Interconnected Systems of Fliess Operators

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Abstract

Given two analytic nonlinear input-output systems represented as Fliess operators, four system interconnections are considered in a unified setting: the parallel connection, product connection, cascade connection and feedback connection. In each case, the corresponding generating series is produced, when one exists, and conditions for convergence of the corresponding Fliess operator are given. In the process, an existing notion of a *composition product* for formal power series is generalized to the multivariable setting, and its set of known properties is expanded. In addition, the notion of a *feedback product* for formal power series is introduced and characterized.

1 Introduction

Let $I = \{0, 1, \ldots, m\}$ denote an alphabet and I^* the set of all words over I. A formal power series in I is any mapping of the form $I^* \mapsto \mathbb{R}^\ell$, and the set of all such mappings will be denoted by $\mathbb{R}^\ell \langle\langle I \rangle\rangle$. For each $c \in \mathbb{R}^\ell \langle\langle I \rangle\rangle$, one can formally associate a corresponding m-input, ℓ-output operator F_c in the following manner. Let $p \geq 1$ and $a < b$ be given. For a measurable function $u : [a, b] \to \mathbb{R}^m$, define $\|u\|_p = \max\{\|u_i\|_p : 1 \leq i \leq m\}$, where $\|u_i\|_p$ is the usual L_p-norm for a measurable real-valued function, u_i, defined on $[a, b]$. Let $L_p^m[a, b]$ denote the set of all measurable functions defined on $[a, b]$ having a finite $\| \cdot \|_p$ norm. With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define inductively for each $\eta \in I^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\phi = 1$, and

$$E_{i_k i_{k-1} \cdots i_1}[u](t, t_0) = \int_{t_0}^{t} u_{i_k}(\tau) E_{i_{k-1} \cdots i_1}[u](\tau, t_0)\, d\tau,$$

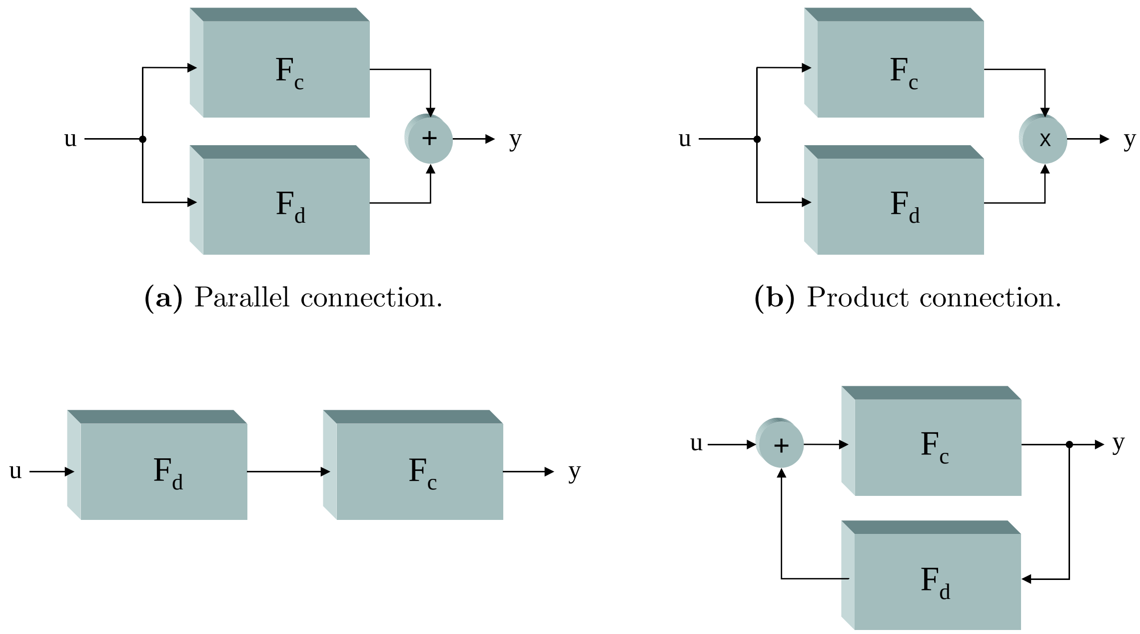
where $u_0(t) \equiv 1$. The input-output operator corresponding to c is then

$$F_c[u](t) = \sum_{\eta \in I^*} (c, \eta)\, E_\eta[u](t, t_0),$$

which is referred to as a *Fliess operator*. In the classical literature where these operators first appeared [4, 5, 6, 8, 9, 10], it is normally assumed that there exists real numbers $K > 0$ and $M \geq 1$ such that

$$|(c, \eta)| \leq K M^{|\eta|} |\eta|!, \qquad \forall\, \eta \in I^*, \tag{1.1}$$

where $|z| = \max\{|z_1|, |z_2|, \ldots, |z_\ell|\}$ when $z \in \mathbb{R}^\ell$, and $|\eta|$ denotes the number of symbols in η. These growth conditions on the coefficients of c insure that there exist positive real numbers R and T_0 such that for all piecewise continuous u with $\|u\|_\infty \leq R$ and $T \leq T_0$, the series defining F_c converges uniformly and absolutely on $[t_0, t_0 + T]$. Under such conditions the power series c is said to be *locally convergent*. More recently, it was shown in [7] that the growth condition (1.1) also



(a) Parallel connection. **(b)** Product connection.

(c) Cascade connection. **(d)** Feedback connection.

Figure 1: Elementary system interconnections.

implies that F_c constitutes a well defined operator from $B_p^m(R)[t_0,\, t_0+T]$ into $B_q^\ell(S)[t_0,\, t_0+T]$ for sufficiently small $R, S, T > 0$, where the numbers $p, q \in [1, \infty]$ are conjugate exponents, i.e., $1/p + 1/q = 1$ with $(1, \infty)$ being a conjugate pair by convention.

