The Wave Equation as a Port-Hamiltonian System, and a Finite Dimensional Approximation

V. Talasila, G. Golo, A.J. van der Schaft
University of Twente, Faculty of Mathematical Sciences,
P.O. Box 217, 7500 AE Enschede
The Netherlands
Email: t.viswanat,g.golo,a.j.vanderschaft@math.utwente.nl

Abstract

The problem of approximating a distributed parameter system with *free* boundary conditions is solved for the 2-dimensional wave equation. To this end we first model the wave equation as a distributed-parameter port-Hamiltonian system. Then we employ the idea that it is natural to use *different* finite elements for the approximation of different geometric variables (forms) describing a distributed-parameter system, to spatially discretize the system and we show that we obtain a finite-dimensional port-Hamiltonian system, which also preserves the conservation laws.

1 Introduction

From a control and interconnection point of view it is quite essential to describe a distributed-parameter system with varying boundary conditions inducing energy exchange through the boundary, since in many applications the interaction with the environment or with other systems (e.g. measurement or actuation) takes place via the boundary of the system. However, the treatment of distributed-parameter Hamiltonian systems wherein the geometric structure of the system is described via Poisson structures, seems mostly focused on systems with infinite spatial domain or on systems with boundary conditions such that the energy exchange through the boundary is zero. Moreover, it is not obvious how to incorporate non-zero energy flow in this framework. This fundamental problem of giving a Hamiltonian formulation for classes of distributed-parameter systems with a bounded domain and with energy-exchange through its boundary, has been solved in [1],[2], by introducing the concept of Stokes-Dirac structures. The resulting system models in this framework are infinite-dimensional Port-Hamiltonian systems. In this framework, a complex system (lumped parameter or distributed) is viewed as composing several subsystems, interconnected to each other by an power-conserving interconnection structure; which mathematically corresponds to a Dirac structure. Examples of interconnection structures are Kirchhoff's laws, Newton's third law, mass balance laws, Amperes laws, Faradays laws etc.

From the simulation point of view, a similar problem exists. Indeed, numerical methods for the solution of PDE's usually assume that the boundary conditions are given. However, as mentioned before, in the framework of control and interconnection this becomes a serious

limitation. This fundamental problem of retaining the energetic-port structure after discretization, has been solved for classes of one-dimensional distributed-parameter systems in [3]. In this paper we solve this problem for classes of two-dimensional systems. We specifically solve the example of the two-dimensional wave equation, although the same procedure holds equally well for other two-dimensional distributed-parameter port-Hamiltonian systems. We will show in this paper that the intrinsic Hamiltonian formulation of the wave equation suggests finite-element methods which result in finite-dimensional approximations which are again port-Hamiltonian systems. We prefer to work in the setting of differential geometry. Indeed, as we argue in this paper, differential forms are the natural candidates (both from a geometric as well as a physical point of view) for representing the physical variables of the system. Considering a specific example for a wave equation, we show that the velocity of a two-dimensional vibrating membrane (say) which is excited by a force acting on it's boundary (which is one-dimensional), is an object that can be defined at points and hence qualifies as a function (or, a zero-form). Also we show that it makes sense from a physical and a geometric point of view to consider elastic stress (and therefore elastic strain) as defined along paths and hence they should be represented by one-forms, while the motion of a mass (describing the kinetic momentum) finds it's natural setting along surfaces, thus kinetic momentum is represented by a two-form. However, it can be observed that most work done in numerical analysis and modeling, concentrates on thinking of waves in terms of scalar or vector fields. That is to say that not much attention is paid to the geometric and physical structure of the physical variables, the equations (PDE's) are simply considered as either scalar or vector fields, and simulations are done for these types of structures. So the 'natural' structure of the equations, which are contained in the physical variables, is not often utilized. Moreover the bulk of finite-element modeling has consisted mostly in adapting to vector-fields, methods which work well for scalar fields, see ([4]). This is understandable because representations in coordinates are often preferred to vector expressions. Indeed, for the practicing programmer who has to deal with such objects as finite-element shape functions, it would seem that a co-ordinate system is required for their manipulation. However, *the situation will be different if one can define geometrical objects that are to differential forms what finite-element interpolating functions are to scalar fields.* This point of view was stressed in [4]. These desired geometrical objects are the so-called Whitney forms, which were introduced more than 40 years ago, [5]. In this paper, we introduce more general geometric objects than the type of Whitney forms used in [4], although for illustrating our computations we use the Whitney kind of geometric elements as in [4]. Furthermore, we also solve the problem of having to retain the energetic-port structure in this framework. This implies that our discretization procedure results in open-dynamical systems. The language of differential forms seems to be crucial, in describing the variables of a system, both from the modeling and the simulation points of view. Indeed, this language, and the framework of finite-dimensional port-Hamiltonian systems [6], were instrumental in the modeling of distributed-parameter systems via a Hamiltonian formulation, with the energetic-port point of view. For simulation, this language opens up many interesting vistas. Indeed, this ap-

proach has enabled the preserving of the energetic-port structure even after the discretization. Also, the preserving of the physical properties of the distributed-parameter system seem to be quite straightforward in this setting.

First we model the n-dimensional wave equation as a port-Hamiltonian system in Section 2, and we also illustrate the physical and geometrical considerations that we take in defining the geometric structure of the variables. In Section 3 we illustrate some properties of the distributed-parameter system. In Section 4 we introduce the spatial discretization procedure, and we make a short comment on two different approaches that one may take. Section 5 illustrates the Hamiltonian structure of the discretized model, and the properties of the finite-dimensional model are in Section 6.

