1, \\ & H \in \mathbb{S}^n, \quad H \succeq 0. \end{array}\right\}, \tag{23}$$

which has been obtained in [3]. At this point it is immediate to conclude from Theorem 6 that system (1) is unstable if, and only if, $\nu > 0$.

The above shows that the all stability conditions given in the (time domain) primal Theorem 5 can be given a frequency domain interpretation through the dual Theorem 6.

5 Conclusion

In this paper we have used optimization theory to obtain a dual statement of the stability analysis tool introduced in [1]. The primal conditions are derived using a combination of Lyapunov stability theory with Finsler's Lemma, hence in the time domain, while the dual conditions have been given a frequency domain interpretation. This establishes a complete parallel between mathematical duality and a conceptually appealing time domain versus frequency domain duality.

The results in the paper also show that the methods of [1] and [2, 3] are closely related, the first one working on the (primal) time domain and the second ones on the (dual) frequency domain. This correspondence can be used, for instance, to provide alternative proofs to the rank constraint relaxations considered in [3].

Although we do not present any application of the obtained duality relations in the paper, we foresee that the given interpretations can be of much interest. For instance, a tempting application is to use duality to provide a frequency domain interpretation to stability conditions which are known only in the time domain. This is the case, for instance, for linear systems with Markovian jump parameters, where stability has been characterized exclusively in the time domain [8]. Conversely, results that are based on frequency domain methods, such as the pole placement results of [3], could be given time domain interpretations. In special, we are currently investigating to what extent duality can be used to generalize the concept of pole placement to systems with no clear frequency domain representation, such as time varying uncertain systems and nonlinear systems.

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