Over-determined Systems

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Abstract

Let $R(\partial)$ be a matrix whose entries are linear partial differential operators, with constant coefficients. Let Ω be a non-empty, open, bounded, convex set. We consider the homogeneous system $R(\partial)f = 0$ in a neighborhood of $\overline{\Omega}$, subject to the boundary condition $f = g$ in a neighborhood of $\partial\Omega$. For a given smooth function g, we give a criterion for the (unique) existence of a smooth solution f to this problem. There is an obvious necessary condition: $R(\partial)g = 0$ in a neighborhood of $\partial\Omega$. We characterize the class of differential operators $R(\partial)$ for which the problem is solvable for any g satisfying the necessary condition. A system is over-determined if and only if there exists a unique solution to the problem, for any such Ω and g. We derive an algebraic criterion for this property. Finally, in the case where the solution is non-unique, we consider the possibility of obtaining uniqueness by fixing several components of the desired solution.

1 Introduction

This paper is motivated by the notion of over-determined systems, as discussed in Palamodov's book [6]. Speaking in behavioral terms [8], this property is a strong form of autonomy [10], and it is algebraically characterized by kernel representation matrices that are minor right prime. Since the equivalence of controllability and left prime representation matrices has been discussed quite extensively in the behavioral literature of the last few years, it seems interesting to give an interpretation to the somehow dual situation as well. The systems theoretic significance of the notion is closely related to the extension problem, a boundary value problem in which the values of the desired function are prescribed in a whole neighborhood of the boundary. It turns out that a behavior is over-determined if and only if the extension problem is uniquely solvable for any choice of the boundary data that is compatible with the system laws. The present paper can be seen as a small step towards a treatment of boundary value problems within the behavioral framework. However, it relies strongly on prior work of Palamodov [6] and Malgrange [4].

2 Preliminaries

For an open set $U \subseteq \mathbb{R}^n$, let $\mathcal{E}(U)$ denote the space of (complex-valued) smooth functions on U. Let $\Omega \subset \mathbb{R}^n$ be a non-empty, open, bounded, convex set, and let $K := \overline{\Omega}$ be its closure. Let $\mathcal{E}(K)$ denote the space of all smooth functions in an (open) neighborhood of K, that is,

$$\mathcal{E}(K) = \{f \in \mathcal{E}(U) \mid U \text{ is a neighborhood of } K\}.$$

Let $f, g \in \mathcal{E}(K)$. We write $f \overset{K}{\sim} g$ if there exists a neighborhood of K on which the values of f and g coincide. This defines an equivalence relation on $\mathcal{E}(K)$. Define

$$\mathcal{E}_K := \mathcal{E}(K)/\sim.$$

Similarly, let ∂K denote the boundary of K (or Ω, equivalently) and let $\mathcal{E}(\partial K)$ be the space of smooth functions in a neighborhood of ∂K. We write $f \overset{\partial}{\sim} g$ if two functions $f, g \in \mathcal{E}(\partial K)$ coincide in some neighborhood of ∂K, and we define

$$\mathcal{E}_\partial := \mathcal{E}(\partial K)/\overset{\partial}{\sim}.$$

Since any element of $\mathcal{E}(K)$ can also be considered as an element of $\mathcal{E}(\partial K)$ and since

$$f \overset{K}{\sim} g \quad \Rightarrow \quad f \overset{\partial}{\sim} g$$

there is a natural homomorphism $B_\Omega : \mathcal{E}_K \to \mathcal{E}_\partial$ that maps $[f]_K$ to $[f]_\partial$, which are the equivalence classes of $f \in \mathcal{E}(K)$ with regard to $\overset{K}{\sim}$ and $\overset{\partial}{\sim}$, respectively.

The following fact about smooth functions is fundamental and well-known.