In many applications input-output systems are interconnected in a variety of ways. Given two Fliess operators F_c and F_d, where $c, d \in \mathbb{R}^\ell\langle\langle I \rangle\rangle$ are locally convergent, Figure 1 shows four elementary interconnections. In the case of the cascade and feedback connections it is assumed that $\ell = m$. The general goal of this paper is to describe in a unified manner the generating series for each of the four interconnections shown in Figure 1 and conditions under which it is locally convergent. The parallel connection is the trivial case, and the product connection was analyzed in [11]. They are included for completeness and some of the analysis is applicable to the other two interconnections. It was shown in [3] for the SISO case (i.e., $\ell = m = 1$) that there always exists a series $c \circ d$ such that $y = F_c[F_d[u]] = F_{c\circ d}[u]$, but a multivariable analysis of this *composition product* is apparently not available in the literature, nor are any results about local convergence. So in Section 2 the composition product is first investigated independent of the interconnection problem. It is defined in the multivariable setting and various fundamental properties are presented. Then a condition is introduced under which the composition product preserves both rationality and local convergence. Finally, in preparation for the feedback analysis, it is next shown that the composition product produces a contractive mapping on the set of all formal power series using the familiar ultrametric. In Section 3, the three *nonrecursive* connections: the parallel, product and cascade connections are analyzed primarily by applying the results of Section 2. In Section 4 the feedback connection is considered. Such a system is said to be *well-posed* if the support of c and d each contain at least one word having a nonzero symbol. Otherwise, there is no real recursive structure, and a degenerate case results. If, for example, F_c is a linear operator then formally the solution to the feedback equation

$$y = F_c[u + F_d[y]] \tag{1.2}$$

is

$$y = F_c[u] + F_c \circ F_d \circ F_c[u] + \cdots \tag{1.3}$$

It is not immediately clear that this series converges in any manner, and in particular, converges to another Fliess operator, say $F_{c@d}$ for some $c@d \in \mathbb{R}^m \langle\langle I \rangle\rangle$. When F_c is nonlinear, the problem is further complicated by the fact that a simple series representation (1.3) is not possible. Thus, for the case where the system inputs are being generated by an exosystem which is itself a Fliess operator, a sufficient condition is given under which a unique solution to the feedback equation exists. Then the closed-loop system is characterized in terms of a new Fliess operator when a certain series factorization property is available. This leads to an implicit characterization of the *feedback product*, $c@d$, for formal power series.

2 The Composition Product

The composition product of two series over an alphabet $X = \{x_0, x_1\}$ is defined recursively in terms of the shuffle product [2, 3]. For any $\eta \in X^*$ and $d \in \mathbb{R}\langle\langle X \rangle\rangle$, let

$$\eta \circ d := \begin{cases} \eta & : \ |\eta|_{x_1} = 0 \\ x_0^{k+1}[d \sqcup\!\sqcup (\eta' \circ d)] & : \ \eta = x_0^k x_1 \eta', \ \ k \geq 0, \end{cases} \tag{2.4}$$

where $|\eta|_{x_1}$ denotes the number of symbols in η equivalent to x_1. For $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ the definition is extended to

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\ \eta \circ d. \tag{2.5}$$

For SISO systems it is easily verified that $F_c \circ F_d = F_{c \circ d}$. For the multivariable case it is necessary to consider power series of the form $d : X^* \mapsto \mathbb{R}^m$, where $X = \{x_0, x_1, \ldots, x_m\}$ is an arbitrary alphabet with $m + 1$ letters. In which case, the defining equation (2.4) becomes

$$\eta \circ d = \begin{cases} \eta & : \ |\eta|_{x_i} = 0, \ \ \forall i \neq 0 \\ x_0^{k+1}[d_i \sqcup\!\sqcup (\eta' \circ d)] & : \ \eta = x_0^k x_i \eta', \ \ k \geq 0, \ \ i \neq 0, \end{cases}$$

where $d_i : \xi \mapsto ((d, \xi))_i$, the i-th component of (d, ξ). Observe that in general for

$$\eta = x_0^{n_k} x_{i_k} x_0^{n_{k-1}} x_{i_{k-1}} \cdots x_0^{n_1} x_{i_1} x_0^{n_0}, \tag{2.6}$$

where $i_j \neq 0$ for $j = 1, \ldots, k$, it follows that

$$\eta \circ d = x_0^{n_k+1}[d_{i_k} \sqcup\!\sqcup x_0^{n_{k-1}+1}[d_{i_{k-1}} \sqcup\!\sqcup \cdots x_0^{n_1+1}[d_{i_1} \sqcup\!\sqcup x_0^{n_0}] \cdots]].$$

The following theorem guarantees that the composition product is well defined by insuring that the series (2.5) is summable.

Theorem 2.1 *Given a fixed $d \in \mathbb{R}^m\langle\langle X \rangle\rangle$, the family of series $\{\eta \circ d : \eta \in X^*\}$ is locally finite, and therefore summable.*

Proof: Given an arbitrary $\eta \in X^*$ expressed in the form (2.6), it follows directly that

$$\begin{aligned} \operatorname{ord}(\eta \circ d) &= n_0 + k + \sum_{j=1}^{k} n_j + \operatorname{ord}(d_{i_j}) \\ &= |\eta| + \sum_{j=1}^{|\eta|-|\eta|_{x_0}} \operatorname{ord}(d_{i_j}), \end{aligned} \tag{2.7}$$

where the *order* of c is defined as

$$\operatorname{ord}(c) = \begin{cases} \inf\{|\eta| : \eta \in \operatorname{supp}(c)\} & : \ c \neq 0 \\ \infty & : \ c = 0, \end{cases}$$

and $\operatorname{supp}(c) := \{\eta \in X^* : (c, \eta) \neq 0\}$ denotes the support of c. Hence, for any $\xi \in X^*$

$$\begin{aligned} I_d(\xi) &:= \{\eta \in X^* : (\eta \circ d, \xi) \neq 0\} \\ &\subset \{\eta \in X^* : \operatorname{ord}(\eta \circ d) \leq |\xi|\} \\ &= \{\eta \in X^* : |\eta| + \sum_{j=1}^{|\eta|-|\eta|_{x_0}} \operatorname{ord}(d_{i_j}) \leq |\xi|\}. \end{aligned}$$

Clearly this latter set is finite, and thus $I_d(\xi)$ is finite for all $\xi \in X^*$. This fact implies summability [1]. ■