2 Port-Hamiltonian model of the wave equation

A distributed-parameter port-Hamiltonian system is defined by a 5-tuple $(Z, \mathcal{F}, \mathcal{E}, \mathcal{D}, H)$. Z is the manifold of spatial variables and $\mathcal{F}$ represents the space of rate energy variables, or in the bond-graph notation; flows. Correspondingly there exists the space, $\mathcal{E}$, of co-energy variables, or in the bond-graph notation; efforts. In the lumped-parameter finite-dimensional case the space of flows and the space of efforts simply correspond to a vector space and it's dual; where the duality can be seen as 'power' duality, in the sense that the duality product of an element of the flow space with an element of the effort space results in physical power [7]. In the distributed-parameter case the space of flows is a space of differential forms on the spatial domain, and the space of efforts can be defined to be (see [1]) a dual space (again a space of differential forms) to the space of flows, and with the duality product defined to be equal to physical power. Furthermore, $\mathcal{D}$ is a *Dirac structure* which is defined on a subspace of $\mathcal{F} \times \mathcal{E}$, see below for definition. Essentially, *the Dirac structure captures the natural power-conserving interconnection structure of a system.* Finally, $H \in C^\infty(Z)$ represents the Hamiltonian, or total energy, of the system.

Now we give a simple definition of a Dirac structure, see ([1]) for a more precise definition and further details. We adopt the definition of a system as being composed of a number of energy storing elements and power-dissipating elements connected by a *power-conserving* interconnection structure ([7]). Commonly encountered power-conserving interconnection structures are Kirchhoff's laws, Newton's third law and physical devices like transformers, gyrators, kinematic pairs etc. For example, Kirchhoff's laws can be looked upon as an interconnection structure (a linear subspace of all possible voltages and currents in an electrical network) in which (physically allowable) combinations of the product of voltages and currents result in zero. Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces, with physical power defined as $P = \langle e|f\rangle$, $f \in \mathcal{F}$ and $e \in \mathcal{E}$; these could be the spaces of voltages and currents in an electrical context, or forces and velocities in an mechanical context. $\mathcal{F}$ and $\mathcal{E}$ are spaces that are defined to be dual to each other, and the duality product is physical power. The duality property is quite straightforward in the finite-dimensional lumped parameter case - indeed, let $\mathcal{F}$ be a vector

space, then $\mathcal{E}$ is simply it's usual dual space. In the distributed-parameter case this duality property has been extended to spaces of differential forms which are the natural choices for the spaces of $\mathcal{F}$ and $\mathcal{E}$, see ([1]). The power-conserving interconnection structure in the distributed-parameter case captures the basic balance laws of the system - like Faraday's and Ampere's laws, or mass balance. There exists a canonically defined symmetric bilinear form on the space of *power* variables $\mathcal{F} \times \mathcal{E}$

$$\ll (f^1, e^1), (f^2, e^2) \gg := \langle e^1 | f^2 \rangle + \langle e^2 | f^1 \rangle \tag{2.1}$$

Definition 2.1. *A Dirac structure (i.e. a power-conserving interconnection structure) on a linear space $\mathcal{F}$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\mathcal{D} = \mathcal{D}^{\perp}$, where $\perp$ denotes the orthogonal complement with respect to the symmetric bilinear form.*

Note that the bilinear form $\ll, \gg$ restricted to $\mathcal{D}$ is zero, and this gives the property of power-conservation.
Consider the n-dimensional wave-equation

$$\mu \ddot{w} + E \Delta w = 0, \quad w(z,t) \in \mathbb{R},\ z = (z_1, ..., z_n) \in Z \tag{2.2}$$

where μ is the mass density and E is the Youngs modulus.

Proposition 2.1. *The port-Hamiltonian system*

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \epsilon} \\ \frac{\partial \mathcal{H}}{\partial \rho} \end{bmatrix}, \qquad \begin{bmatrix} v^B \\ \sigma^B \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \epsilon}|_{\partial Z} \\ \frac{\partial \mathcal{H}}{\partial \rho}|_{\partial Z} \end{bmatrix}, \tag{2.3}$$

defines the n-dimensional wave equation 2.2. d is the usual exterior derivative.

Proof. The interconnection structure of (2.3) is an example of the so-called Stokes-Dirac structure, see ([1]). The energy variables of 2.3 are the n-form kinetic momentum $\rho(t, z_1, ..., z_n)$, and the 1-form elastic strain $\epsilon(t, z_1, ..., z_{n-1})(= \frac{\partial w}{\partial z_1} dz_1 + ... + \frac{\partial w}{\partial z_n} dz_n)$. The co-energy variables are then, the 0-form velocity

$$v(t, z_1, ..., z_n) = \frac{\partial \mathcal{H}}{\partial \rho} \tag{2.4}$$

and the $(n-1)$-form stress

$$\sigma(t, z_1, ..., z_{n-1}) = \frac{\partial \mathcal{H}}{\partial \epsilon} \tag{2.5}$$

where $\mathcal{H}$ is the Hamiltonian density defined as

$$\mathcal{H}(\rho, \epsilon) = \frac{1}{2} (\epsilon \wedge \sigma + \rho \wedge v) \tag{2.6}$$

where $\wedge$ is the wedge product for differential forms, and the co-energy variables σ and v are related to the energy variables by the constitutive relations

$$\sigma = E * \epsilon, \quad v = \frac{1}{\mu} * \rho \tag{2.7}$$

where E and μ are the Youngs modulus and the mass density and $*$ denotes the Hodge star operator corresponding to a choice of a Riemannian metric on Z. Substituting $\epsilon = dw$ in the first part of 2.3 we obtain

$$d(\dot{w} - \frac{1}{\mu} * \rho) = 0 \quad \Rightarrow \quad \dot{w} = \frac{1}{\mu} * \rho + f(t) \tag{2.8}$$