Lemma 2.1. *Let $K \subset \mathbb{R}^n$ be compact, and let $U \subseteq \mathbb{R}^n$ be a neighborhood of K. For any $f \in \mathcal{E}(U)$, there exists $g \in \mathcal{E}(\mathbb{R}^n)$ and a neighborhood $U_1 \subseteq U$ of K such that $f|_{U_1} = g|_{U_1}$.*

As a consequence, $\mathcal{E}_K = \mathcal{E}(\mathbb{R}^n)/\overset{K}{\sim}$ and $\mathcal{E}_\partial = \mathcal{E}(\mathbb{R}^n)/\overset{\partial}{\sim}$. Thus, B_Ω is surjective. Moreover, we have

$$\begin{aligned} \ker(B_\Omega) &= \{[f]_K \mid f \in \mathcal{E}(\mathbb{R}^n), [f]_\partial = 0\} \\ &= \{[\phi]_K \mid \phi \in \mathcal{E}(\mathbb{R}^n), \operatorname{supp}(\phi) \subset \Omega\}. \end{aligned}$$

If U is a neighborhood of $\partial\Omega$ on which f vanishes, we obtain ϕ from f by setting to zero its values on $\mathbb{R}^n \setminus (\Omega \cup U)$. As usual, define

$$\mathcal{D}(\Omega) = \{\phi \in \mathcal{E}(\mathbb{R}^n) \mid \operatorname{supp}(\phi) \subset \Omega\} \tag{2.1}$$

(since Ω is bounded, such a support is necessarily compact). As any $z \in \ker(B_\Omega)$ possesses a *unique* representation $z = [\phi]_K$ with $\phi \in \mathcal{D}(\Omega)$, there is an isomorphism $\mathcal{D}(\Omega) \to \ker(B_\Omega)$, $\phi \mapsto [\phi]_K$.

The set $\mathcal{E}(\mathbb{R}^n)$ carries a $\mathcal{P}$-module structure, where

$$\mathcal{P} = \mathbb{C}[s] = \mathbb{C}[s_1, \dots, s_n]$$

is the ring of polynomials in n variables, with complex coefficients. The action of s_i on $\mathcal{E}(\mathbb{R}^n)$ is given by the partial differentiation ∂_i, that is, for $p \in \mathcal{P}$ and $f \in \mathcal{E}(\mathbb{R}^n)$,

$$pf := p(\partial)f = p(\partial_1, \dots, \partial_n)f.$$

Thus also $\mathcal{E}_K$ becomes a $\mathcal{P}$-module, via $p[f]_K = [pf]_K$. When R is a $\mathcal{P}$-matrix (that is, $R \in \mathcal{P}^{r\times q}$ for some integers $r, q \geq 1$), we write

$$\operatorname{im}_{\mathcal{P}}(R) = R\mathcal{P}^q \quad \text{and} \quad \ker_{\mathcal{P}}(R) = \{p \in \mathcal{P}^q \mid Rp = 0\}$$

for the image and kernel of R as a mapping $\mathcal{P}^q \to \mathcal{P}^r$, and

$$\operatorname{im}_{\mathcal{E}_K}(R) = R(\partial)\mathcal{E}_K^q \quad \text{and} \quad \ker_{\mathcal{E}_K}(R) = \{w \in \mathcal{E}_K^q \mid R(\partial)w = 0\} \tag{2.2}$$

denote the image and kernel of R as a mapping $\mathcal{E}_K^q \to \mathcal{E}_K^r$. The set $\mathcal{D}(\Omega)$ from (2.1) is a $\mathcal{P}$-submodule of $\mathcal{E}(\mathbb{R}^n)$. Analogously as in (2.2), we write $\ker_{\mathcal{D}(\Omega)}(R)$ and $\operatorname{im}_{\mathcal{D}(\Omega)}(R)$ for the kernel and image of R as a mapping $\mathcal{D}(\Omega)^q \to \mathcal{D}(\Omega)^r$, etc.