From the definition, it is easily verified that for any series $c, d, e \in \mathbb{R}^m\langle\langle X \rangle\rangle$,

$$(c + d) \circ e = c \circ e + d \circ e,$$

but in general $c \circ (d + e) \neq c \circ d + c \circ e$. A special exception are *linear series.* A series $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle$ is called linear if

$$\operatorname{supp}(c) \subseteq \{\eta \in X^* : \eta = x_0^{n_1} x_i x_0^{n_0}, \ \ i \in \{1, 2, \ldots, m\}, \ \ n_1, n_0 \geq 0\}.$$

Since the shuffle product distributes over addition, given any $\eta = x_0^{n_1} x_i x_0^{n_0}$:

$$\begin{aligned} \eta \circ (d + e) &= x_0^{n_1+1}[(d + e)_i \sqcup\!\sqcup\, x_0^{n_0}] \\ &= x_0^{n_1+1}(d_i \sqcup\!\sqcup\, x_0^{n_0}) + x_0^{n_1+1}(e_i \sqcup\!\sqcup\, x_0^{n_0}) \\ &= \eta \circ d + \eta \circ e. \end{aligned}$$

Therefore,

$$\begin{aligned} c \circ (d + e) &= \sum_{\eta \in I^*} (c, \eta)\ \eta \circ (d + e) \\ &= \sum_{\eta \in I^*} (c, \eta)\ \eta \circ d + (c, \eta)\ \eta \circ e \\ &= c \circ d + c \circ e. \end{aligned}$$

A linear series should not be confused with a rational series. The series

$$c = \sum_{k=0}^{\infty} k!\; x_0^k x_1$$

is linear but not rational, while the bilinear series

$$c = \sum_{k=0}^{\infty} \sum_{i_1,\ldots,i_k=0}^{\infty} (CN_{i_k} \cdots N_{i_1} z_0)\; x_{i_k} \cdots x_{i_1}$$

with $N_i \in \mathbb{R}^{n\times n}$ and $C^T, z_0 \in \mathbb{R}^{n\times 1}$ is rational but not linear.

The set $\mathbb{R}\langle\langle X\rangle\rangle$ forms a metric space under the ultrametric

$$\begin{aligned} \text{dist} \;&:\; \mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle \mapsto \mathbb{R}^+ \cup \{0\} \\ &:\; (c,d) \mapsto \sigma^{\text{ord}(c-d)}, \end{aligned}$$

where $\sigma \in (0,1)$ is arbitrary. The following theorem states that the composition product on $\mathbb{R}\langle\langle X\rangle\rangle \times \mathbb{R}\langle\langle X\rangle\rangle$ is at least continuous in its first argument. The result extends naturally to vector-valued series in a componentwise fashion.

Theorem 2.2 *Let $\{c_i\}_{i\geq 1}$ be a sequence in $\mathbb{R}\langle\langle X\rangle\rangle$ with $\lim_{i\to\infty} c_i = c$. Then $\lim_{i\to\infty}(c_i \circ d) = c \circ d$.*

Proof: Define the sequence $n_i = \text{ord}(c_i - c)$ for $i \geq 1$. Since c is the limit of the sequence $\{c_i\}_{i\geq 1}$, $\{n_i\}_{i\geq 1}$ must have a monotone increasing subsequence $\{n_{i_j}\}$. Now observe that

$$\text{dist}(c_i \circ d, c \circ d) = \sigma^{\text{ord}((c_i - c)\,\circ d)}$$

and

$$\begin{aligned} \text{ord}((c_{i_j} - c) \circ d) &= \text{ord}\left(\sum_{\eta \in \text{supp}(c_{i_j} - c)} (c_{i_j} - c, \eta)\; \eta \circ d \right) \\ &\geq \inf_{\eta \in \text{supp}(c_{i_j} - c)} \text{ord}(\eta \circ d) \\ &= \inf_{\eta \in \text{supp}(c_{i_j} - c)} |\eta| + (|\eta| - |\eta|_{x_0})\; \text{ord}(d) \\ &\geq n_{i_j}. \end{aligned}$$

Thus, $\text{dist}(c_{i_j} \circ d, c \circ d) \leq \sigma^{n_{i_j}}$ for all $j \geq 1$, and the theorem is proven. ■

It is shown in [3] by counter example that the composition product is *not* a rational operation. That is, the composition of two rational series does not in general produce a rational series. But in [2], it is shown in the SISO case that special classes of rational series produce rational series when the composition product is applied. The following definition is the multivariable extention of this essential property, and the corresponding rationality proof is not significantly different.

Definition 2.1 *A series $c \in \mathbb{R}\langle\langle X\rangle\rangle$ is* ***limited relative to x_i*** *if there exists an integer $N_i \geq 0$ such that*

$$\sup_{\eta \in \text{supp}(c)} |\eta|_{x_i} \leq N_i.$$

If c is limited relative to x_i for every $i = 1, 2, \ldots, m$ then c is ***input-limited****. In such cases, let $N_c := \sum_i N_i$. A series $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$ is input-limited if each component series, c_j, is input-limited for $j = 1, 2, \ldots, \ell$. In this case, $N_c := \max_j N_{c_j}$.*

It is shown next that the composition product will preserve local convergence if its first argument is input-limited.

Theorem 2.3 *Suppose $c, d \in \mathbb{R}^m \langle\langle X \rangle\rangle$ are locally convergent series with growth constants K_c, M_c and K_d, M_d, respectively. If c is input-limited then $c \circ d$ is locally convergent with*

$$|(c \circ d, \nu)| < K_c K_d^{N_c} (N_c + 1)\ (M(N_c + 1))^{|\nu|} |\nu|!, \quad \forall \nu \in X^*, \tag{2.8}$$

where $M = \max(M_c, M_d)$ and $N_c \geq 1$.

The proof of this result requires the following lemma.

Lemma 2.1 *[11] Suppose $c, d \in \mathbb{R}\langle\langle X \rangle\rangle$ are locally convergent series with growth constants K_c, M_c and K_d, M_d, respectively. Then $c \sqcup\!\sqcup\, d$ is locally convergent with*

$$|(c \sqcup\!\sqcup\, d, \nu)| \leq K_c K_d M^{|\nu|} (|\nu| + 1)!, \quad \forall \nu \in X^*, \tag{2.9}$$

where $M = \max(M_c, M_d)$.