Set $f(t) = 0$ [1]. Next we write the second part of 2.3 as $*\dot{\rho} = - * d(E * \epsilon)$ and substitute $*\rho = \mu\dot{w}$. This yields (due to $\epsilon = dw$)

$$\mu\ddot{w} + E(*d * d)w = 0 \tag{2.9}$$

The codifferentiation $\delta : \Omega^{n-k}(Z) \to \Omega^{n-k-1}(Z)$ is a map from the space of $(n-k)$-forms to the space of $(n-k-1)$-forms, and is defined as $\delta = - * d*$, hence

$$(*d * d)w = -(\delta \circ d)w = -(\delta \circ d + d \circ \delta)w = \Delta w, \quad \text{since } \delta w = 0 \tag{2.10}$$

□

In 2.3 the second expression is the interconnection structure of the boundary variables. B denotes boundary, with v^B and σ^B denoting the boundary flow and effort variables, which are simply the velocity $v = \frac{\partial \mathcal{H}}{\partial \rho}$ and the stress $\sigma = \frac{\partial \mathcal{H}}{\partial \epsilon}$ restricted to the boundary. In this paper we consider the 2-dimensional wave equation, so $n = 2$ in (2.3), hence the kinetic momentum is a 2-form, the elastic strain remains a 1-form, the elastic stress is a 1-form, and velocity remains a 0-form.

Remark 2.1. *In this remark we explain the reasons for assuming the particular geometric structures of the physical variables of our model. Let us consider a 2-dimensional vibrating object, like a vibrating membrane for example. We assume that the force is applied on the* boundary *of the system. And we consider the wave motion in only a single direction, i.e. we consider what is commonly known as* transversal waves*. We also make the basic (and not unreasonable) assumption that the velocity along every point of an infinitesimal part of the membrane is the same. The velocity of an infinitesimal part moving in only one direction, is then an vector field along that direction. Now, we can always identify a vector field with a function by a canonical mapping. Hence we will assume the velocity to be identified with a 0-form (function). The kinetic momentum on the other hand* cannot *be assumed to be the same at every point of an infinitesimal area. This is because the momentum depends on the mass density function in that infinitesimal area. Hence the momentum is an object that can be only defined on an infinitesimal area. Objects that can be integrated over 2-dimensional spaces are 2-forms, implying that the kinetic momentum is a 2-form. Since we assumed that*

[1] $f(t)$ is the motion of the rigid body (membrane in our case) as a whole, and it does not say anything about deformations. Just imagine a longitudinal wave in a vibrating string which is moving vertically up and down as a whole; this will not contribute anything to the behavior resulting from the deformations that we are interested in. Hence we set $f(t) = 0$.

the force acting on the membrane is on the boundary, hence the elastic stress must be defined as the force acting along an interval on the boundary of the infinitesimal area. Hence the stress is a 1-form. Due to the fact that the stress is applied on 1-dimensional spatial intervals, the deformation of the infinitesimal part occurs along intervals, hence strain will be defined as the ratio of change in length to the original length, implying that the strain is also a 1-form. From this discussion we can see that the variables of the distributed-parameter system indeed have different physical and geometric properties. Hence it is quiet important that we use different different finite-elements for approximating different variables, and indeed, this point of view has also been stressed in [4].

3 Some properties of the port-Hamiltonian model

- The port variables of the interconnection structure are: $(\dot{\rho}(t,z_1,z_2), v(t,z_1,z_2))$ (kinetic port),
  $(\dot{\epsilon}(t,z_1,z_2), \sigma(t,z_1,z_2))$ (potential port) and the boundary port $(v^B(t,z_1,z_2), \sigma^B(t,z_1,z_2))$.
  Ignoring all time and spatial arguments, the total power is

$$P^{total} := \int_Z \big(\dot{\rho} \wedge v\big) + \int_z \big(\dot{\epsilon} \wedge \sigma\big) - \int_{\partial Z} \big(\sigma^B \wedge v^B\big) \tag{3.11}$$

  The first term in (3.11) represents the power in the kinetic domain, the second term represents the power in the elastic potential domain, and the last term is the power through the boundary. The net power of the interconnection should be zero, indeed

$$P^{total} = -\int_Z \big(d\sigma \wedge v\big) + \int_z \big(dv \wedge \sigma\big) - \int_{\partial Z} \big(\sigma^B \wedge v^B\big) \tag{3.12}$$

  Using the following properties of the exterior derivative and the wedge product: $d(\beta \wedge \alpha) = d\beta \wedge \alpha + (-1)^{n-q}\beta \wedge d\alpha$, and $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$ where in our case we have $n = 2$, $q = 1$, $k = 1$ and $l = 0$; we obtain

$$P^{total} = \int_Z d(\sigma \wedge v) - \int_{\partial Z} \big(\sigma^B \wedge v^B\big) = \text{(by Stokes theorem)} \quad \int_{\partial Z} \big(\sigma \wedge v\big) - \int_{\partial Z} \big(\sigma^B \wedge v^B\big) = 0 \tag{3.13}$$

  Note that in the above result we have used the following: $\sigma|_{\partial Z} = \sigma^B$ and $v|_{\partial Z} = v^B$.

- The port-Hamiltonian system (2.3) has the following energy balance law

$$\frac{dH(\rho,\epsilon)}{dt} = \int_{\partial Z} \sigma^B \wedge v^B \tag{3.14}$$

  where H is the Hamiltonian, defined as $H(\rho,\epsilon) = \int_Z \mathcal{H}(\rho,\epsilon)$. (3.14) expresses that the rate of increase in energy inside the domain Z is equal to the rate of supplied

energy through the boundary. This property directly leads to the property of *passivity*: integrating (3.14) with respect to time we have

$$\begin{aligned} H(\rho(T),\epsilon(T)) - H(\rho(0),\epsilon(0)) &= \int_0^T \int_{\partial Z} \sigma^B \wedge v^B \\ \Rightarrow \int_0^T s(\sigma^B(t), v^B(t)) &\geq -H(\rho(0),\epsilon(0)) \end{aligned} \tag{3.15}$$

where $s(\sigma^B(t), v^B(t)) = \int_{\partial Z} \sigma^B \wedge v^B$ is the supplied power through the boundary. (3.15) implies that the input-output maps of the system for every $(\rho(0), \epsilon(0))$ are passive.

