Cones of trajectories as subsets of linear systems: the autonomous case

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Abstract

In this paper we introduce the concept of conical behaviour, a complete time invariant cones of trajesctories. We focus our attention on the cones as subsets of autonomous linear system. We give geometrical conditions that such cones admit a non negative state realization. In addition we consider the closed sets of trajectories of a autoregressive scalar model on wich are imposed inequalities constraints. We provide some criteria for a class of these cones to be represented by a non negative state realization.

1 Introduction

In most pratical application the designer has to face to physical and technological constraints that lead to a set of inequality involving the variables of the model. Think of the next models The set of non negative trajectories of an autoregressive model

$$\mathbf{w}(t+n) = c_{n-1}\mathbf{w}(t+n-1) + \cdots + c_0\mathbf{w}(t),\ c_i \in \mathbb{R} \quad \mathbf{w}(t) \geq 0\ .$$

The set of non decreasing trajectories of an autoregressive model

$$\mathbf{w}(t+n) = c_{n-1}\mathbf{w}(t+n-1) + \cdots + c_0\mathbf{w}(t),\ c_i \in \mathbb{R} \quad \mathbf{w}(t+1) \geq \mathbf{w}(t)\ .$$

We can meet a more geberal model

$$C\mathbf{w}|_{[t,t+n]} \leq 0 \quad C \in \mathbb{R}^{m\times n}$$

in economy, byology, and other scientific fields. We introduce the next definitions tu study also such models.

2 Definitions and first properties

In this section we provide the definition of conical behaviour, some basic mathematical tools, and the firs properties. Given a trajectory $\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$, we recall that $\sigma\mathbf{w}$

$$\sigma\mathbf{w}(t) = \mathbf{w}(t+1),\ t \in \mathbb{Z}_+$$

Definition 2.1. $\mathcal{C} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ *is a time invariant conical behaviour if*

1. $\forall \mathbf{w}_1$ *and* $\mathbf{w}_2 \in \mathcal{C}$*,* $\mathbf{w}_1 + \mathbf{w}_2 \in \mathcal{C}$
2. $\forall \mathbf{w}_1$ *and* $\alpha \in \mathbb{R}_+$*,* $\alpha \mathbf{w} \in \mathcal{C}$*.*
3. $\sigma \mathbf{w} \in \mathcal{C} \ \forall \mathbf{w} \in \mathcal{C}$

1) and 2) say that $\mathcal{C}$ is a *cone*, and 3) say that $\mathcal{C}$ is *time invariant.*
$\mathcal{C}$ is complete if

$$\mathbf{w}|_I \in \mathcal{C}|_I \quad \forall I \subset \mathbb{Z}_+ \Rightarrow \mathbf{w} \in \mathcal{C} \ .$$

It can be shown (as in [W1] Prop. 4) that if $\mathcal{C}$ is colsed under the pointwiese convergence topology, then it is complete. In this paper we only consider time invariant and closed behaviour, henceforth, without confusion, we speak of conical behaviour meaning time invariant and closed behaviour. .

After J.C. Willems [W1] we give the next definition.

Definition 2.2. *A conical behaviour* $\mathcal{C} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ *is autonomous if there is a map* $f(\cdot)$*, and an integer* $t' \geq 0$ *such that, given any trajectory* $\mathbf{w} \in \mathcal{C}$

$$f : \mathbf{w}|_{[0,t']} \mapsto \mathbf{w}_{[t'+1,+\infty)} \ .$$

In addition $f(\cdot)$ *has the following properties.*
1) $f(\mathbf{w}_1|_{[0,t']} + \mathbf{w}_2|_{[0,t']}) = f(\mathbf{w}_1|_{[0,t']}) + f(\mathbf{w}_2|_{[0,t']})$*, where* $\mathbf{w}_1$ *and* $\mathbf{w}_2 \in \mathcal{C}$*.*
2) $f(\alpha \mathbf{w}|_{[0,t']}) = \alpha f(\mathbf{w}|_{[0,t']})$*, for* $\alpha \geq 0$ *and* $\mathbf{w} \in \mathcal{C}$*.*

Given a conical behaviour $\mathcal{C}$, it can be enclosed in the smallest complete time invariant linear behaviour which we denote $\operatorname{span}\mathcal{C}$.
The next propsition gives some characterizations of $\operatorname{span}\mathcal{C}$ for a given autonomous conical behaviour $\mathcal{C}$.