Proof of Theorem 2.3: The proof has two main parts. Only the SISO case is considered here for brevity. First it is shown that for any $\eta \in X^*$ in the form of equation (2.6) (which is no restriction):

$$|(\eta \circ d, \nu)| \leq \left(K_d^k \frac{M_d^{-|\eta|}}{n_0!(n_1 + 1)! \cdots (n_k + 1)!} \right) M_d^{|\nu|} |\nu|!, \quad \forall \nu \in X^*. \tag{2.10}$$

For any set of integers $n_j \geq 0$, $j \geq 0$, define the set of words

$$\eta_{j+1} = x_0^{n_{j+1}} x_1 \eta_j, \quad \eta_0 = x_0^{n_0}. \tag{2.11}$$

The upperbound (2.10) is proven for any given η by verifying inductively that it holds for every word η_j, $j \geq 0$. Observe that for $j = 0$:

$$|(\eta_0 \circ d, \nu)| = |(\eta_0, \nu)| = \begin{cases} 1 & : \ \nu = \eta_0 \\ 0 & : \ \nu \neq \eta_0 \end{cases}$$

and

$$\frac{M_d^{-|\eta_0|}}{n_0!} M_d^{|\nu|} |\nu|! = \begin{cases} 1 & : \ \nu = \eta_0 \\ \frac{M_d^{|\nu| - |\eta_0|}}{n_0!} |\nu|! & : \ \nu \neq \eta_0. \end{cases}$$

Hence, the inequality is satisfied in the trivial case. Now suppose the result is true up to some given integer $j - 1 \geq 0$. From the definition of the composition product

$$|(\eta_j \circ d, \nu)| = \left| (x_0^{n_j + 1} [d \sqcup\!\sqcup\, (\eta_{j-1} \circ d)], \nu) \right|.$$

Applying Lemma 2.1 gives

$$|(d \sqcup (\eta_{j-1} \circ d), \nu)| \leq K_d \left(K_d^{j-1} \frac{M_d^{-|\eta_{j-1}|}}{n_0!(n_1+1)! \cdots (n_{j-1}+1)!} \right) M_d^{|\nu|}(|\nu|+1)!.$$

Therefore,

$$\begin{aligned} |(\eta_j \circ d, \nu)| &= \begin{cases} \left| \left(d \sqcup (\eta_{j-1} \circ d), \left(x_0^{n_j+1}\right)^{-1}(\nu)\right) \right| & : \quad \nu = x_0^{n_j+1}\nu' \\ 0 & : \quad \text{otherwise} \end{cases} \\ &\leq \left(K_d^j \frac{M_d^{-|\eta_{j-1}|}}{n_0!(n_1+1)! \cdots (n_{j-1}+1)!} \right) M_d^{|\nu|-(n_j+1)}(|\nu|-n_j)!. \end{aligned}$$

Now using the fact that for any integer $n \geq 1$, $\binom{n}{k} \geq k+1$ when $k = 0, 1, \ldots, n-1$, it follows that

$$(|\nu| - n_j)! \leq \frac{|\nu|!}{(n_j+1)!}, \quad 0 \leq n_j \leq |\nu| - 1.$$

Thus,

$$|(\eta_j \circ d, \nu)| \leq \left(K_d^j \frac{M_d^{-|\eta_j|}}{n_0!(n_1+1)! \cdots (n_j+1)!} \right) M_d^{|\nu|}|\nu|!$$

since $n_j < |\nu|$ is necessary for $(\eta_j \circ d, v)$ to be nonzero. Therefore, the inequality holds for all $j \geq 0$, or equivalently, for any $\eta \in X^*$

Now if c is input-limited the theorem is proven in the second main step as follows:

$$\begin{aligned} |(c \circ d, \nu)| &= \left| \sum_{\eta \in \mathrm{supp}(c) \cap I_d(\nu)} (c,\eta)(\eta \circ d, \nu) \right| \\ &\leq \sum_{i=0}^{|\nu|} \sum_{k=0}^{min(i,N_c)} \sum_{\substack{\eta \in \mathrm{supp}(c) \\ n_0+n_1+\ldots+n_k+k=i}} \left[K_c M_c^{|\eta|} |\eta|! \right] \cdot \\ &\quad \left[\left(K_d^k \frac{M_d^{-|\eta|}}{n_0!(n_1+1)! \cdots (n_k+1)!} \right) M_d^{|\nu|}|\nu|! \right] \\ &\leq K_c K_d^{N_c} M^{|\nu|} |\nu|! \sum_{i=0}^{|\nu|} \sum_{k=0}^{N_c} \\ &\quad \sum_{\substack{\eta \in \mathrm{supp}(c) \\ n_0+n_1+\ldots+n_k+k=i}} \left(\frac{|\eta|!}{n_0!(n_1+1)! \cdots (n_k+1)!} \right) \\ &\leq K_c K_d^{N_c} M^{|\nu|} |\nu|! \sum_{i=0}^{|\nu|} \sum_{k=0}^{N_c} (k+1)^i \\ &\leq K_c K_d^{N_c} \frac{N_c+2}{2} M^{|\nu|} |\nu|! \sum_{i=0}^{|\nu|} (N_c+1)^i \\ &\leq K_c K_d^{N_c} \frac{N_c+2}{2N_c} M^{|\nu|} |\nu|! \, (N_c+1)^{|\nu|+1} \\ &\leq K_c K_d^{N_c} (N_c+1) M^{|\nu|} (N_c+1)^{|\nu|} |\nu|!, \end{aligned}$$

assuming in this last step that $N_c > 1$. The inequality for the power sum $S^i(n) := \sum_{k=1}^n k^i \leq n^i[\frac{n+1}{2}]$, $n, i > 0$ has also been used in the development. To produce the result when $N_c = 1$, simply note that

$$\sum_{i=0}^{|\nu|}\sum_{k=0}^{1}\sum_{\substack{\eta\in\text{supp}(c)\\ n_0+n_1+\ldots+n_k+k=i}}(n_0, n_1+1, \ldots, n_k+1)! = \sum_{i=0}^{|\nu|} 1 + (2^i - 1) \leq 2^{|\nu|+1}.$$