- We obtain two conservation laws, respectively for the total kinetic momentum and the total elastic strain. We obtain these by integrating the first part of (2.3) over Z, i.e.

$$\begin{aligned} \frac{d}{dt}\left(\int_Z \rho(t, z_1, z_2) \right) &= \int_Z -d\sigma(t, z_1, z_2) = -\int_{\partial Z} \sigma(t, z_1, z_2), \\ \frac{d}{dt}\left(\int_Z \epsilon(t, z_1, z_2) \right) &= \int_Z dv(t, z_1, z_2) = \int_{\partial Z} v(t, z_1, z_2) \end{aligned} \tag{3.16}$$

  The first expression in (3.16) represents that the change of total kinetic momentum is equal to the stress integrated over the boundary, and the second represents that the change of total elastic strain is equal to the velocity integrated over the boundary, see [1].

4 Spatial Discretization of the wave equation

The objective of this section and the next is: to first discretize the power-conserving interconnection structure and obtain a finite-dimensional power-conserving interconnection structure, then to discretize the constitutive relations (2.7) and thus the Hamiltonian , and to obtain a finite-dimensional Hamiltonian structure. So all this would result in a finite-dimensional port-Hamiltonian system. And finally to make sure that the conservation laws of section 3 are preserved in the finite-dimensional model also. We adopt the point of view stressed in [4] that it is natural to use different finite-elements for different geometric objects.

The 2-dimensional spatial manifold is split into $n+1$ triangular grids. Due to the spatial compositionality properties of port-Hamiltonian systems [7], the approximation procedure is performed to only one cell, and for simplicity of notation we use the first cell. For the ease of illustration we assume a right-angled triangular grid. Denote the nodes of this triangular grid as a, b, c and the edges as ab, bc, ca and the spatial cell as $\mathcal{Z}$. Again for ease of illustration, we identify the node a with the coordinates $(0,0)$, b with $(1,0)$ and c with $(0,1)$.

The boundary variables are the 0-form velocity and the 1-form elastic stress. The velocity on the boundary is approximated on the nodes a, b, c as follows

$$v_a^B = v_a(t,a),\ v_b^B = v_b(t,b),\ v_c^B = v_c(t,c) \tag{4.17}$$

where v_a^B represents the 0-form velocity evaluated at the node $a = (0, 0)$, and so on. The strain on the boundary is approximated as

$$\sigma_{ab}^B = \sigma_{ab}(t, ab),\ \sigma_{bc}^B = \sigma_{bc}(t, bc),\ \sigma_{ca}^B = \sigma_{ca}(t, ca) \tag{4.18}$$

where σ_{ab}^B represents the 1-form strain evaluated on the edge ab, and so on.

The 0-form velocity is approximated as follows

$$v(t, z_1, z_2) = v_a(t)\mathbf{v_a}(z_1, z_2) + v_b(t)\mathbf{v_b}(z_1, z_2) + v_c(t)\mathbf{v_c}(z_1, z_2) \tag{4.19}$$

where the spatial functions $\mathbf{v}_{(.)}(z_1, z_2)$ need to satisfy the following constraints[2]

$$\mathbf{v}_x(y) = \begin{cases} 0 & \text{if } x \neq y, \quad x \text{ and } y \text{ are the nodes } (a, b, c) \text{ of the grid,} \\ 1 & \text{if } x = y \end{cases} \tag{4.20}$$

Also, the 0-form velocity cannot be approximated with either a single zero-form or with two zero forms, see ([3]) for more details. The 1-form rate elastic strain is approximated as

$$\dot{\epsilon}(t, z_1, z_2) = \dot{\epsilon}_{ab}(t)\epsilon_{ab}(z_1, z_2) + \dot{\epsilon}_{bc}(t)\epsilon_{bc}(z_1, z_2) + \dot{\epsilon}_{ca}(t)\epsilon_{ca}(z_1, z_2) \tag{4.21}$$

where the spatial functions $\epsilon_{(.)}(z_1, z_2)$ need to satisfy[3]

$$\int_S \epsilon_s = \begin{cases} 0 & \text{if } s \neq S, \quad \text{where } S, s \text{ are the edges } (ab, bc, ca) \text{ of the grid,} \\ 1 & \text{if } s = S \end{cases} \tag{4.22}$$

The 1-form elastic stress is similarly approximated as

$$\sigma(t, z_1, z_2) = \sigma_{ab}(t)\sigma_{ab}(z_1, z_2) + \sigma_{bc}(t)\sigma_{bc}(z_1, z_2) + \sigma_{ca}(t)\sigma_{ca}(z_1, z_2) \tag{4.23}$$

where the spatial functions $\sigma_{(.)}(z_1, z_2)$ satisfy a similar constraint as in 4.22. Next, the 2-form rate kinetic momentum is approximated as

$$\dot{\rho}(t, z_1, z_2) = \dot{\rho}(t)\rho(z_1, z_2) \tag{4.24}$$

where $\rho(z_1, z_2)$ satisfies the constraint[4]

$$\int_{\mathcal{Z}} \rho(z_1, z_2) = 1, \quad \mathcal{Z} \text{ is the spatial cell.} \tag{4.25}$$

There are 2 ways in which we can proceed from here:

[2]Due to this constraint $v_x(t, x)$ coincides with $v_x(t)$ at the node x.

[3]Due to this constraint $\sigma_S(t, S)$ coincides with $\sigma_S(t)$ on the edge S.