Proposition 2.1. *Suppose* $\mathcal{C} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ *a conical behaviour, then* $\mathcal{C}$ *is autonomous if and only if* $\operatorname{span}\mathcal{C}$ *is an autonomous complete lienar behaviour*

Proof Let $\mathcal{C}$ be autonomous, then a conical map $f : \mathbb{R}^{(t'+1)q} \to (\mathbb{R}^q)^{\mathbb{Z}_+}$ exists, for a fixed non–negative integer t', such that

$$f(\mathbf{w}|_{[0,t`]}) = \mathbf{w}|_{[t`+1,+\infty)} \ .$$

Given a $\mathbf{w} \in \operatorname{span}\mathcal{C}$, then $\mathbf{w} = \mathbf{u}` - \mathbf{u}``$ for some suitable $\mathbf{u}`$ abd $\mathbf{u}``$. We can define a linear map $f_{\text{span}}(\cdot)$ as follows

$$f_{\text{span}}(\mathbf{w}) = f(\mathbf{u}') - f(\mathbf{u}'') \ .$$

$f_{\mathrm{span}}(\cdot)$ is well–defined, in fact if $\mathbf{u}' - \mathbf{u}'' = \mathbf{v}' - \mathbf{v}''$ then, by the additivity of $f(\cdot)$, we have that $f_{\mathrm{span}}(\mathbf{u}' - \mathbf{u}'') = f_{\mathrm{span}}(\mathbf{v}' - \mathbf{v}'')$. $\mathrm{span}\,\mathcal{C}|_{[t'+1,+\infty)}$ is finite dimensional as well $\mathcal{C}$, then $\mathcal{C}$ is closed i.e. ([W1] Prop. 4) complete.
On the other hand if $\mathrm{span}\,\mathcal{C}$ is autonomous, then there is a linear map $g(\cdots)$ such that

$$g : \mathbf{w}_{[0,t']} \mapsto \mathbf{w}_{[t'+1,+\infty)}, \quad \mathbf{w} \in \mathrm{span}\,\mathcal{C} \, .$$

If we restrict the map to $\mathcal{C}_{[0,t']}$ we obtain the result. □

A cone $\mathfrak{C}$ in a real vector space is *pointed* if it doesn't contain a non trivial vector space.
It can be shown ([SW] (2.10.5)) that every cone $\mathfrak{C}$ can be decomposed into the direct sum of a vector space and a pointed cone.

Definition 2.3. *The conical behaviour $\mathcal{C}$ is pointed if*

$$\mathbf{w} \ \textit{and} \ -\mathbf{w} \in \mathcal{C} \Rightarrow \mathbf{w} = \mathbf{0} \, .$$

Being an autonomous conical behaviour a subset of a finite dimensional vector space, the next proposition immediately follows.

Proposition 2.2. *If $\mathcal{C} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ is an autonomous conical behaviour then*

$$\mathcal{C} = \mathcal{C}_p \oplus \mathcal{B}_a \, .$$

$\mathcal{C}_p$ is an autonomous pointed conical behaviour and $\mathcal{B}_a$ is an autonomous linear behaviour.