■

When c and d are both linear series, the growth condition (2.8) is known to be conservative. Representing the composition product as a convolution sum and using the fact that $\sum_{k=0}^n \binom{n}{k}^{-1} < 3$ for any $n \geq 0$, it can be shown that a tighter bound is

$$|(c \circ d, \nu)| < K_c K_d M^{|\nu|} |\nu|!, \quad \forall \nu \in X^*. \tag{2.12}$$

It is conjectured that in the general case perhaps the tighter bound

$$|(c \circ d, \nu)| < K_c K_d^{N_c} M^{|\nu|} |\nu|!, \quad \forall \nu \in X^* \tag{2.13}$$

applies.

The metric space $(\mathbb{R}\langle\langle X\rangle\rangle, \text{dist})$ is known to be complete [1]. Given a fixed $c \in \mathbb{R}\langle\langle X\rangle\rangle$, consider the mapping $\mathbb{R}\langle\langle X\rangle\rangle \mapsto \mathbb{R}\langle\langle X\rangle\rangle : d \mapsto c \circ d$. The section is concluded by showing that this mapping is always a contraction on $\mathbb{R}\langle\langle X\rangle\rangle$, i.e.,

$$\text{dist}(c \circ d, c \circ e) < \text{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X\rangle\rangle.$$

The focus is on the SISO case where any $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be written unambiguously in the form

$$c = c_0 + c_1 + \cdots,$$

where $c_k \in \mathbb{R}\langle\langle X\rangle\rangle$ has the property that $\eta \in \text{supp}(c_k)$ only if $|\eta|_{x_1} = k$. Some of the series c_k may be the zero series. When $c_0 = 0$, c is referred to as being *homogeneous*. When $c_k = 0$ for $k = 0, 1, \ldots, l-1$ and $c_l \neq 0$ then c is called *homogeneous of order l*. In this setting consider the following theorem.

Theorem 2.4 *For any $c_k \in \mathbb{R}\langle\langle X\rangle\rangle$ with $X = \{x_0, x_1\}$*

$$\text{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k \cdot \text{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X\rangle\rangle.$$

Proof: The proof is by induction for the nontrivial case where $c_k \neq 0$. First suppose $k = 0$. From the definition of the composition product it follows directly that $\eta \circ d = \eta$ for all $\eta \in \text{supp}(c_0)$. Therefore,

$$c_0 \circ d = \sum_{\eta\in\text{supp}(c_0)} (c_0, \eta)\, \eta \circ d = \sum_{\eta\in\text{supp}(c_0)} (c_0, \eta)\, \eta = c_0,$$

and

$$\begin{aligned} \text{dist}(c_0 \circ d, c_0 \circ e) &= \text{dist}(c_0, c_0) = 0 \\ &\leq \sigma^0 \cdot \text{dist}(d, e). \end{aligned}$$

Now fix any $k \geq 0$ and assume the claim is true for all $c_0, c_1, \ldots, c_k$. In particular, this implies that

$$\mathrm{ord}(c_k \circ d - c_k \circ e) \geq k + \mathrm{ord}(d - e). \tag{2.14}$$

For any $j \geq 0$, words in $\mathrm{supp}(c_j)$ have the form η_j as defined in equation (2.11). Observe then that

$$\begin{aligned} c_{k+1} \circ d - c_{k+1} \circ e &= \sum_{\eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\ \eta_{k+1} \circ d - (c_{k+1}, \eta_{k+1})\ \eta_{k+1} \circ e \\ &= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\ \Big[{x_0}^{n_{k+1}+1}[d \sqcup\!\sqcup [\eta_k \circ d]] - \\ &\qquad {x_0}^{n_{k+1}+1}[e \sqcup\!\sqcup [\eta_k \circ e]] \Big] \\ &= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\ \Big[{x_0}^{n_{k+1}+1}[d \sqcup\!\sqcup [\eta_k \circ d]] - \\ &\qquad {x_0}^{n_{k+1}+1}[d \sqcup\!\sqcup [\eta_k \circ e]] \Big] + \Big[{x_0}^{n_{k+1}+1}[d \sqcup\!\sqcup [\eta_k \circ e]] - \\ &\qquad {x_0}^{n_{k+1}+1}[e \sqcup\!\sqcup [\eta_k \circ e]] \Big] \\ &= \sum_{\eta_k, \eta_{k+1} \in X^*} (c_{k+1}, \eta_{k+1})\ \Big[{x_0}^{n_{k+1}+1}[d \sqcup\!\sqcup [\eta_k \circ d - \eta_k \circ e]] + \\ &\qquad {x_0}^{n_{k+1}+1}[(d-e) \sqcup\!\sqcup [\eta_k \circ e]] \Big] \end{aligned}$$

using the fact that the shuffle product distributes over addition. Next, applying the identity (2.7) and the inequality (2.14) with $c_k = \eta_k$, it follows that

$$\begin{aligned} \mathrm{ord}(c_{k+1} \circ d - c_{k+1} \circ e) &\geq \min \Big\{ \inf_{\eta_{k+1} \in \mathrm{supp}(c_{k+1})} n_{k+1} + 1 + \mathrm{ord}(d) + k + \mathrm{ord}(d-e), \\ &\qquad \inf_{\eta_{k+1} \in \mathrm{supp}(c_{k+1})} n_{k+1} + 1 + \mathrm{ord}(d-e) + |\eta_k| + k \cdot \mathrm{ord}(e) \Big\} \\ &\geq k + 1 + \mathrm{ord}(d-e), \end{aligned}$$

and thus,

$$\mathrm{dist}(c_{k+1} \circ d, c_{k+1} \circ e) \leq \sigma^{k+1} \cdot \mathrm{dist}(d, e).$$

Hence, $\mathrm{dist}(c_k \circ d, c_k \circ e) \leq \sigma^k \cdot \mathrm{dist}(d, e)$ holds for any $k \geq 0$. ■

Applying the above theorem leads to following result.