[4]Due to this constraint $\rho(t, z_1, z_2)$ coincides with $\rho(t)$ on the triangular grid

- First we approximate the 2-form rate kinetic momentum as done before, and then we compute the time functions of the lower order forms (strain, stress and velocity). This procedure has been illustrated in ([3]) in the case of the 1-dimensional wave equation - the Telegraphers equations. The motivation for this comes from physical considerations as we explain now: let us assume that the mass density of a 2-dimensional vibrating membrane (say) is non-homogenous. Then by an appropriate choice of the spatial function of the approximating kinetic momentum variable (and with the constraint of (4.25)) we can exactly match the total kinetic momentum $\rho(t)$ of the cell with our approximation. However, if we start from the 0-form velocity, and compute the higher forms - then it is not clear how to incorporate the material properties (like mass density distribution, or elastic properties) into the approximations.

- However, the advantage with starting with the lowest order form (0-form in our case) and then constructing the time functions of the (higher forms) 1-forms and the 2-form, is in computation. Indeed, as the procedure used in this paper will indicate, all we need to do is to define the approximating function for the 0-form velocity with it's constraints as in (4.20), and we *immediately* obtain the higher forms which also *immediately* satisfy the required constraints as in (4.22) and (4.25). In the other procedure, one needs to be careful in order that the lower order forms satisfy the necessary constraints, and the computation can get a little difficult for certain choices of approximating functions (see the procedure used in [3]).

In this paper we adopt the second approach, where we employ the fact that on a given mesh of tetrahedra there exists a family of piecewise polynomial differential forms which can be described as *finite-element bases for differential forms* [4]. Let the velocity at each node of the triangular grid be approximated by a general first order expression: $\mathbf{v}_{(.)} = k_1 z_1 + k_2 z_2 + k_3$. Using the conditions in (4.20) we can easily compute the velocities at the three nodes to be

$$\mathbf{v}_a = -z_1 - z_2 + 1, \quad \mathbf{v}_b = z_1, \quad \mathbf{v}_c = z_2. \tag{4.26}$$

The associated 1-form elastic strains on each edge are then computed using the expressions

$$\epsilon_{ab} = \mathbf{v}_a d\mathbf{v}_b - \mathbf{v}_b d\mathbf{v}_a, \quad \epsilon_{bc} = \mathbf{v}_b d\mathbf{v}_c - \mathbf{v}_c d\mathbf{v}_b \quad \epsilon_{ca} = \mathbf{v}_c d\mathbf{v}_a - \mathbf{v}_a d\mathbf{v}_c \tag{4.27}$$

Note that this manner of computing the 1-form from the 0-form immediately implies the constraints (4.22). Using (4.26), we obtain from the previous equation:

$$\epsilon_{ab} = (1 - z_2)dz_1 + z_1 dz_2, \quad \epsilon_{bc} = z_1 dz_2 - z_2 dz_1, \quad \epsilon_{ca} = -z_2 dz_1 + (z_1 - 1)dz_2 \tag{4.28}$$

For the present paper, we make the assumption: $\sigma_S(z_1, z_2) = \epsilon_S(z_1, z_2)$, where S is any edge ab, bc, ca. Finally the 2-form is calculated as follows

$$\rho(z_1, z_2) = 2(\mathbf{v}_a d\mathbf{v}_b \wedge d\mathbf{v}_c + \mathbf{v}_b d\mathbf{v}_c \wedge d\mathbf{v}_a + \mathbf{v}_c d\mathbf{v}_a \wedge d\mathbf{v}_b) \tag{4.29}$$

Again note how the constraints are immediately present in this construction. Using (4.26), we obtain from the previous equation:

$$\rho(z_1, z_2) = 2(dz_1 \wedge dz_2) \tag{4.30}$$

where $\wedge$ is the wedge product for differential forms. Next we compute the relationships among the time functions. Using 2.3, we have that $\rho(t, z_1, z_2) = -d\sigma(t, z_1, z_2)$ and $\epsilon(t, z_1, z_2) = dv(t, z_1, z_2)$. From these two expressions and using (4.26), (4.28), (4.30) and $\sigma_S(z_1, z_2) = \epsilon_S(z_1, z_2)$ we can easily derive the time relations as

$$\dot{\rho}(t) = \sigma_{ab}(t) + \sigma_{bc}(t) + \sigma_{ca}(t) \tag{4.31}$$

and

$$\begin{aligned} \dot{\epsilon}_{ab}(t) &= -v_a(t) + v_b(t) \\ \dot{\epsilon}_{bc}(t) &= -v_b(t) + v_c(t) \\ \dot{\epsilon}_{ca}(t) &= -v_c(t) + v_a(t) \end{aligned} \tag{4.32}$$

Since we are interested in working in the framework of network modeling, we need to identify the ports of the discretized interconnection structure. For that we first write down the total power of the discretized cell. The net power of the cell is

$$P = \int_{\mathcal{Z}} \big(\rho \wedge v + \epsilon \wedge \sigma\big) - \int_{\partial \mathcal{Z}} \sigma^B \wedge v^B \tag{4.33}$$

Expanding (4.33) by using (4.26), (4.28), (4.30) and the boundary variables we obtain the following expression for the power (the argument 't' is omitted):

$$\begin{aligned} &\dot{\rho}(\frac{v_a + v_b + v_c}{3}) + \dot{\epsilon}_{ab}(\frac{\sigma_{bc} - \sigma_{ca}}{6}) + \dot{\epsilon}_{bc}(\frac{\sigma_{ca} - \sigma_{ab}}{6}) + \dot{\epsilon}_{ca}(\frac{\sigma_{ab} - \sigma_{bc}}{6}) \\ &-\sigma^B_{ab}(\frac{v^B_a + v^B_b}{2}) - \sigma^B_{bc}(\frac{v^B_b + v^B_c}{2}) - \sigma^B_{ca}(\frac{v^B_c + v^B_a}{2}) \end{aligned} \tag{4.34}$$