Two facts will be crucial in the following: First, since Ω is non-empty, open and convex, the set $\mathcal{D}(\Omega)$ is a faithfully flat $\mathcal{P}$-module [4, 6, 7]. This means that for $\mathcal{P}$-matrices R and M

$$\operatorname{im}_{\mathcal{P}}(R) = \ker_{\mathcal{P}}(M) \quad \Leftrightarrow \quad \operatorname{im}_{\mathcal{D}(\Omega)}(R) = \ker_{\mathcal{D}(\Omega)}(M).$$

Secondly, if U is non-empty, open, and convex, then $\mathcal{E}(U)$ is an injective $\mathcal{P}$-module [1, 4, 6]. This means that for $\mathcal{P}$-matrices R and M

$$\ker_{\mathcal{P}}(R^T) = \operatorname{im}_{\mathcal{P}}(M^T) \quad \Rightarrow \quad \operatorname{im}_{\mathcal{E}(U)}(R) = \ker_{\mathcal{E}(U)}(M). \tag{2.3}$$

In fact, (2.3) is an equivalence [5], but this will not be needed in the following. Instead, we shall prove the following variant.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be non-empty, open, bounded, and convex. Let $K = \overline{\Omega}$. Then $\mathcal{E}_K$ is an injective $\mathcal{P}$-module.*

Proof: Let M and R be $\mathcal{P}$-matrices with $\ker_{\mathcal{P}}(R^T) = \operatorname{im}_{\mathcal{P}}(M^T)$. We need to show that $\operatorname{im}_{\mathcal{E}_K}(R) \supseteq \ker_{\mathcal{E}_K}(M)$ (the other inclusion is trivial since $MR = 0$). Suppose that $[f]_K \in \ker_{\mathcal{E}_K}(M)$, that is, $f \in \mathcal{E}(\mathbb{R}^n)^r$ and $M(\partial)f = 0$ in some neighborhood U of K. Then there exists an open convex set U_1 such that $K \subset U_1 \subseteq U$ and $f|_{U_1} \in \ker_{\mathcal{E}(U_1)}(M)$. Since $\mathcal{E}(U_1)$ is injective, we have $f|_{U_1} \in \operatorname{im}_{\mathcal{E}(U_1)}(R)$, that is, $f|_{U_1} = R(\partial)g$ for some $g \in \mathcal{E}(U_1)^q$. Since $[f]_K = [f|_{U_1}]_K = R(\partial)[g]_K$, we conclude that $[f]_K \in \operatorname{im}_{\mathcal{E}_K}(R)$. □

3 Over-determined systems

Let $R \in \mathcal{P}^{r \times q}$ be a polynomial matrix, and let $\Omega \subset \mathbb{R}^n$ be as in Section 2, with $K = \overline{\Omega}$. Consider the over-determined boundary value problem for $f, g \in \mathcal{E}(\mathbb{R}^n)^q$

$$\begin{aligned} R(\partial)f &= 0 \quad \text{in some neighborhood of } K \\ f &= g \quad \text{in some neighborhood of } \partial K. \end{aligned}$$

More precisely, the problem may be formulated as follows. Consider the local *behavior* [8]

$$\begin{aligned} \mathcal{B}_K = \ker_{\mathcal{E}_K}(R) &= \{w \in \mathcal{E}_K^q \mid R(\partial)w = 0\} \\ &= \{[f]_K \mid f \in \mathcal{E}(\mathbb{R}^n)^q, R(\partial)f = 0 \text{ in some neighborhood of } K\} \end{aligned}$$

which is the solution space, in $\mathcal{E}_K^q$, of the linear constant-coefficient partial differential operator $R(\partial)$. Let $g \in \mathcal{E}(\mathbb{R}^n)^q$ be given.

Extension problem: Given $v = [g]_\partial \in \mathcal{E}_\partial^q$, does there exist $w = [f]_K \in \mathcal{B}_K$ such that $B_\Omega w = v$, that is, $f \overset{\partial}{\sim} g$?