Theorem 2.5 *If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ with $X = \{x_0, x_1\}$ then for any series c_0'*

$$\mathrm{dist}((c_0' + c) \circ d, (c_0' + c) \circ e) = \mathrm{dist}(c \circ d, c \circ e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle. \tag{2.15}$$

If c is homogeneous of order $l \geq 1$ then

$$\mathrm{dist}(c \circ d, c \circ e) \leq \sigma^l \cdot \mathrm{dist}(d, e), \quad \forall d, e \in \mathbb{R}\langle\langle X \rangle\rangle. \tag{2.16}$$

Proof: The equality is proven first. Since the metric dist is shift-invariant:

$$\begin{aligned} \mathrm{dist}((c_0' + c) \circ d, (c_0' + c) \circ e) &= \mathrm{dist}\left(c_0' \circ d + c \circ d, c_0' \circ e + c \circ e\right) \\ &= \mathrm{dist}\left(c_0' + c \circ d, c_0' + c \circ e\right) \\ &= \mathrm{dist}\left(c \circ d, c \circ e\right). \end{aligned}$$

The inequality is proven next by first selecting any fixed $l \geq 1$ and showing inductively that it holds for any partial sum $\sum_{i=l}^{l+k} c_i$ where $k \geq 0$. When $k = 0$ Theorem 2.4 implies that

$$\mathrm{dist}(c_l \circ d, c_l \circ e) \leq \sigma^l \cdot \mathrm{dist}(d, e).$$

If the result is true for partial sums up to any fixed k then using the ultrametric property

$$\mathrm{dist}(d, e) \leq \max\{\mathrm{dist}(d, f), \mathrm{dist}(f, e)\}, \quad \forall d, e, f \in \mathbb{R}\langle\langle X \rangle\rangle,$$

it follows that

$$\begin{aligned}
&\mathrm{dist}\left(\left(\sum_{i=l}^{l+k+1} c_i\right) \circ d, \left(\sum_{i=l}^{l+k+1} c_i\right) \circ e\right) \\
&= \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ e + c_{l+k+1} \circ e\right) \\
&\leq \max\left\{\mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ e\right),\right. \\
&\qquad \left.\mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d + c_{l+k+1} \circ e, \left(\sum_{i=l}^{l+k} c_i\right) \circ e + c_{l+k+1} \circ e\right)\right\} \\
&= \max\left\{\mathrm{dist}(c_{l+k+1} \circ d, c_{l+k+1} \circ e), \mathrm{dist}\left(\left(\sum_{i=l}^{l+k} c_i\right) \circ d, \left(\sum_{i=l}^{l+k} c_i\right) \circ e\right)\right\} \\
&\leq \max\left\{\sigma^{l+k+1} \cdot \mathrm{dist}(d, e), \sigma^l \cdot \mathrm{dist}(d, e)\right\} \\
&\leq \sigma^l \cdot \mathrm{dist}(d, e).
\end{aligned}$$

Hence, the result holds for all $k \geq 0$. Finally the theorem is proven by noting that $c = lim_{k \to \infty} \sum_{i=l}^{l+k} c_i$ and using the continuity of the composition product proven in Theorem 2.2 and the metric dist.

■

The final result of this section is given below.

Theorem 2.6 *For any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ with $X = \{x_0, x_1\}$, the mapping $d \mapsto c \circ d$ is a contraction on $\mathbb{R}\langle\langle X \rangle\rangle$.*

Proof: Choose any series $d, e \in \mathbb{R}\langle\langle X \rangle\rangle$. If c is homogeneous of order $l \geq 1$ then the result follows directly from equation (2.16). Otherwise, observe that via equation (2.15):

$$\begin{aligned}
\mathrm{dist}(c \circ d, c \circ e) &= \mathrm{dist}\left(\left(\sum_{l=1}^{\infty} c_i\right) \circ d, \left(\sum_{l=1}^{\infty} c_i\right) \circ e\right) \\
&\leq \sigma \cdot \mathrm{dist}(d, e) \\
&< \mathrm{dist}(d, e).
\end{aligned}$$

■

3 The Nonrecursive Connections

In this section the generating series are produced for the three nonrecursive interconnections and their local convergence is characterized.

Theorem 3.1 *Suppose $c, d \in \mathbb{R}^{\ell}\langle\langle I \rangle\rangle$ are locally convergent power series. Then each nonrecursive interconnected input-output system shown in Figure 1 (a)-(c) has a Fliess operator representation generated by a locally convergent series as indicated:*

1. $F_c + F_d = F_{c+d}$
2. $F_c \cdot F_d = F_{c \,\sqcup\!\sqcup\, d}$
3. $F_c \circ F_d = F_{c \circ d}$, *where* $\ell = m$, *and* c *is input-limited.*

Proof:
1. Observe that

$$\begin{aligned} F_c[u](t) + F_d[u](t) &= \sum_{\eta \in I^*} [(c,\eta) + (d,\eta)]\, E_\eta[u](t,t_0) \\ &= F_{c+d}[u](t). \end{aligned}$$

Since c and d are locally convergent, define $M = max\{M_c, M_d\}$. Then for any $\eta \in I^*$ it follows that

$$\begin{aligned} |(c+d,\eta)| &= |(c,\eta) + (d,\eta)| \\ &\leq (K_c + K_d) M^{|\eta|} |\eta|! \end{aligned}$$

or $c + d$ is locally convergent.
2. In light of the componentwise definition of the shuffle product, it can be assumed here without loss of generality that $\ell = 1$. Thus,

$$\begin{aligned} F_c[u](t) F_d[u](t) &= \sum_{\eta \in I^*} (c,\eta) E_\eta[u](t,t_0) \sum_{\xi \in I^*} (c,\xi) E_\xi[u](t,t_0) \\ &= \sum_{\eta,\xi \in I^*} (c,\eta)(d,\xi)\; E_\eta[u](t,t_0) E_\xi[u](t,t_0) \\ &= \sum_{\eta,\xi \in I^*} (c,\eta)(d,\xi)\; E_{\eta \,\sqcup\!\sqcup\, \xi}[u](t,t_0) \\ &= F_{c \,\sqcup\!\sqcup\, d}[u](t). \end{aligned}$$