Expression 4.34 will be used for the identification of the ports of the spatially discretized cell. The boundary ports are then identified as

$$\begin{aligned} &(\sigma^B_{ab}(t), V^B_{ab}(t)) = \big(\sigma^B_{ab}(t), \frac{v^B_a(t) + v^B_b(t)}{2}\big), \quad (\sigma^B_{bc}(t), V^B_{bc}(t)) = \big(\sigma^B_{bc}(t), \frac{v^B_b(t) + v^B_c(t)}{2}\big), \\ &(\sigma^B_{ca}(t), V^B_{ca}(t)) = \big(\sigma^B_{ca}(t), \frac{v^B_c(t) + v^B_a(t)}{2}\big) \end{aligned} \tag{4.35}$$

The ports for the kinetic domain are

$$(\dot{\rho}(t), \bar{V}(t)) = \left(\dot{\rho}(t), \frac{v_a(t) + v_b(t) + v_c(t)}{3}\right) \tag{4.36}$$

Let

$$\epsilon_1(t) = \frac{\epsilon_{ab}(t) - \epsilon_{ca}(t)}{2}, \quad \epsilon_2(t) = \frac{\epsilon_{bc}(t) - \epsilon_{ca}(t)}{2} \tag{4.37}$$

Using 4.32 and the boundary port variables, we can write $\frac{d\epsilon_1(t)}{dt}$ and $\frac{d\epsilon_2(t)}{dt}$ as

$$\begin{aligned}\dot{\epsilon}_1(t) &= -V_{ab}^B(t) + 2V_{bc}^B(t) - V_{ca}^B(t) \\ \dot{\epsilon}_2(t) &= -2V_{ab}^B(t) + V_{bc}^B(t) + V_{ca}^B(t)\end{aligned} \tag{4.38}$$

Then the ports for the potential domain are

$$\begin{aligned}(\dot{\epsilon_1}, \sigma_1) &= \left(-V_{ab}^B(t) + 2V_{bc}^B(t) - V_{ca}^B(t), \frac{\sigma_{bc}(t) - \sigma_{ca}(t)}{3} \right) \\ (\dot{\epsilon_2}, \sigma_2) &= \left(-2V_{ab}^B(t) + V_{bc}^B(t) + V_{ca}^B(t), \frac{\sigma_{ca}(t) - \sigma_{ab}(t)}{3} \right)\end{aligned} \tag{4.39}$$

Now we can write the interconnection structure of the discretized cell as

$$\underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1/3 & 1/3 \\ 0 & 0 & -1 & 1/3 & 0 & -1/3 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} \bar{V} \\ \sigma_1 \\ \sigma_2 \\ -\sigma_{ab}^B \\ -\sigma_{bc}^B \\ -\sigma_{ca}^B \end{bmatrix}}_{\mathbf{e}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \dot{\rho} \\ \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ V_{ab}^B \\ V^B bc \\ V_{ca}^B \end{bmatrix}}_{\mathbf{f}} = 0 \tag{4.40}$$

We define the following subspace

$$\mathcal{D}_d = \{(\mathbf{f}, \mathbf{e}) \in \mathbb{R}^{12} : \mathbf{E}\mathbf{e} + \mathbf{F}\mathbf{f} = 0\} \tag{4.41}$$

Proposition 4.1. *$\mathcal{D}_d$ is a finite-dimensional Dirac structure with respect to the bilinear form defined in (2.1)*

Proof. $\mathcal{D}_d$ is a finite-dimensional Dirac structure with respect to the bilinear form given in (2.1) if and only if the following two conditions are satisfied

- Rank Condition: $\text{rank}[\mathbf{E} \quad \mathbf{F}] = 6$
- Zero net power: $\mathbf{E}\mathbf{F}^T + \mathbf{F}\mathbf{E}^T = 0$

These two conditions are immediately satisfied for the matrices $\mathbf{E}$ and $\mathbf{F}$ obtained in (4.40). □

The above procedure holds for the remaining n cells, and for each of the spatially discretized cells we can derive a finite-dimensional Dirac structure. Due to the compositionality properties of finite-dimensional Dirac structures (see [7]) we can interconnect these Dirac structure and this will result in a Dirac structure - thus preserving the natural power-conserving property of the system.

5 Discretization of the Constitutive Relations

The approximation of the flow variables, *rate* kinetic momentum and *rate* elastic strain, has been shown in the previous section. Then, the kinetic momentum and elastic strain will be approximated as

$$\begin{aligned}\rho(t, z_1, z_2) &= \rho(t)\, \rho(z_1, z_2),\\ \epsilon(t, z_1, z_2) &= \epsilon_{ab}(t)\epsilon_{ab}(z_1, z_2) + \epsilon_{bc}(t)\epsilon_{bc}(z_1, z_2) + \epsilon_{ca}(t)\epsilon_{ca}(z_1, z_2)\end{aligned} \tag{5.42}$$

Now we derive the Hamiltonian structure of the finite-dimensional model. The kinetic energy of the cell can be defined in two ways

$$H^\rho(t) = \frac{1}{2}\int_{\mathcal{Z}} \rho \wedge v, \quad \text{or} \quad H^\rho(t) = \frac{1}{2}\int_{\mathcal{Z}} \rho \wedge *\frac{\rho}{\mu(z_1, z_2)} \tag{5.43}$$

The second term expands as follows

$$H^\rho_1(t) = \frac{\rho^2(t)}{2}\int_{\mathcal{Z}} \frac{\rho(z_1, z_2) \wedge *\rho(z_1, z_2)}{\mu(z_1, z_2)} = \frac{\rho^2(t)}{2\, M} \tag{5.44}$$

where $M^{-1} = \int_{\mathcal{Z}} \frac{\rho(z_1,z_2)\wedge *\rho(z_1,z_2)}{\mu(z_1,z_2)} = 2$, since we assume in this paper that $\mu(z_1, z_2) = 1$ and we use $*\rho(z_1, z_2) = *(2\, dz_1 \wedge dz_2) = 2$. The first term expands as

$$\begin{aligned}H^\rho_2(t) &= \int_{\mathcal{Z}} \rho(t)\rho(z_1, z_2) \wedge \left[v_a(t)\, \mathbf{v}_a(z_1, z_2) + v_b(t)\, \mathbf{v}_b(z_1, z_2) + v_c(t)\, \mathbf{v}_c(z_1, z_2)\right]\\ &= \frac{\rho(t)}{2}\left(\frac{v_a(t) + v_b(t) + v_c(t)}{3}\right)\end{aligned} \tag{5.45}$$