There is an obvious necessary condition: v has to satisfy the system laws locally, that is, $R(\partial)v = 0$, or equivalently, $R(\partial)g = 0$ in some neighborhood of ∂K. Let $\mathcal{B}_\partial$ denote the set of all $v \in \mathcal{E}_\partial^q$ that satisfy this necessary condition. We may write

$$\mathcal{B}_\partial = \{v \in \mathcal{E}_\partial^q \mid R(\partial)v = 0\}.$$

Note that if $v = [g]_\partial \in \mathcal{B}_\partial$, then there exists $\phi \in \mathcal{D}(\Omega)^r$ such that $[R(\partial)g]_K = [\phi]_K$.

Lemma 3.1. *Let $v = [g]_\partial \in \mathcal{B}_\partial$ be given, and let $\phi \in \mathcal{D}(\Omega)^r$ be such that $[R(\partial)g]_K = [\phi]_K$. The extension problem is solvable if and only if $R(\partial)\chi = \phi$ has a solution $\chi \in \mathcal{D}(\Omega)^q$.*

Proof: Suppose that the extension problem is solvable, that is, there exists $w = [f]_K \in \mathcal{B}_K$ with $B_\Omega w = v$. Then $R(\partial)[g - f]_K = [\phi]_K$ and $[g - f]_\partial = 0$, hence $[g - f]_K = [\chi]_K$ for some $\chi \in \mathcal{D}(\Omega)^q$. Hence $R(\partial)\chi = \phi$ in a neighborhood of K, and hence everywhere. Conversely, let $R(\partial)\chi = \phi$ for some $\chi \in \mathcal{D}(\Omega)^q$. Set $w = [g - \chi]_K$. Then $R(\partial)w = 0$ and $B_\Omega w = v$, that is, w solves the extension problem. □

Using this lemma, we can now give a criterion for the extension problem to be solvable for any $v \in \mathcal{B}_\partial$. This means that $B_\Omega : \mathcal{B}_K \to \mathcal{B}_\partial$ is surjective. For this, we need the following notion.

Definition 3.1. A $\mathcal{P}$-matrix R is called a *right syzygy matrix* if there exists a $\mathcal{P}$-matrix M such that $\mathrm{im}_\mathcal{P}(R) = \ker_\mathcal{P}(M)$.

Theorem 3.1. *The extension problem has a solution for any Ω as above, and for any $v \in \mathcal{B}_\partial$ if and only if R is a right syzygy matrix.*

Proof: If R is a right syzygy matrix, say, $\mathrm{im}_{\mathcal{P}}(R) = \ker_{\mathcal{P}}(M)$, then we have $\mathrm{im}_{\mathcal{D}(\Omega)}(R) = \ker_{\mathcal{D}(\Omega)}(M)$, since $\mathcal{D}$ is a flat $\mathcal{P}$-module. Let $v = [g]_\partial \in \mathcal{B}_\partial$ be given, and let $\phi \in \mathcal{D}(\Omega)^r$ be such that $[R(\partial)g]_K = [\phi]_K$. Since $MR = 0$, this implies that $M(\partial)[\phi]_K = 0$, that is, $M(\partial)\phi = 0$ in some neighborhood of K. Since $\phi \in \mathcal{D}(\Omega)^r$, $M(\partial)\phi = 0$ everywhere, and thus $\phi \in \ker_{\mathcal{D}(\Omega)}(M) = \mathrm{im}_{\mathcal{D}(\Omega)}(R)$. Then $\phi = R(\partial)\chi$ for some $\chi \in \mathcal{D}(\Omega)^q$, and we are finished according to Lemma 3.1.