Applying Lemma 2.1 and the fact that $2^n \leq n+1$, $n \geq 0$ gives

$$|(c \,\sqcup\!\sqcup\, d, \eta)| \leq K_c K_d (2M)^{|\eta|} |\eta|!.$$

Thus, $c \,\sqcup\!\sqcup\, d$ is locally convergent.
3. For any monomial $\eta \in I^*$ and $d \in \mathbb{R}^m\langle\langle I \rangle\rangle$ the corresponding Fliess operators are

$$\begin{aligned} F_\eta[u](t) &= E_\eta[u](t,t_0) \\ F_d[u](t) &= \sum_{\xi \in I^*} (d,\xi) E_\xi[u](t,t_0). \end{aligned}$$

Therefore,

$$(F_\eta \circ F_d[u])(t) = E_\eta\left[F_d[u]\right](t,t_0).$$

If $|\eta| = |\eta|_{x_0}$ then

$$\begin{aligned}(F_\eta \circ F_d[u])(t) &= E_\eta[u](t,t_0) = F_\eta[u](t) \\ &= F_{\eta\circ d}[u](t).\end{aligned}$$

If, on the other hand, $\eta = \underbrace{0\cdots 0}_{k \text{ times}} i\eta'$ with $i \neq 0$ then

$$\begin{aligned}(F_\eta \circ F_d[u])(t) &= E_{\underbrace{0\cdots0}_{k\text{ times}} i\eta'}[F_d[u]](t,t_0) \\ &= \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}}_{k+1 \text{ times}} F_{d_i}[u](\tau)E_{\eta'}[F_d[u]](\tau,t_0)\, d\tau_1\ldots d\tau_{k+1} \\ &= \underbrace{\int_{t_0}^{t}\cdots\int_{t_0}^{\tau_2}}_{k+1 \text{ times}} F_{d_i \sqcup\!\sqcup (\eta'\circ d)}[u](\tau,t_0)\, d\tau_1\ldots d\tau_{k+1} \\ &= F_{\underbrace{0\cdots0}_{k+1\text{ times}}[d_i \sqcup\!\sqcup (\eta'\circ d)]}[u](t) \\ &= F_{\eta\circ d}[u](t).\end{aligned}$$

Thus,

$$\begin{aligned}(F_c \circ F_d[u])(t) &= \sum_{\eta\in I^*}(c,\eta)E_\eta[F_d[u]](t,t_0) \\ &= \sum_{\eta\in I^*}(c,\eta)F_{\eta\circ d}[u](t) \\ &= \sum_{\eta\in I^*}(c,\eta)\left[\sum_{\nu\in I^*}(\eta\circ d,\nu)E_\nu[u](t,t_0)\right] \\ &= \sum_{\nu\in I^*}\left[\sum_{\eta\in I^*}(c,\eta)(\eta\circ d,\nu)\right]E_\nu[u](t,t_0) \\ &= \sum_{\nu\in I^*}(c\circ d,\nu)E_\nu(t,t_0) \\ &= F_{c\circ d}[u](t).\end{aligned}$$

Local convergence of $c \circ d$ under the stated conditions was proven in Theorem 2.3. ■

It should be noted that c being input-limited is only a *sufficient* condition for the composition product to produce a locally convergent series. If in Theorem 2.3 it is instead assumed that both c and d have finite Lie rank, then the mappings F_c and F_d each have a finite dimensional analytic state space realization, and therefore so does the mapping $F_c \circ F_d$. The classical literature then provides that the generating series $c\circ d$ must be locally convergent [10]. An example of this situation is given below. This suggests the possibility that just as the composition of two analytic functions is again analytic, the composition of two Fliess operators may *always* produce another well-defined Fliess operator. But showing directly that $c \circ d$ is always locally convergent is presently an open problem.

Example 3.1 Consider the state space system

$$\begin{aligned} \dot{z} &= z^2 u, \quad z(0) = 1 \\ y &= z. \end{aligned}$$

It is easily verified that $y = F_c[u]$ where the only nonzero coefficients are $(c, \underbrace{1 \cdots 1}_{k}) = k!$, $k \geq 0$. So c is not input-limited. But the mapping $F_{c \circ c}$ clearly has the analytic state space realization,

$$\begin{aligned} \dot{z}_1 &= z_1^2 u, \quad z_1(0) = 1 \\ \dot{z}_2 &= z_2^2 z_1, \quad z_2(0) = 1 \\ y &= z_2, \end{aligned}$$

and therefore the generating series $c \circ c$ must be locally convergent. □

4 The Feedback Connection

The general goal of this section is to determine when there exists a y which satisfies the feedback equation (1.2), and in particular, when does there exist a generating series e so that $y = F_e[u]$ over some appropriate input set. In the latter case, the feedback equation becomes equivalent to

$$F_e[u] = F_c[u + F_{d \circ e}[u]],$$

and the *feedback product* is defined by $c@d = e$. To make the analysis simpler, it is assumed through out that the inputs u are supplied by an exosystem which is itself a Fliess operator as shown in Figure 2. That is, $u = F_b(v)$ for some locally convergent series b. In this setting, a sufficient

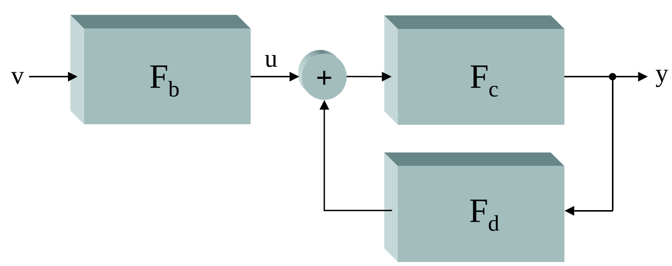


Figure 2: A feedback configuration with a Fliess operator exosystem providing the inputs.

condition is given under which a unique solution y of the feedback equation (1.2) is known to exist, and y is characterized as the output of a new Fliess operator when a certain series factorization property is available. This leads to an implicit characterization of the feedback product $c@d$.

