STATE FEEDBACK MIXED H^2 / H^∞ PROBLEM FOR LINEAR SYSTEMS WITH FINITE JUMPS

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Abstract

The paper considers the mixed H^2/H^∞ problem for linear systems with jumps. There are two main results proved in the paper. The first one provides an evaluation of the norm induced by the inputs of exponentially stable systems with jumps. This evaluation is useful in design problems when a stabilising controller minimising the induced mixed H^2/H^∞ norm is required. The second result gives the solution of the state feedback mixed H^2/H^∞ problem.

Keywords: Linear systems with finite jumps, Stabilisation, Mixed H^2/H^∞ control problem, State feedback, Riccati type systems with jumps.

1 Introduction

It is already a known fact that jumps systems play an important role in control theory. A general introduction in the problem of the analysis and design of such systems can be found in [5]. Applications include the control problem of sampled-data systems which are intensively used in the control of continuous-time plants using digital devices.

Among the major recent results in this area we mention the Bounded-Real-Lemma type result given in [6], and the H^2 and H^∞ theories developed in [2], [3] and [4]. In the present paper a new result concerning the mixed H^2/H^∞ problem is given. A similar problem has received much attention in the control literature and solutions have been derived for the continuous-time case in different papers (*e.g.* [1]). In this paper a specific development for linear systems with jumps is given. Hence in Section 3 the expression of the performance index corresponding to the mixed H^2/H^∞ problem is derived. Based on this result, in Section 4 the state-feedback mixed H^2/H^∞ problem is solved. Its solution is expressed as function of a game-theoretic Riccati type system with jumps.

2 Notations and definitions

Consider the following system with jumps:

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + B_{oc} w_{oc}(t) + B_{1c} w_{1c}(t), \ \ t \neq ih \\
&x(ih^{+}) = A_d x(ih) + B_{od} w_{od}(i) + B_{1d} w_{1d}(i), \ i = 0,1,.. \\
&y(t) = Cx(t) + Dw_{1c}(t) \\
&y_d(i) = C_d x(ih) + D_d w_{1d}(i)
\end{aligned}
\tag{2.1}
$$

where $x \in \mathbf{R}^n$ denotes the state vector, w_{1c} is the continuous bounded in power input, w_{oc} is a continuous white noise, w_{1d} is the discrete-time bounded in power input and w_{od} denotes a discrete-time white noise. The outputs y_c and y_d are continuous-time and discrete-time, respectively. It is assumed that the white noise signals have unit spectral densities. In (2.1), $h > 0$ denotes the sampling period. By definition, the state $x(t)$ is left continuous and right discontinuous with finite jumps at instants ih. It is also assumed that the white noise signals are independent of the control variables w_{1c} and w_{1d}. A definition based on the state transition matrix of exponential stability (ES) of linear systems with jumps can be found in [7]. The following known result provides necessary and sufficient conditions for exponential stability.

Proposition 2.1 *The system (2.1) is ES if and only if* $\left\| A_d e^{Ah} \right\| < 1$.

If the white noise components in (2.1) are missing, namely if $B_{oc} = 0$ and $B_{od} = 0$ then the obtained system which will be denoted by G_1 has only the inputs w_{1c} and w_{1d}. In the situation when G_1 is ES, one can consider its associated input-output operator $\mathbf{G_1}$ defined as:

$$
\begin{gathered}
\mathbf{G_1} : L^2[0,\infty) \times \ell^2 \mapsto L^2[0,\infty) \times \ell^2 \\
(w_{1c}, w_{1d}) \mapsto (y, y_d)
\end{gathered}
$$

where $L^2[0,\infty)$ and ℓ^2 denote the standard Lebesgue spaces of the square integrable $\mathbf{R}^k$ valued functions over $[0,\infty)$ and over $\{0,1,...\}$, respectively.