Conversely, let M be a $\mathcal{P}$-matrix with $\ker_{\mathcal{P}}(R^T) = \mathrm{im}_{\mathcal{P}}(M^T)$. Then, since $\mathcal{E}_K$ is injective,

$$\mathrm{im}_{\mathcal{E}_K}(R) = \ker_{\mathcal{E}_K}(M). \tag{3.4}$$

Since $\mathcal{D}(\Omega)$ is a faithfully flat $\mathcal{P}$-module, it suffices to show that

$$\ker_{\mathcal{D}(\Omega)}(M) \subseteq \mathrm{im}_{\mathcal{D}(\Omega)}(R)$$

(the other inclusion is trivial since $MR = 0$). Let $\phi \in \ker_{\mathcal{D}(\Omega)}(M)$ be given. Then $[\phi]_K \in \ker_{\mathcal{E}_K}(M)$. Equation (3.4) implies that $[\phi]_K = R(\partial)[g]_K$ for some $g \in \mathcal{E}(\mathbb{R}^n)^q$. Define $v = [g]_\partial \in \mathcal{E}_\partial^q$. Since $R(\partial)v = [R(\partial)g]_\partial = B_\Omega[R(\partial)g]_K = B_\Omega[\phi]_K = 0$, we have $v \in \mathcal{B}_\partial$. By assumption, the extension problem is solvable, and hence, according to Lemma 3.1, $\phi = R(\partial)\chi$ for some $\chi \in \mathcal{D}(\Omega)^q$. Thus $\phi \in \mathrm{im}_{\mathcal{D}(\Omega)}(R)$. □

The next question is: When is $B_\Omega : \mathcal{B}_K \to \mathcal{B}_\partial$ even a bijection?

Corollary 3.1. *The extension problem possesses a* unique *solution for any Ω as above and any $v \in \mathcal{B}_\partial$ if and only if R is a right syzygy matrix with full column rank.*

Proof: The matrix R has full column rank if and only if $\ker_{\mathcal{P}}(R) = 0$, which is equivalent to $\ker_{\mathcal{D}(\Omega)}(R) = 0$, since $\mathcal{D}(\Omega)$ is a faithfully flat $\mathcal{P}$-module. We show that $\ker_{\mathcal{D}(\Omega)}(R) = 0$ is equivalent to the uniqueness of the solution to the extension problem. Let $\ker_{\mathcal{D}(\Omega)}(R) = 0$ and suppose that w_1, w_2 are such that $R(\partial)w_1 = R(\partial)w_2 = 0$ and $B_\Omega w_1 = B_\Omega w_2 = v$. Then $w := w_1 - w_2$ satisfies $R(\partial)w = 0$ and $B_\Omega w = 0$, that is, $w = [\phi]_K$ for some $\phi \in \mathcal{D}(\Omega)^q$. This means that $R(\partial)\phi = 0$ in some neighborhood of K, and hence everywhere. Thus $\phi \in \ker_{\mathcal{D}(\Omega)}(R)$ which implies that $\phi = 0$ and hence $w_1 = w_2$. Conversely, if $\ker_{\mathcal{D}(\Omega)}(R) \neq 0$, then there exists $0 \neq \phi \in \mathcal{D}(\Omega)^q$ with $R(\partial)\phi = 0$. Now if w solves the extension problem, so does $w + [\phi]_K \neq w$. □

Remark 3.1. A polynomial matrix with full column rank is a right syzygy matrix if and only if the greatest common divisor of its maximal minors is one [9].

Definition 3.2. A full column rank polynomial matrix whose maximal minors are coprime, is called *minor right prime.* If R is minor right prime, the associated global behavior

$$\mathcal{B} = \{w \in \mathcal{E}(\mathbb{R}^n)^q \mid R(\partial)w = 0\}$$

is called *over-determined.* This signifies that for any $\Omega \subset \mathbb{R}^n$ (non-empty, open, bounded, and convex) and any function that satisfies the system law in a neighborhood of $\partial\Omega$, there exists a unique extension to all of $K = \overline{\Omega}$.