Henceforth our attention is devoted to the pointed conical behaviour. **All the conical behaviour we will consider are pointed**

We remind that a cone $\mathfrak{C}$, as a subset of a real vector space $\mathfrak{U}$, is a *polyhedral cone* if it is given by the intersection of a finite number of closed (under a chosen topology) half speces. If $\mathfrak{U}$ is finite dimensional, then (under the usual topology), from the theorem of Minkowski [SW](2.8.6), there are $p_1, \ldots, p_{n_+} \in \mathfrak{U}$ such that

$$v = \sum_{i=1}^{n_+} \alpha_i p_i \quad \alpha_i \in \mathbb{R}_+ \quad \forall v \in \mathfrak{C}.$$

Given a pointed cone $\mathfrak{C}$, we can choose the previous n–upla $(p_1, \ldots, p_{n_+})$ such that p_i are *non–negatively indipendent* [BFH], we write indipendent$_+$, i.e. there is not a non trivial n–upla $\beta_1, \ldots, \beta_n$ of non negative scalar such that $\beta_k p_k = \sum_{i=1,\, i\neq k}^{n_+} \alpha_i p_i$, $0 \leq k \leq n_+$. We say that (p_i) is a *non–negative basis*, we write base$_+$, for $\mathfrak{C}$. Every basis$_+$ of a pointed polyhedral cone $\mathfrak{C}$ in a finite dimensional vector space has the same number of elements, see Theorem (3.1)(4)[BFH].

It is note that every autonomous complete time invariant linear system $(\mathbb{R}^q, \mathbb{T}, \mathcal{B})$ admits such a minimal state rappresentation

$$(S) \qquad \begin{aligned} \sigma\mathbf{s} &= F\mathbf{s} \\ \mathbf{w} &= H\mathbf{s} \end{aligned} \, ,$$

$\mathbf{s} \in (\mathbb{R}^n)^{\mathbb{Z}_+}$ is the state trajectory and $\mathcal{B}$ is the external behaviour of $\mathcal{B}_s = \{\mathbf{w}, \mathbf{s}\} \in (\mathbb{R}^{q \times n_s})^{\mathbb{Z}_+}$ where $\mathbf{w}, \mathbf{s}$ satisfy (S). An aim of this paper is to provide a similar description of

$$\mathcal{C}_s = \left\{ (\mathbf{w}, \mathbf{s}) \in \mathcal{B}_s : \quad \mathbf{w} \in \mathcal{C} \right\} .$$

An autonomous conical behaviour $\mathcal{C}$ admits a non negative state realization, (S_+), if $\mathcal{C}$ is the external behaviour of $\mathcal{C}_s$, the set of the pairs $(\mathbf{w}, \mathbf{s})$ satisfying

$$(S_+) \qquad \begin{array}{r} \sigma \mathbf{x} = A \mathbf{s} \\ \mathbf{w} = C \mathbf{s} \end{array} ,$$

where $\mathbf{x} \in \mathbb{R}^n_+$.
Now, in a geometrical view, we give the necessary and sufficient conditions that an autonomous conical behaviour $\mathcal{C}$ admits a $(S)_+$ realization.

Proposition 2.3. *Let the autonomous conical behaviour $\mathcal{C} \in (\mathbb{R}^q)^{\mathbb{Z}_+}$ be polyhedral and pointed, then the next statement follow.*

i) *$\mathcal{C}$ admits such state rappresentation*

$$\begin{array}{r} \sigma \mathbf{x} = A \mathbf{x} \\ \mathbf{w} = C \mathbf{x} \end{array} ,$$

$A \in \mathbb{R}^{n_+ \times n_+}$, *and* $\mathbf{x} \in ({\mathbb{R}_+}^{n_+})^{\mathbb{Z}_+}$.

ii) *Given a minimal state realizaton for* $\operatorname{span} \mathcal{C}$

$$\begin{array}{r} \sigma \mathbf{s} = F \mathbf{s} \\ \mathbf{z} = H \mathbf{s} \end{array} ,$$

$F \in \mathbb{R}^{n \times n}$. *$A$ and C can be chosen such that*

$$C = HP \text{ and } FP = PA ,$$

where the n_+ columns of P provide a basis$_+$ for the proper polyhedral cone

$$\mathfrak{C}_s = \left\{ \mathbf{s}(0) : \quad F\mathbf{s} \in \mathcal{C} \right\} .$$