3 Performance analysis pf hybrid linear systems

The purpose of the developments in this section is to compute:

$$
J_0 = \sup_w E\left[\|y\|^2_{L^2} + \|y_d\|^2_{\ell^2} - \gamma^2 \left(\|w_{1c}\|^2_{L^2} + \|w_{1d}\|^2_{\ell^2} \right) \right] \tag{3.1}
$$

where $w = (w_{1c}, w_{1d}) \in L^2(0,\infty) \times \ell^2$ and $\gamma > \|\mathbf{G}_1\|$, with $\mathbf{G_1}$ defined above. Remark that if w_{1c} and w_{1d} are ignored then the norm induced by w_{oc} and w_{od} is just the H^2 norm of the linear systems with jumps.

The main result of this section is given by the following theorem:

Theorem 3.1 *The optimum in (3.1) is given by*

$$J_0 = Tr\left(B_{od}{}^T X\left(ih^+\right)B_{od}\right) + \frac{1}{h}\int_{0^+}^{h} Tr\left(B_{oc}{}^T X(t) B_{oc}\right)dt \qquad (3.2)$$

where $X(t)$ *denotes the stabilising solution of the Riccati type equations:*

$$-\dot{X}(t) = A^T X(t) + X(t)A + \left(X(t)B_{1c} + C^T D\right)^{-1}\left(\gamma^2 I - D^T D\right)\left(B_{1c}{}^T X(t) + D^T C\right) + C^T C \qquad (3.3)$$

at $i \neq ih$*, and*

$$\begin{aligned} X(ih) = {} & A_d{}^T X\left(ih^+\right)A_d + \left(A_d{}^T X(ih)B_{1d} + C_d{}^T D_d\right)\left(\gamma^2 I - D_d{}^T D_d - B_{1d}{}^T X\left(ih^+\right)B_{1d}\right)^{-1} \\ & \times\left(A_d{}^T X(ih)B_{1d} + C_d{}^T D_d\right) + C_d{}^T C_d, \ i = 0,1, \ldots \end{aligned} \qquad (3.4)$$

namely, $X(t)$ *satisfies the following two conditions:*

a) $\gamma^2 I - D_d{}^T D_d - B_d{}^T P\left(ih^+\right)B_d > 0, \ i \in \{0,1,\ldots\}$, *and*
b) The linear system with jumps:

$$\begin{aligned} \dot{x}(t) &= \left[A + B_{1c}\left(\gamma^2 I - D^T D\right)^{-1}\left(D^T C + B_{1c}{}^T X(t)\right)\right]x(t) \\ x\left(ih^+\right) &= A_d + B_{1d}\left(\gamma^2 I - D_d{}^T D_d - B_{1d}{}^T X\left(ih^+ B_{1d}\right)\right)^{-1} \\ &\quad \times\left(B^T{}_{1d} X\left(ih^+\right)A_d + D_d{}^T D_d\right)x\left(ih^+\right) \end{aligned}$$

is ES.

Proof. Let differentiate using the Itô's formula the function $x^T(t)X(t)x(t)$, where $X(t)$ denotes the stabilising solution of the system of Riccati equations with jumps (3.3) and (3.4):

$$\begin{aligned} d\left(x^T X x\right) = {} & [x^T \dot{X} x + \left(Ax + B_{1c} w_{1c}\right)^T X x + x^T X\left(Ax + B_{1c} w_{1c}\right) + Tr\left(B_{oc}{}^T X B_{oc}\right)]dt \\ & + \left(B_{oc}{}^T X x + x^T X B_{oc}\right)dw_{oc} \end{aligned} \qquad (3.5)$$

where the dependence with respect to *t* has been omitted in order to simplify the writing.
Integrating the above equation and taking into account (3.3) one obtains:

$$\begin{aligned} \int_{ih^+}^{(i+1)h} d\left(x^T X x\right) = {} & \int_{ih^+}^{(i+1)h} [-x^T\left(\gamma^{-2} X B_{1c} B_{1c}{}^T X + C_d{}^T C_d\right)x + Tr\left(B_{oc}{}^T X B_{oc}\right)]dt \\ & + \left(B_{oc}{}^T X x + x^T X B_{oc}\right)dw_{oc} \end{aligned} \qquad (3.6)$$