Remark 3.2. The notion of over-determined systems is intrinsically multidimensional. Since a univariate ($n = 1$) matrix is minor right prime if and only if it possesses a polynomial left inverse, the only one-dimensional over-determined behavior is the trivial behavior $\mathcal{B} = \{0\}$. For two-dimensional systems ($n = 2$), a behavior is over-determined if and only if it is finite-dimensional.

Example 3.1. The Cauchy-Riemann equations for a function of two complex variables (considered as four real variables) are represented by the matrix

$$R = \begin{bmatrix} s_1 & -s_2 \\ s_2 & s_1 \\ s_3 & -s_4 \\ s_4 & s_3 \end{bmatrix} \in \mathbb{R}[s_1, s_2, s_3, s_4]^{4\times 2}.$$

This matrix is minor right prime, and hence the Cauchy-Riemann equations for functions of two complex variables are over-determined, see e.g. [2].

If R has full column rank, the characterization given in Lemma 3.1 possesses an equivalent form in terms of Fourier transforms.

Theorem 3.2. *Let R have full column rank. The extension problem possesses a (unique) solution for any given Ω as above and any $v = [g]_\partial \in \mathcal{B}_\partial$ if and only if the equation*

$$R(i\xi)\psi(\xi) = \mathcal{F}(\phi)(\xi)$$

has a (unique) solution ψ that is an entire *function of $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. Here $\mathcal{F}$ denotes the Fourier transform and $\phi \in \mathcal{D}(\Omega)^r$ is such that $[R(\partial)g]_K = [\phi]_K$.*

The proof requires the following arguments.

Remark 3.3. Let $0 \neq p \in \mathcal{P}$ and let ν be a smooth function with compact support. If $\psi(\xi) = \mathcal{F}(\nu)(\xi)/p(i\xi)$ is an entire function, then ψ is the Fourier transform of some smooth function with compact support. In [3, Lemma 3.4.2], this fact is proven for distributions with compact support. The result itself, and its validity for smooth functions, is a consequence of the Paley-Wiener theorem, see e.g. [3, Theorem 1.7.7].

Let $0 \neq p \in \mathcal{P}$ and let μ be a smooth function with compact support. Then [3, Lemma 3.4.3]

$$\text{supp}(\mu) \subseteq \text{ch}(\text{supp}(p(\partial)\mu))$$

where $\text{ch}(\cdot)$ denotes the convex hull of a set. In particular, since Ω is convex, $p(\partial)\mu \in \mathcal{D}(\Omega)$ implies $\mu \in \mathcal{D}(\Omega)$.

Proof of Theorem 3.2: In view of Lemma 3.1, we need to prove that $R(\partial)\chi = \phi$ has a solution $\chi \in \mathcal{D}(\Omega)^q$ if and only if $R(i\xi)\psi(\xi) = \mathcal{F}(\phi)(\xi)$ has an entire solution ψ.

If $R(\partial)\chi = \phi$ with $\chi \in \mathcal{D}(\Omega)^q$, then

$$\mathcal{F}(R(\partial)\chi)(\xi) = R(i\xi)\mathcal{F}(\chi)(\xi) = \mathcal{F}(\phi)(\xi)$$

and the result follows from the fact that the Fourier transform of a compact support function is an entire function of ξ.

Conversely, let $R(i\xi)\psi(\xi) = \mathcal{F}(\phi)(\xi)$ with an entire function ψ. Since R has full column rank, there exists a polynomial matrix X and a polynomial $p \neq 0$ such that

$$XR = pI.$$

Therefore

$$X(i\xi)R(i\xi)\psi(\xi) = p(i\xi)\psi(\xi) = X(i\xi)\mathcal{F}(\phi)(\xi) = \mathcal{F}(X(\partial)\phi)(\xi).$$