It is strightforward to show

$$P = P_1 [I_n | S] = [p_1 | P_2] .$$

For A we give that factorization

$$\begin{array}{cc} & \begin{array}{cc} n & n_+ - n \end{array} \\ & \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \begin{array}{l} n \\ n_+ - n \end{array} \end{array} .$$

A matrix S *exists such that*

$$A_{12} = -SA_{22} + (A_{11} + SB_{21})S \ .$$

There is a non singular square matrix T *such that*

$$A = T^{-1} \begin{pmatrix} F & 0 \\ * & * \end{pmatrix} T.$$

Moreover $F = P_1(A_{11} + SA_{21})P_1^{-1}$.

iv) The columns of the matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

provide a basis$_+$ *for the polyhedral cone* $\mathcal{C}|_{[0,n-1]}$.

Proof For span $\mathcal{C}$ we have a minimal state raprresentation

$$\begin{aligned} \sigma \mathbf{s} &= F\mathbf{s} \\ \mathbf{z} &= H\mathbf{s} \end{aligned} ,$$

where the pair (F,G) be observable, $F \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{q\times n_s}$.
Let the *state cone*

$$\mathfrak{C}_s = \Big\{ \mathbf{s}(0) : \quad F\mathbf{s} \in \mathcal{C} \Big\} \ ,$$

$\mathfrak{C}_s$ is a full cone in $\mathbb{R}^{n_s}$ from the construction.
Being $\mathcal{C}$ a plyhedral pointed cone in span $\mathcal{C}$, a finite dimensional real vector space, from Minkowski theorem and previous proposition (see also [SW]), it admits a base$_+$ $(\mathbf{p}_1, \ldots, \mathbf{p}_{n_+})$. From the obsevability of the pair (F,H) we have that $(\mathbf{p}_1|_{[0,n-1]}, \ldots, \mathbf{p}_n|_{[0,n-1]})$ is a base$_+$ for $\mathcal{C}|_{[0,n-1]}$, that follows from the fact that $\mathbf{w} = \mathbf{0}$ iff $\mathbf{w}|_{[0,n-1]} = \mathbf{0}|_{[0,n-1]}$. Put

$$O = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

and $\mathbf{p}_i|_{[0,n_s-1]} = O\mathbf{s}(0)_i$, $1 \le i \le n_+$. Defining O a bijection between span $\mathcal{C}|_{[0,n-1]}$ and the state space, $(\mathbf{s}_1(0), \ldots, \mathbf{s}_n(0))$ is a base$_+$ for $\mathfrak{C}_s$. Let $P = [\mathbf{s}(0)_1, \ldots, \mathbf{s}(0)_n] \in \mathbb{R}^{n_s\times n}$, following the theorem 4.2 [BFH] the proof is completed. $\square$

Remark 2.1. *It is clear that if $\mathcal{C}$ admits a (S_+) rappresentation if polyhedral.*
In a geometrical view n is the number of the edges of the state cone not only with respect to a particular state ralization for $\operatorname{span}\mathcal{C}$*, this follows from the equivalence of minimal state realization for* $\operatorname{span}\mathcal{C}$.

3 A model

In this section we restrict our attention to the conical behaviours that satisfies such a model

$$(P) \qquad \begin{array}{l} \mathbf{w}(t+n) = c_{n-1}\mathbf{w}(t+n-1) + \cdots + c_0\mathbf{w}(t) \\ K^T\mathbf{w}|_{[0,n-1]} \leq 0 \end{array} \qquad , \qquad c_0 \neq 0$$

namely an autoregressive model where the inequality constraints act on the trajectories only in a finite time interval. Denoting with $\operatorname{span}_+ K$ the cone generated by the columns of K, we suppose $\operatorname{span}_+ K$ a full and pointed (polyhedral) cone. It is clear that $\mathcal{C}$, the set of trajectories satisfying (P) is a closed autonomous conical behaviour.

Let λ_i be the roots of minimal polynomial of a square matrix F with m_i their multiplicity. We say that F satisfies the *Perron conditions* if his spectrum has the next properties.