On the other hand, since w_{od} is allowed to be independent of w_{1d}, using (3.4) it follows that:

$$
\begin{aligned}
&\int_{ih^+}^{(i+1)h} d\left(x^T Xx\right) = x^T\left((i+1)h\right)X\left((i+1)h\right)x\left((i+1)h\right) - x^T\left(ih^+\right)X\left(ih^+\right)x\left(ih^+\right) \\
&= x^T\left((i+1)h\right)X\left((i+1)h\right)x\left((i+1)h\right) - x^T(ih)X(ih)x(ih) + x^T(ih)C_d{}^T C_d x(ih) \\
&-\gamma^2 w_{1d}{}^T(i)w_{1d}(i) - 2x^T\left(ih^+\right)B_{od}w_{od}(i) - 2w_{1d}{}^T(i)B_{1d}{}^T X\left(ih^+\right)B_{od}w_{od}(i) \\
&- w_{0d}{}^T(i)B_{od}{}^T X\left(ih^+\right)B_{0d}w_{od}(i) + P(i)
\end{aligned}
\tag{3.7}
$$

where $P(i) \geq 0$ is defined as:

$$
\begin{aligned}
P(i) := &[x^T(ih)A_d{}^T X\left(ih^+\right)B_{1d}\left(\gamma^2 I - D_d{}^T D_d - B_{1d}{}^T X\left(ih^+\right)B_{1d}\right)^{-1} - w_{1d}{}^T] \\
&\times\left(\gamma^2 I - D_d{}^T D_d - B_{1d}{}^T X\left(ih^+\right)B_{1d}\right) \\
&\times[\left(\gamma^2 I - D_d{}^T D_d - B_{1d}{}^T X(ih)B_{1d}\right)^{-1} B_{1d}{}^T X\left(ih^+\right)A_d x(ih) - w_{1d}]
\end{aligned}
$$

Further, the right side terms of (3.6) and (3.7) have been equalised and then the mean has been applied to the obtained equality based on the fact that

$$
E\int_{ih^+}^{(i+1)h}\left(B_{oc}{}^T Xx + x^T XB_{oc}\right)dw_{oc} = 0.
$$

Then, changing the limits of integration to 0 and $T = kh + r$ with $r \in (0, h)$ and making $k \to \infty$ one obtains (3.2).

4 State feedback mixed H^2 / H^∞ problem

Consider the linear system with jumps:

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + B_{oc}w_{oc}(t) + B_{1c}u_{1c}(t) + B_{2c}u_{2c}(t),\ t \neq ih \\
&x\left(ih^+\right) = A_d x(ih) + B_{od}w_{od}(i) + B_{2d}u_{2d}(i),\ i = 0,1,.. \\
&y_{1c}(t) = Cx(t) + Du_{2c}(t) \\
&y_{1d}(i) = u_{2d}(i) \\
&y_{2c}(t) = x(t) \\
&y_{2d}(i) = x(ih)
\end{aligned}
\tag{4.1}
$$

where u_{1c} is an exogenous input, u_{2c}, u_{2d} are control variables, y_{1c} is the regulated output and y_{2c}, y_{2d} are measured outputs. For the simplicity of the computations the following orthogonality assumption is made:

$$
D^T\left[C \quad D\right] = \left[0 \quad I\right] \tag{4.2}
$$

Notice that although (4.1) is not the most general form of a system with jumps, it covers the sampled-data systems special form ([3]).

The problem consists in finding the gains $F(t), t \neq ih$ and $F_d(i), i = 0,1,\ldots$ such that the resulting system obtained by taking $u_{2c}(t) = F(t)x(t), t \neq ih$ and $u_{2d}(i) = F_d(i)x(ih), i = 0,1,..$has the following properties:

i) It is ES;

ii) The H^∞ norm of the resulting system with jumps obtained by ignoring the white noises $w_{oc}(t)$ and $w_{od}(i)$, respectively is less than a specified $\gamma > 0$;

iii) The performance index (3.2) is minimized.