By assumption, $\psi(\xi) = \mathcal{F}(X(\partial)\phi)(\xi)/p(i\xi)$ is an entire function of ξ. According to the previous remark, this implies that ψ is the Fourier transform of a smooth function with compact support, say $\psi = \mathcal{F}(\mu)$. Then $R(\partial)\mu = \phi$ and thus $p(\partial)\mu = X(\partial)\phi \in \mathcal{D}(\Omega)^q$. This implies that $\chi := \mu \in \mathcal{D}(\Omega)^q$, again by the previous remark. □

4 Over-determined input-output structures

Let us return to the situation where the solution to the extension problem is not necessarily unique. This non-uniqueness can be characterized in more detail as follows.

For this, let Ω and K be as usual, and consider again $\mathcal{B}_K = \{w \in \mathcal{E}_K^q \mid R(\partial)w = 0\}$ with $R \in \mathcal{P}^{r\times q}$. Set $p := \operatorname{rank}(R)$. There exists (up to permutation) a partition

$$R = \begin{bmatrix} -Q & P \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

in which $P \in \mathcal{P}^{r\times p}$ has full column rank. Set $m := q - p$. By construction, $Q = PH$ for some rational $p \times m$ matrix H.

Definition 4.1. Such a partition is called an *input-output (I/O) structure* of

$$\mathcal{B}_K = \{(u, y) \in \mathcal{E}_K^m \times \mathcal{E}_K^p \mid P(\partial)y = Q(\partial)u\}. \tag{4.5}$$

The matrix H is called *transfer matrix* of the I/O structure.

Remark 4.1. This notion is due to Oberst [5] and it is motivated by the following observation: The sub-vector u of w is a vector of free variables in $\mathcal{B}$, that is, an "input". More precisely, for any $u \in \mathcal{E}_K^m$, there exists $y \in \mathcal{E}_K^p$ such that $(u, y) \in \mathcal{B}_K$, that is, $P(\partial)y = Q(\partial)u$. This follows from the injectivity of $\mathcal{E}_K$: Let $\ker_{\mathcal{P}}(P^T) = \operatorname{im}_{\mathcal{P}}(N^T)$ for some polynomial matrix N. Then $\operatorname{im}_{\mathcal{E}_K}(P) = \ker_{\mathcal{E}_K}(N)$. Since $NQ = NPH = 0$, we have $Q(\partial)u \in \operatorname{im}_{\mathcal{E}_K}(P)$ for any $u \in \mathcal{E}_K^m$.

Suppose that $\mathcal{B}_K$ is a behavior with input-output structure (4.5). Then the signal sub-vector u is free. If we fix u, then there are, in general, many "outputs" y with $(u, y) \in \mathcal{B}_K$. Additional constraints may be imposed on y in order to obtain uniqueness. The I/O extension problem asks whether fixing the values of y in a neighborhood of the boundary, say $B_\Omega y = x$, is an admissible such constraint. If yes, then the boundary data x play the role of "state" in this setting.

I/O extension problem: Given $u \in \mathcal{E}_K^m$ and $x \in \mathcal{E}_\partial^p$, does there exist $y \in \mathcal{E}_K^p$ such that

$$P(\partial)y = Q(\partial)u \quad \text{and} \quad B_\Omega y = x?$$

Note that since P has full column rank, there can be at most one such y. Hence, uniqueness is not an issue here. A necessary condition for existence is easily found: $P(\partial)x = Q(\partial)B_\Omega u$. Let $\mathcal{B}_\partial$ denote the set of all pairs $(B_\Omega u, x) \in \mathcal{E}_\partial^q$ with this property, that is,

$$\mathcal{B}_\partial = \{(B_\Omega u, x) \in \mathcal{E}_\partial^m \times \mathcal{E}_\partial^p \mid P(\partial)x = Q(\partial)B_\Omega u\}.$$