Perron condituions

- Let $\lambda_1 = \rho(F)$, $\rho(F) = \max_i |\lambda_i|$ is the spectral radius of F.
- If $|\lambda_i| = \rho(F)$ then $m_i \leq m_1$

The spectrum of F satisfies the "Perron Schaefer Condition" if the next hold.

Perron–Schaefer condituions

- The spectrum of F satisfies the Perron conditions.
- Each eigenvalue λ_i such that $|\lambda_i| = \rho(F)$ and $m_i = m_1$ is equal to λ_1 times a root of unity.

Before poroceding we give the next proposition.

Proposition 3.1. *Let $\mathcal{C}$ be the set of trajectories $\mathbf{w}$ that satsfies the (S) model*

$$\begin{aligned} \sigma\mathbf{s} &= F\mathbf{s} \\ \mathbf{w} &= H\mathbf{w} \end{aligned} \quad ,$$

$F \in \mathbb{R}^{n\times n}$, and such that $K^T\mathbf{w}|_{[0,n-1]} \leq 0$.
If O is the observability matrix, $\mathcal{C}$ is pointed, and admits a non–negative state realization (S_+)

$$\begin{aligned}\mathbf{x}(t+1) =& A\mathbf{x}(t)\\ \mathbf{w}(t) =& C\mathbf{x}(t)\end{aligned}\,, \quad \mathbf{x}(t) \geq 0\ .$$

iff the cone (cl is the closure)

$$\mathfrak{C} = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ O^TK \,\middle|\, F^TO^TK \,\middle|\, \cdots \,\middle|\, (F^T)^iO^TK\ \right]$$

is polyhedral and full.
$\operatorname{span}\mathcal{C} = \ker \sigma^n - c_{n-1}\sigma^{n-1} - \cdots c_0$ *iff $\mathfrak{C}$ is pointed.*

Proof By the duality of cones the state cone

$$\mathfrak{C}_s = \left\{\mathbf{s}(0) : \quad K^TOF^i\mathbf{s}(0) \leq 0\right\}$$

is a polyhedral pointed cone iff

$$\mathfrak{C} = cl\operatorname{span}_+ \lim_{i\to+\infty} \operatorname{span}_+ \left[\ K \,\middle|\, O^TK \,\middle|\, \cdots \,\middle|\, (O^T)^iK\ \right] \quad [1]$$

if full and polyhefral cone. Let the columns of P be a basis$_+$ for $\mathfrak{C}_s$. We put $C = HP$ and A the non negative solution of the matrix equation

$$FP = PA\ .$$

In addition, if $\mathfrak{C}$ is pointed, by the duality, $\mathfrak{C}_s$ is full in the state space $\mathbb{R}^n$, therefore

$$\operatorname{span}\mathcal{C} = \left\{\mathbf{w} : \quad \mathbf{w}(t+n) = c_{n-1}\mathbf{w}(n-1) + \cdots c_0\mathbf{w}(0)\right\}\ . \quad \Box$$

A lemma follows. □

Lemma 3.1. *Let $F \in \mathbb{R}^{n\times n}$ satisfy the Perron conditions, and let (F^T, k_1), $k_1 \in \mathbb{R}^n$, be a reachable pair, then the cone*

$$\mathfrak{C} = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ k_1 \,\middle|\, F^Tk_1 \,\middle|\, \cdots \,\middle|\, (F^T)^ik_1\ \right]$$

is proper (closed, full and pointed).

Proof See [FB] lemma 2 after [OMK]. □

Here above we return to the foregoing (P) model

(P)

$$\begin{aligned}&\mathbf{w}(t+n) = c_{n-1}\mathbf{w}(t+n-1) + \cdots + c_0\mathbf{w}(t)\\ &K^T\mathbf{w}|_{[0,n-1]} \leq 0\end{aligned}\ .$$

[1] $\mathfrak{C}_s = \{s : \quad v^Ts \leq 0\ \forall v \in \mathfrak{C}\}$.