Remark that $X(t)$ in (3.2) is in the mixed H^2/H^∞ design problem the stabilising solution of the Riccati norm type system (3.3) and (3.4) associated with the resulting system obtained with $u_{2c}(t) = F(t)x(t), t \neq ih$ and $u_{2d}(i) = F_d(i)x(ih), \; i = 0,1,..$

The main result of this section is:

Theorem 4.1 *The solution of the state feedback mixed* H^2/H^∞ *problem associated with the system (4.1) is given by:*

$$F(t) = -B_{2c}{}^T X(t), \; t \neq ih \tag{4.3}$$

$$F_d(i) = -E^{-1}(ih^+) B_{2d}{}^T X(ih^+) A_d, \; i = 0,1,\ldots \tag{4.4}$$

where

$$E(ih^+) := I + B_{2d}{}^T X(ih^+) B_{2d} \tag{4.5}$$

and $X(.)$ *is the* h*-periodic stabilising solution of the game-theoretic Riccati type system with jumps:*

$$-\dot{X}(t) = A^T X(t) + X(t)A + X(t)\left(\gamma^{-2} B_{1c} B_{1c}{}^T - B_{2c} B_{2c}{}^T\right) X(t) + C^T C \tag{4.6}$$

$$X(ih) = A_d{}^T X(ih^+) A_d - A_d{}^T X(ih^+) B_{2d} E^{-1}(ih^+) B_{2d}{}^T X(ih^+) A_d \tag{4.7}$$

Proof. Consider the cost function

$$\begin{aligned} J(x_0, u_{1c}, u_{2c}, u_{2d}) = \int_{0^+}^{h} \left(\|Cx(t) + Du_{2c}(t)\|_2{}^2 - \gamma^2 \|u_{1c}(t)\|_2{}^2 \right) dt \\ + \|u_{2d}(0)\|_2{}^2 \end{aligned} \tag{4.8}$$

Then replacing $C^T C$ from (4.6) into (4.8) and using the orthogonality condition (4.2) one directly obtains for $\tilde{u}_{1c}(t) = \gamma^{-2} B_{1c}{}^T X(t)x(t), \; t \neq ih$:

$$J(x_0,\tilde{u}_{1c},u_{2c},u_{2d})=x^T(0^+)X(0^+)x(0^+)+\int_{0^+}^{h}\left(u_{2c}(t)+B_{2c}{}^T X(t)x(t)\right)^T\left(u_{2c}(t)+B_{2c}X(t)x(t)\right)dt \\ +\left\|u_{2d}(0)\right\|_2{}^2 \tag{4.9}$$

Further, taking into account the second equation (4.1) with $w_{od}=0$, (4.7) and expression (4.5) of $E(ih^+)$ one obtains that:

$$x^T(0^+)X(0^+)x(0^+)+\left\|u_{2d}(0)\right\|_2{}^2=x_0{}^T X(0)x_0+\left[u_{2d}(0)+E^{-1}(0^+)B_{2d}{}^T X(0^+)A_d\right]^T E(0^+) \\ \times\left[u_{2d}(0)+E^{-1}(0^+)B_{2d}{}^T X(0^+)A_d\right] \tag{4.10}$$

where $x_0:=x(0)$ and therefore

$$J(x_0,\tilde{u}_{1c},u_{2c},u_{2d})=x_0{}^T X(0)x_0-x^T(h)X(h)x(h)+P_{1c}(h)+P_{1d}(h) \tag{4.11}$$

with

$$P_{1c}(h):=\int_{0^+}^{h}\left(u_{2c}(t)+B_{2c}{}^T X(t)x(t)\right)^T\left(u_{2c}(t)+B_{2c}X(t)x(t)\right)dt\geq 0$$
$$P_{1d}(h):=\left[u_{2d}(0)+E^{-1}(0^+)B_{2d}{}^T X(0^+A_d)\right]^T E(0^+)\left[u_{2d}(0)+E^{-1}(0^+)B_{2d}{}^T X(0^+A_d)\right]\geq 0 \tag{4.12}$$