Theorem 4.1 *Let* $b, c, d \in \mathbb{R}\langle\langle I \rangle\rangle$ *with* $I = \{0, 1\}$. *Then:*

1. The mapping

$$\begin{aligned} S &: \mathbb{R}\langle\langle I \rangle\rangle \mapsto \mathbb{R}\langle\langle I \rangle\rangle \\ &: \tilde{e}_i \mapsto \tilde{e}_{i+1} = c \circ (b + d \circ \tilde{e}_i) \end{aligned}$$

has a unique fixed point $\tilde{e}$*.*

2. If b*,* c*,* d *and* $\tilde{e}$ *are locally convergent then the feedback equation (1.2) has the unique solution* $y = F_{\tilde{e}}[v]$ *for any admissible* v*.*

3. If $\tilde{e} = e \circ b$ *for some locally convergent series* e *then* $c@d = e$*.*

Proof:
1. The mapping S is a contraction since via Theorem 2.6:

$$\begin{aligned} \operatorname{dist}(S(\tilde{e}_i), S(\tilde{e}_j)) &< \operatorname{dist}(b + d \circ \tilde{e}_i, b + d \circ \tilde{e}_j) \\ &= \operatorname{dist}(d \circ \tilde{e}_i, d \circ \tilde{e}_j) \\ &< \operatorname{dist}(\tilde{e}_i, \tilde{e}_j). \end{aligned}$$

Therefore, the mapping S has a unique fixed point $\tilde{e}$, that is,

$$\tilde{e} = c \circ (b + d \circ \tilde{e}).$$

2. From the stated assumptions concerning b, c, d and $\tilde{e}$ it follows that

$$\begin{aligned} F_{\tilde{e}}[v] &= F_{c \circ (b + d \circ \tilde{e})}[v] \\ &= F_c[F_b[v] + F_d[F_{\tilde{e}}[v]]] \end{aligned}$$

for any admissible v. Therefore equation (1.2) has the unique solution $y = F_{\tilde{e}}[v]$.
3. Since e is locally convergent

$$y = F_{\tilde{e}}[v] = F_e[F_b[v]] = F_e[u],$$

thus $c@d = e$. ∎

This result suggests several open problems. Are there conditions on b, c, and d alone which will insure that $\tilde{e}$ above is locally convergent ? When does there exist a factorization of the form $\tilde{e} = e \circ b$, where e is locally convergent ? Can the theorem be generalized to the case where the inputs are simply from an L_p space and not filtered through a Fliess operator a priori ? Some insight into these questions is provide by the following examples.

Example 4.1 Suppose $c, d \in \mathbb{R}\langle\langle I \rangle\rangle$ are locally convergent. If c is also a linear series, one can formally write using equation (1.3)

$$c@d = c + \sum_{k=1}^{\infty} (c \circ d)^{\circ k} \circ c, \tag{4.17}$$

where $c^{\circ k}$ denotes k copies of c composed $k-1$ times. In light of Theorem 2.2, $c@d$ is well defined as long as the family of series $\{(c \circ d)^{\circ k} : k \geq 1\}$ is summable, and it is easily verified that

$$((c \circ d)^{\circ k}, \nu) = 0, \ \ \forall k > |\nu|.$$

So for this special case an application of Theorem 4.1 can be avoided for concluding that $c@d$ is well defined. Now from Theorem 2.3, $c \circ d$ is locally convergent. If the (conjectured) growth condition (2.13) holds then it follows immediately that

$$|((c \circ d)^{\circ k}, \nu)| \leq K_{c\circ d}^{k} M^{|\nu|} |\nu|!, \quad \forall \nu \in I^*,$$

and thus,

$$\begin{aligned} |(c@d, \nu)| &\leq K_c M_c^{|\nu|} |\nu|! + \sum_{k=1}^{|\nu|} K_c K_{c\circ d}^{k} M^{|\nu|} |\nu|! \\ &\leq K_c \left(\sum_{k=0}^{|\nu|} K_{c\circ d}^{k} \right) M^{|\nu|} |\nu|!, \quad \forall \nu \in I^*. \end{aligned}$$

If, for example, $K_{c\circ d} > 1$ then

$$|(c@d, \nu)| \leq K_c \frac{K_{c\circ d}}{K_{c\circ d} - 1} (K_{c\circ d} M)^{|\nu|} |\nu|!, \quad \forall \nu \in I^*.$$

Thus, for a linear series c, the closed-loop system can be described by the Fliess operator $F_{c@d}$ with $c@d$ given by (4.17), if $c \circ d$ satisfies the growth condition (2.13). If, in addition, d is linear then $c \circ d$ always satisfies the bound (2.13) (c.f. (2.12)) and in fact $K_{c\circ d} = K_c K_d$. □

Example 4.2 Consider a *generalized series* δ with the defining property that δ is the identity element for the composition product, i.e., $c \circ \delta = \delta \circ c = c$ for any $c \in \mathbb{R}\langle\langle X \rangle\rangle$. Then (mapping the symbols $x_i \mapsto i$) $F_\delta[u] = u$ for any u, and a unity feedback system has the generating series $c@\delta$. Setting $b = 0$ in Figure 2 (or effectively setting $u \equiv 0$), a self-excited feedback loop is described by $F_{\tilde{e}}[v]$, where $\tilde{e} = c \circ \tilde{e}$ and $\tilde{e} = e \circ 0$. In this case $\tilde{e} = \lim_{k\to\infty} c^{\circ k}$ and $e = c@d = \tilde{e}$. From Theorem 4.1, $c@\delta$ is well defined in general. Analogous to the situation with the composition product, if c has finite Lie rank then $c@\delta$ will always be locally convergent. For example, when $c = 1 + x_1$ it is easy verified that $c@\delta = \sum_{k\geq 0} x_0^k$ so that $F_{c@\delta}[0](t) = e^t$ for $t \geq 0$. When $c = 1 + 2x_1 + 2x_1^2$ it follows that $c@\delta = \sum_{k\geq 0} (k+1)!\, x_0^k$ and $F_{c@\delta}[0](t) = 1/(1-t)^2$ for $0 \leq t < 1$. □

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