From 5.44 and 5.45, and using the identification of ports as done in Section 4 we have the following relation

$$\left(\frac{v_a(t) + v_b(t) + v_c(t)}{3}\right) = \bar{V} = \frac{\rho(t)}{M} \tag{5.46}$$

The potential energy of the cell is also defined in two ways

$$H^\epsilon(t) = \frac{1}{2}\int_{\mathcal{Z}} \epsilon \wedge \sigma \quad \text{or} \quad H^\epsilon(t) = \frac{1}{2}\int_{\mathcal{Z}} \epsilon \wedge \frac{*\epsilon}{Y} \tag{5.47}$$

The first term expands as

$$\frac{1}{2}\int_{\mathcal{Z}} \Big[\epsilon_{ab}(t)\epsilon_{ab}(z_1, z_2)+\epsilon_{bc}(t)\epsilon_{bc}(z_1, z_2)+\epsilon_{ca}(t)\epsilon_{ca}(z_1, z_2)\Big] \wedge \Big[\sigma_{ab}(t)\sigma_{ab}(z_1, z_2)+\sigma_{bc}(t)\sigma_{bc}(z_1, z_2)+\sigma_{ca}(t)\sigma_{ca}(z_1, z_2)\Big] \tag{5.48}$$

$$= \frac{1}{2}\left[\epsilon_{ab}(t)\left(\frac{\sigma_{bc}(t) - \sigma_{ca}(t)}{6}\right) + \epsilon_{bc}(t)\left(\frac{\sigma_{ca}(t) - \sigma_{ab}(t)}{6}\right) + \epsilon_{ca}(t)\left(\frac{\sigma_{ab}(t) - \sigma_{bc}(t)}{6}\right)\right] \tag{5.49}$$

By the identification of ports as done in Section 4, we rewrite this as

$$H^\epsilon_1(t) = \frac{1}{2}\Big[\epsilon_1\sigma_1 + \epsilon_2\sigma_2\Big] \tag{5.50}$$

The second term expands as

$$\begin{aligned}\frac{1}{2}\Bigg[&\epsilon_{ab}(t)\Bigg(\epsilon_{ab}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ab}\wedge *\epsilon_{ab}}{E^1_{ab}(z_1,z_2)}+\epsilon_{bc}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ab}\wedge *\epsilon_{bc}}{E^2_{ab}(z_1,z_2)}+\epsilon_{ca}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ab}\wedge *\epsilon_{ca}}{E^3_{ab}(z_1,z_2)}\Bigg)\\ &+\epsilon_{bc}(t)\Bigg(\epsilon_{ab}(t)\int_{\mathcal{Z}}\frac{\epsilon_{bc}\wedge *\epsilon_{ab}}{E^1_{bc}(z_1,z_2)}+\epsilon_{bc}(t)\int_{\mathcal{Z}}\frac{\epsilon_{bc}\wedge *\epsilon_{bc}}{E^2_{bc}(z_1,z_2)}+\epsilon_{ca}(t)\int_{\mathcal{Z}}\frac{\epsilon_{bc}\wedge *\epsilon_{ca}}{E^3_{bc}(z_1,z_2)}\Bigg)\\ &+\epsilon_{ca}(t)\Bigg(\epsilon_{ab}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ca}\wedge *\epsilon_{ab}}{E^1_{ca}(z_1,z_2)}+\epsilon_{bc}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ca}\wedge *\epsilon_{bc}}{E^2_{ca}(z_1,z_2)}+\epsilon_{ca}(t)\int_{\mathcal{Z}}\frac{\epsilon_{ca}\wedge *\epsilon_{ca}}{E^3_{ca}(z_1,z_2)}\Bigg)\Bigg]\end{aligned}\tag{5.51}$$

where the $E^{(.)}_{(.)}$ are the Youngs modulus along any particular edge of the triangular grid. Using the following relations

$$\begin{aligned}*\,\epsilon_{ab} &= *\big[(1-z_2)dz_1+z_1dz_2\big] = (1-z_2)dz_2 - z_1dz_1\\ *\,\epsilon_{bc} &= *\big[z_1dz_2 - z_2dz_1\big] = -z_1dz_1 - z_2dz_2\\ *\,\epsilon_{ca} &= *\big[-z_2dz_1+(z_1-1)dz_2\big] = -z_2dz_2+(1-z_1)dz_1\end{aligned}\tag{5.52}$$

we can easily show that (5.51) reduces to

$$H^{\epsilon}_2(t) = \frac{5}{9}\Big[\epsilon_1^2+\epsilon_2^2-\epsilon_1\epsilon_2\Big]\tag{5.53}$$

where $\epsilon_1(t) = \frac{\epsilon_{ab}(t)-\epsilon_{ca}(t)}{2}$ and $\epsilon_2(t) = \frac{\epsilon_{bc}(t)-\epsilon_{ca}(t)}{2}$ as in 4.37
From 5.50 and 5.53 we can derive the following

$$\begin{aligned}\sigma_1 &= \frac{5}{9}\Big(2\epsilon_1-\epsilon_2\Big)\\ \sigma_2 &= \frac{5}{9}\Big(2\epsilon_2-\epsilon_1\Big)\end{aligned}\tag{5.54}$$