Proposition 3.2. *Let the* (P) *model*

$$\begin{aligned}&\mathbf{w}(t+n) = c_{n-1}\mathbf{w}(t+n-1) + \cdots + c_0\mathbf{w}(t)\\ &K^T\mathbf{w}|_{[0,n-1]} \leq 0, \quad c_0 \neq 0\end{aligned}$$

have the two properties.

i) Let λ_i *be the roots of* $p(z) = z^n - c_{n-1}z^{n-1} - \cdots c_0$ *with their multiplicity* m_i *such that* $|\lambda_{i+1}| \leq |\lambda_i|$*, and let* $\lambda_1 > 0$*. If* $|\lambda_i| = \lambda_1$ *we also suppose* $m_i \leq m_1$*.*

ii) Let k_1 *the first column of* K *a non null vector such that*

$$\mathfrak{C} = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ k_1 \ \middle|\ F^T k_1 \ \middle|\ \cdots \ \middle|\ (F^T)^i k_1\ \right] = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ K \ \middle|\ F^T K \ \middle|\ \cdots \ \middle|\ (F^T)^i K\ \right] .$$

Then $\mathcal{C}$ *is a pointed conical behaviour and* $\operatorname{span}\mathcal{C} = \ker(\sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0)$*.*

Proof We write the minimal state realization for $\ker(\sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0)$

$$\begin{aligned}\sigma\mathbf{s} &= F\mathbf{s}\\ \mathbf{w} &= h^T\mathbf{s}\end{aligned}$$

where:

$$F = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ c_0 & c_1 & \cdots & c_{n-1}\end{bmatrix} \qquad \begin{aligned} h^T &= [1\ 0 \cdots 0] \\ \mathbf{s}(t) &= (\mathbf{w}(t), \ldots, \mathbf{w}(t+n_s))^T \end{aligned} .$$

Becouse of $c_0 \neq 0$, (F, k_1) is reachable. From this fact, $i)$ and the previous lemma we have that the cone

$$\mathfrak{C} = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ k_1 \ \middle|\ F^T k_1 \ \middle|\ \cdots \ \middle|\ (F^T)^i k_1\ \right]$$

is proper. But the observability matrix O is equal to I_n, therefore, from $ii)$,

$$\mathfrak{C} = cl \lim_{i\to+\infty} \operatorname{span}_+ \left[\ O^T K \ \middle|\ F^T O^T K \ \middle|\ \cdots \ \middle|\ (F^T)^i O^T K\ \right] .$$

By the duality of cones $\mathfrak{C}_s = \{\mathbf{s}(0) : \mathbf{w} = H\mathbf{s} \in \mathcal{C}\}$ is proper, and them, as in Prop. 3.1, we prove that $\operatorname{span}\mathcal{C} = \ker(\sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0)$. $\square$.

The following proposition gives a sufficient condition that $\mathcal{C}$ corresponding to (P) admits a (S_+) rappresentation.

Proposition 3.3. *Let the* (P) *model with* $i)$*,* $ii)$ *as in the previous proposition, and the next property holds*

iii) $z^n - c_{n-1} - \cdots - c_0$ *dovides the polynomial* $z^{n_+} - \alpha_{n_+-1}z^{n_+-1} - \cdots - \alpha_0$*,* $\alpha_i \geq 0$*.*

Then $\operatorname{span}\mathcal{C} = \ker(\sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0)$*, and* $\mathcal{C}$ *admits a* (S_+) *realization.*

Proof As in the previous prop we buid the mininimal state realization. $\mathfrak{C}$, as in Prop. 3.1, is equal to

$$\mathrm{span}_+ \left[\; O^T K \;\middle|\; F^T O^T K \;\middle|\; \cdots \;\middle|\; (F^T)^{n_+ -1} O^T K \; \right] \;,$$

then is polyhedral. From Prop. 3.1. and Prop. 3.2. we have completed the proof. □