Consider now the stabilising state feedback gains $\hat{F}(t), t\neq ih$ and $\hat{F}_d(ih), i=0,1,\ldots$ such that the resulting system with jumps obtained with $\hat{u}_2(t)=\hat{F}(t)\hat{x}(t),\ t\neq ih$ and $\hat{u}_{2d}(i)=\hat{F}_d(ih)\hat{x}(ih)$, $i=0,1,\ldots$ has the H^∞ norm less than γ. Then using the norm Riccati type equations (3.3) and (3.4) written for the resulting system, direct computations give:

$$J(x_0,u_{1c},\hat{u}_{2c},\hat{u}_{2d})=x_0{}^T\hat{X}(0)x_0-\hat{x}^T(h)\hat{X}(h)\hat{x}(h)-P_2(u_{1c}) \tag{4.13}$$

where

$$P_2(u_{1c})=\int_{o^+}^{h}\left(\gamma u_{1c}-\gamma^{-1}B_1{}^T\hat{X}(t)\hat{x}(t)\right)^T\left(\gamma u_{1c}-\gamma^{-1}B_1{}^T\hat{X}(t)\hat{x}(t)\right)dt\geq 0$$

and $\hat{X}(.)$ is the stabilising solution of the Riccati system (3.3), (3.4) corresponding to the closed loop system. Taking in (4.11) $u^*{}_{2c}(t)=-B_{2c}{}^T X(t)x(t)$, $t\in(0,h]$ and $u_{2d}{}^*(0)=-E^{-1}(0^+)B_{2d}{}^T X(0^+)A_d$, from (4.11) and (4.13) one obtains:

$$x_0{}^T X(0)x_0-x^T(h)X(h)x(h)\leq x_0{}^T\hat{X}(0)x_0-\hat{x}^T(h)\hat{X}(h)\hat{x}(h)-P_2(\tilde{u}_{1c}). \tag{4.14}$$

Define now the cost function:

$$J_i(x_0, u_{1c}, u_{2c}, u_{2d}) = \int_{0^+}^{ih} \left(\| Cx(t) + Du_{2c}(t) \|_2^{\,2} - \gamma^2 \| u_{1c}(t) \|_2^{\,2} \right) dt + \sum_{k=0}^{i-1} \| u_{2d}(k) \|_2^{\,2} \tag{4.15}$$

Since $X(.)$ and $\hat{X}(.)$ are stabilising solutions of the Riccati type systems (4.6), (4.7) and (3.3), (3.4) respectively, it follows that:

$$\lim_{i \to \infty} x(ih) = \lim_{i \to \infty} \hat{x}(ih) = 0$$

and therefore (4.14) together with (4.15) gives:

$$x_0^{\,T} X(0) x_0 \le x_0^{\,T} \hat{X}(0) x_0 - \sum_{k=0}^{\infty} P_2(\tilde{u}_{1c}(k))$$

from which it follows that

$$X(0) \le \hat{X}(0) \tag{4.16}$$

Further, repeating the same reasoning for the cost function:

$$J_t(x_0, u_{1c}, u_{2c}, u_{2d}) = \int_{t}^{ih} \left(\| Cx(t) + Du_{2c}(t) \|_2^{\,2} - \gamma^2 \| u_{1c}(t) \|_2^{\,2} \right) dt + \sum_{k=0}^{i-1} \| u_{2d}(k) \|_2^{\,2}$$

with $t \in (0, h)$, one obtains that

$$X(t) \le \hat{X}(t) \tag{4.17}$$

for all $t \in (0, h)$. Since for the state feedback gains (4.3) and (4.4) the norm Riccati type system (3.3), (3.4) coincides with the game-theoretic Riccati system (4.6), (4.7) it follows that the corresponding state feedback controls are stabilising and the resulting system with jumps has the norm less than γ. Moreover from (4.16) and (4.17) it results that the minimum of the cost function (3.2) corresponding to the mixed H^2 / H^∞ problem is minimized by the stabilizing solution of (4.6), (4.7) and thus the proof ends.

5 Conclusions

The result stated in Theorem 3.1 is useful in control problems of linear systems with jumps in which a stabilising controller is required such that the cost function J_0 is minimised. Such developments has been performed for the case of two-input, two-output systems with finite jumps. Based on this result Theorem 4.1 provides a solution to the mixed H^2/H^∞ state-feedback control problem.

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