Theorem 4.1. *The I/O extension problem has a solution for any $u \in \mathcal{E}_K^m$ and any $x \in \mathcal{E}_\partial^p$ with $(B_\Omega u, x) \in \mathcal{B}_\partial$ if and only if P is a right syzygy matrix. Then the solution is uniquely determined.*

Proof: By the previous remark, $Q(\partial)u \in \operatorname{im}_{\mathcal{E}_K}(P)$, i.e., there exists $y_1 \in \mathcal{E}_K^p$ with $P(\partial)y_1 = Q(\partial)u$. Put $\tilde{y} := y - y_1$, then the I/O extension problem is equivalent to the following extension problem: Given $x - B_\Omega y_1 \in \mathcal{E}_\partial^p$, with $P(\partial)(x - B_\Omega y_1) = 0$, find $\tilde{y} \in \mathcal{E}_K^p$ with $P(\partial)\tilde{y} = 0$ and $B_\Omega \tilde{y} = x - B_\Omega y_1$. Therefore the statement can be proven similarly as Theorem 3.1. □

Definition 4.2. In the situation of Theorem 4.1, we call the associated global behavior

$$\mathcal{B} = \{(u, y) \in \mathcal{E}(\mathbb{R}^n)^m \times \mathcal{E}(\mathbb{R}^n)^p \mid P(\partial)y = Q(\partial)u\} \tag{4.6}$$

a system with *over-determined I/O structure*.

Remark 4.2. Let (4.6) be a system with over-determined I/O structure. Then for all Ω as above, and for all $u \in \mathcal{D}(\Omega)^m$, there exists a unique $y \in \mathcal{D}(\Omega)^p$ such that $P(\partial)y = Q(\partial)u$. Moreover, the transfer matrix of an over-determined I/O structure is polynomial (rather than rational).

Corollary 4.1. *Let $R \in \mathcal{P}^{r \times q}$. The behavior $\mathcal{B} = \{w \in \mathcal{E}(\mathbb{R}^n)^q \mid R(\partial)w = 0\}$ possesses an over-determined input-output structure if and only if R possesses a minor right prime sub-matrix P with* $\operatorname{rank}(R) = \operatorname{rank}(P)$.

Finally, we address the problem of solving the I/O extension problem for a specific choice of the input and boundary data. For this, let $h \in \mathcal{E}(\mathbb{R}^n)^p$ be given.

Theorem 4.2. *Let* Ω *and* K *be as usual. Let* $u \in \mathcal{E}_K^m$ *and* $x = [h]_\partial \in \mathcal{E}_\partial^p$ *be such that* $(B_\Omega u, x) \in \mathcal{B}_\partial$. *Then* $[P(\partial)h]_K - Q(\partial)u = [\phi]_K$ *for some* $\phi \in \mathcal{D}(\Omega)^r$. *The following are equivalent:*

1. *The I/O extension problem has a (unique) solution.*
2. $P(\partial)\chi = \phi$ *has a (unique) solution* $\chi \in \mathcal{D}(\Omega)^p$.
3. $P(i\xi)\psi(\xi) = \mathcal{F}(\phi)(\xi)$ *has a (unique) entire solution* ψ.

Proof: In all three assertions, the uniqueness follows from the fact that P has full column rank. The equivalence of 2 and 3 is shown as in the proof of Theorem 3.2. We prove the equivalence of 1 and 2.

If y is such that $P(\partial)y = Q(\partial)u$ and $B_\Omega y = x = [h]_\partial$, then $P(\partial)([h]_K - y) = [\phi]_K$ and $B_\Omega([h]_K - y) = 0$. Thus $[h]_K - y = [\chi]_K$ for some $\chi \in \mathcal{D}(\Omega)^p$, and $P(\partial)\chi = \phi$.

Conversely, let $P(\partial)\chi = \phi$, then $P(\partial)[\chi]_K = [\phi]_K = [P(\partial)h]_K - Q(\partial)u$ and hence $y := [h - \chi]_K$ solves the I/O extension problem. □

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