It is note (see [TS]) that a square matrix maps a cone into itsel iff satisfies the Perron Schaefer conditions. Let $\mathcal{C}$ corresponding to the (P) model have the properties $i)$ and ii of Prop. 3.2.. Given an observable pair (F, H) for a minimal state realization of $\mathrm{span}\,\mathcal{C}$, $\mathcal{C}$ is polyhedral (pointed), i.e. admits a basis$_+$, if $\mathfrak{C}_s = \{\mathbf{s}(0) : H\mathbf{s} \in \mathcal{C}\}$ is a proper polyhedral cone. Hence F have to satisfy the Perron Schaefer conditions. Then the roots λ_i of $z^n - c_{n-1} - \cdots - c_0$ are such that satisfy $i)$ (Prop 3.1.) and the next property.

$iv)$ If $|\lambda_i| = \lambda_1$ and $m_i = m_1$ then λ_i is equal to λ_1 times a root of unity..

It is note (see also [MK] ASppendix I) that if $iv)$ holds then, for a suitable positive integer r, the below limits exist.

$$F^{(h)} = \lim_{i \to +\infty} \frac{F^{h+ir}}{(h+ir)^{m_1 - 1} \lambda_1^{h+ir}} \quad 0 \le h < r \;.$$

By the convexity of

$$\mathfrak{C} = cl \lim_{i \to +\infty} \mathrm{span}_+ \left[\; k_1 \;\middle|\; F^T k_1 \;\middle|\; \cdots \;\middle|\; (F^T)^i k_1 \; \right] \;,$$

$v \in \mathfrak{C}$ iff

$$v = \sum_i \alpha_i (F^T)^i k_1 + \sum_{j=0}^{r-1} \beta_j (F^{(j)})^T k_1 \quad \alpha_i \ge 0 \;, \beta_j \ge 0 \;.$$

In particular $\mathfrak{C}$ is polyhedral if exists a n^* such that $v \in \mathfrak{C}$ iff

$$v = \sum_{i=0}^{n^*-1} \alpha_i F^i k_1 + \sum_{j=0}^{r-1} \beta_j F^{(j)} k_1 \quad \alpha_i \ge 0 \;, \beta_j \ge 0 \;.$$

Proposition 3.4. *Let $\mathcal{C}$ be the conical behaviour corresponding to (P). Let $i)$, $ii)$ and $iv)$ hold. Giving the observable pair (F, H) corresponding to a minimal state realization for* $\ker \sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0$ *we also have that the next holds.*

$v)$ There is a positive integer n^* such that

$$F^{n^*} = \sum_{i=0}^{n^*-1} \alpha_i F^i + \sum_{j=0}^{r-1} \beta_j F^{(j)} \quad \alpha_i \ge 0 \;, \beta_j \ge 0 \;,$$

where r and $F^{(j)}$ are as above.

Then $\mathcal{C}$ admits a (S_+) realization, and $\mathrm{span}\,\mathcal{C} = \ker(\sigma^n - c_{n-1}\sigma^{n-1} - \cdots - c_0)$.

Proof The proof of the proposition following directly from Prop.3.1. and the previous considerations. □

Remark 3.1. *Left–multiply vi) by F^r we have (see also [MK] Appendix I) $F^{r+n^*} = F^r(\cdots)+\lambda_1^r\big(\sum_{j=0}^{r-1}\beta_j F^{(j)}\big)$, therefore $(F^r-\lambda_1^r)(F^{n^*}-\sum_{i=0}^{n^*-1}\alpha_i F^i$, i.e $z^n-c_{n-1}z^{n-1}-\cdots-c_0$ divides*

$$(z^r-\lambda_1^r)\Big(z^{n^*}-\sum_{i=0}^{n^*-1}\alpha_i z^i\Big)\ ,\quad \alpha_i\geq 0\ ,$$

for some suitable α_i. We refer to [FB] and similar papers (see the bibliography).

4 Concluding remark

In this paper we have intended to provided the fisrt tools for a coherent and appealing framework to approch to a large class of mathematical models and studing them. The geometrical and algebraic way are nested each other in a unic strategy of investigation. In this framework the non negative systems can be involved and other linear system with inequality constraints.

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