Now it is clear that (5.46) and (5.54) can be rewritten as

$$\bar{V} = \frac{\partial\mathcal{H}_d(\rho,\epsilon_1,\epsilon_2)}{\partial\rho},\quad \sigma_1 = \frac{\partial\mathcal{H}_d(\rho,\epsilon_1,\epsilon_2)}{\partial\epsilon_1},\quad \sigma_2 = \frac{\partial\mathcal{H}_d(\rho,\epsilon_1,\epsilon_2)}{\partial\epsilon_2}\tag{5.55}$$

where $\mathcal{H}_d(\rho,\epsilon_1,\epsilon_2)$, the total energy of the cell, is given by

$$\mathcal{H}_d(\rho,\epsilon_1,\epsilon_2) = H^{\rho}(t)+H^{\epsilon}(t) = \quad \rho^2(t)+\frac{5}{9}\Big(\epsilon_1^2(t)+\epsilon_2^2(t)-\epsilon_1(t)\,\epsilon_2(t)\Big)\tag{5.56}$$

6 Properties of the finite-dimensional model

- We have shown that the interconnection structure of the finite-dimensional model is a Dirac structure, see Proposition(4.1). Also, the constitutive relations of the finite-dimensional model obey the usual Hamiltonian formalism, as in (5.56) and (5.55), then the finite-dimensional model (4.40), (5.42), (5.55) and (5.56) represents a finite-dimensional port-Hamiltonian system. Due to the power-conservation property of the

Dirac structure, defined by the vanishing of the bilinear form when restricted to $\mathcal{D}$, any distributed-parameter port-Hamiltonian system satisfies along it's trajectories the energy balance (3.14). In the finite-dimensional case the vanishing of the bilinear form when restricted to the finite-dimensional Dirac structure $\mathcal{D}_d$ is given as

$$\frac{dH_d(\rho(t),\epsilon_1(t),\epsilon_3(t))}{dt} - \int_{\partial\mathcal{Z}} \left(\sigma^B(t) \wedge v^B(t)\right) = 0 \tag{6.57}$$

(6.57) represents the energy balance law of the finite-dimensional system which has the same form as that for the distributed-parameter system. H_d is the total Hamiltonian of the finite-dimensional system. From this energy balance law one can then derive the property of passivity as follows: integrate (6.57) with respect to time, we obtain

$$\begin{aligned} &H_d(\rho(T),\epsilon_1(T),\epsilon_3(T)) - H_d(\rho(0),\epsilon_1(0),\epsilon_3(0)) = \int_0^T \int_{\partial\mathcal{Z}} \sigma^B(t) \wedge v^B(t) \\ &\Rightarrow \int_0^T s_d(\sigma^B, v^B) \geq -H_d(\rho(0),\epsilon_1(0),\epsilon_3(0)) \end{aligned} \tag{6.58}$$

where $s_d(\sigma^B, v^B) = \int_{\partial\mathcal{Z}} \sigma^B(t) \wedge v^B(t)$ is the supplied power. The above equation is the property of passivity, and has the same form as for the distributed-parameter system.

- Two simple dynamic invariants, kinetic momentum conservation and elastic strain conservation, were highlighted in (3.16). For the finite-dimensional model we have these two dynamic invariants also. Indeed, kinetic momentum conservation is simply (4.31) rewritten as follows

$$\dot{\rho}(t) = \sigma_a^B b(t) + \sigma_b^B c(t) + \sigma_c^B a(t) \tag{6.59}$$

Finite dimensional elastic strain conservation is (4.32) rewritten as

$$\dot{\epsilon}_{ab}(t) = -v_a^B(t) + v_b^B(t), \quad \dot{\epsilon}_{bc}(t) = -v_b^B(t) + v_c^B(t), \quad \dot{\epsilon}_{ca}(t) = -v_c^B(t) + v_a^B(t) \tag{6.60}$$

7 Conclusions

A port-Hamiltonian model was derived for the n-dimensional wave equation, which is a Hamiltonian formulation with a bounded domain and with energy-exchange through it's boundary. A simple argument was given for the importance of using differential forms in the description of the physical variables of the system. This argument can indeed be extended to classes of one and two-dimensional distributed-parameter port-Hamiltonian systems.

Numerical methods for the solution of PDE's usually assume that the boundary conditions are given. However, this is a serious limitation in the framework of control and network modeling. Hence the need to approximate distributed-parameter systems while retaining the

energetic-port structure. This problem has been solved by using the framework of [4], wherein simple Whitney forms were used for approximating the differential forms. An discussion was presented indicating that in this method it is not very clear how to preserve the material properties (like mass density distribution or elastic properties). In a future paper, we intend to prove that using the approach of [3], we not only preserve the energetic port-structure after discretization, but we can also easily preserve the material properties. Some conservation laws have been preserved using the discretization scheme in this paper. Indeed, as we can easily observe, the preservation of these conservation laws are an immediate consequence of the discretization procedure and the structure of port-Hamiltonian systems.

References

[1] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics, to be published.*

[2] B.M. Maschke and A.J. van der Schaft. Port controlled hamiltonian representation of distributed parameter systems. *Proc. IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, Princeton University, Editors N.E. Leonard, R. Ortega*, pages 28–38, 2000.

[3] G. Golo, V. Talasila, A.J. van der Schaft, and B.M. Maschke. Hamiltonian discretizations of the telegrapher's equations. *Automatica, To be submitted*, 2002.

[4] A. Bossavit. Differential forms and the computation of fields and forces in electromagnetism. *European Journal of Mechanics, B/Fluids*, 10(5):474–488, 1991.

[5] Whitney H. *Geometric Integration Theory.* Princeton University Press, 1957.

[6] A.J. van der Schaft and B.M. Maschke. The hamiltonian formulation of energy conserving physical systems with external ports. *Archiv fur Electronik und Ubertragungstechnik, pp. 362-371*, 49:362–371, 1995.

[7] A.J. van der Schaft. *The Mathematics of Systems and Control: From Intelligent Control to Behavioral Systems*, chapter Interconnection and Geometry. Foundations Systems and Control Groningen, 1999.