Gap Metric Robustness of Adaptive Controllers

Mark French
Department of Electronics and Computer Science,
University of Southampton,
SO17 1BJ, UK
mcf@ecs.soton.ac.uk
http://www.ecs.soton.ac.uk/~mcf

Abstract

We consider the construction of adaptive controllers for minimum phase linear systems which achieve non-zero robustness margins in the sense of the (linear) $L^2[0,\infty)$ gap metric. The gap perturbations may be more constrained for larger disturbances and for larger parametric uncertainty. Working within the framework of the nonlinear gap metric [3], universal adaptive controllers are first given achieving this goal for first order plants, and the results are then generalised to relative degree one, minimum phase plants.

1 Introduction

The study of the robustness of adaptive controllers has a long and perhaps infamous history. In the early eighties it was observed that the adaptive designs of the time had limited robustness properties. Closed loops could become unstable even in the presence of small disturbances and innocuous looking classes of unmodelled dynamics. Specifically Rohrs [5] showed that many of the existing designs became unstable even when applied to a first order plant with a pair of unmodelled conjugate poles far out in the left hand plane. These observations gave a great impetus in the 1980s-90s to the study of robust adaptive control [4].

There have also been recent developments in nonlinear control theory which can be utilized to address robust adaptive control problems. In particular, the gap metric (first introduced into control theory by Zames and El-Sakkary [6], [1]) has been generalised to a nonlinear setting in the key fundamental paper [3]. This paper therefore provides a new framework in which to address the problem of robust adaptive control. A great advantage of the robustness framework of [3] is that the existence of (nonlinear) non-zero robustness margins can be reduced to proving the existence of a certain closed loop gain function, and further, that the natural uncertainty descriptions are those commonly considered in robust control theory.

In [3] two standard parametric adaptive controllers are considered, and both are shown to have zero-robustness margins in the sense of the margins defined in that paper. Supporting numerical evidence and series expansions of the closed loop solutions suggested that these designs indeed have no robustness to simple but arbitrarily small gap perturbations.

The fundamental question we address is whether it is possible to construct adaptive controllers with non-zero robustness margins. By answering this question in the affirmative, we develop a class of robust adaptive controllers which are robust to both perturbations of the plant in the gap metric and to bounded L^2 disturbances. We show (perhaps contrary to expectation), that it is possible to construct a universal adaptive controller for a first order plant (which can be arbitrarily unstable), whilst maintaining robustness in a gap metric sense. The gap perturbations may be more constrained for larger disturbances and for larger parametric uncertainty. A similar result is obtained for minimum phase plants of relative degree one. Interestingly, the controllers obtained are essentially standard parametric adaptive controllers, but with a change in a growth rate in the adaptive law.

The results reported in this paper therefore construct adaptive controllers with non-zero gap robustness margins. Since the gap metric induces the graph topology which is the fundamental description in which to investigate robustness of closed loops, the results in this paper represent the start of a seemingly natural approach to robust adaptive control.

2 Background

Throughout this paper follow the notation of [3], hence the material in this section is strongly based on Section II of that paper. We consider causal plants $\mathcal{P}\colon \mathcal{U} \to \mathcal{Y}_e$ and causal controllers $\mathcal{C}\colon \mathcal{Y} \to \mathcal{U}$, where $\mathcal{U}$, $\mathcal{Y}$, $\mathcal{Y}_e$ are the signal spaces $L^2(\mathbb{R}_+, \mathbb{R})$, $L^2(\mathbb{R}_+, \mathbb{R})$ and $L^{2,e}(\mathbb{R}_+, \mathbb{R})$ respectively. Our central concern is with the system of equations:

$$\begin{aligned} y_1 &= Pu_1 \\ u_2 &= Cy_2 \\ y_0 &= y_1 + y_2 \\ u_0 &= u_1 + u_2, \end{aligned} \tag{2.1}$$

which corresponds to the classical feedback configuration of a plant and controller.

Such a feedback configuration is denoted by $[P, C]$, and is said to be well posed if the relation

$$H_{P,C}\colon \mathcal{W} \to \mathcal{W} \times \mathcal{W} \quad : \quad \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \tag{2.2}$$

is a well defined, injective and causal operator.

A general causal operator between normed spaces $F\colon \mathcal{X}_1 \to \mathcal{X}_2$, is said to be stable if it has a finite induced norm, ie.

$$\|F\| = \sup_{\|x\| \neq 0} \frac{\|Fx\|}{\|x\|} < \infty. \tag{2.3}$$

For nonlinear operators, this notion of stability can be excessively restrictive, so we relax the notion to the existence of as gain function. In this context, the operator $F\colon \mathcal{X}_1 \to \mathcal{X}_2$ is said to be gain-function (gf)-stable if there exists a nonlinear gain function

$$\gamma\colon [0,\infty) \to [0,\infty), \quad : \quad \gamma(r) = \sup_{\|x\|\le r} \|Fx\|.^{1} \tag{2.4}$$

A closed loop $[P, C]$ is said to be stable (resp. gf. stable) if $H_{P,C}$ is stable (resp. gf. stable). Corresponding to the plant operator $\mathcal{P}$ is a subset of $\mathcal{W}$, called the graph of the plant $\mathcal{G}_P$, which is defined as follows:

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \ : \ u \in \mathcal{U},\ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}. \tag{2.5}$$

Note that in general $\mathcal{G}_P \neq \mathcal{W}$. Similarly the graph of the controller operator $\mathcal{C}$ is defined as:

$$\mathcal{G}_C = \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \ : \ Cy \in \mathcal{U}, y \in \mathcal{Y} \right\} \subset \mathcal{W}, \tag{2.6}$$

and in general $\mathcal{G}_C \neq \mathcal{W}$.

A summation operator is defined on the cartesian product of the graphs $\mathcal{M} = \mathcal{G}_P$, $\mathcal{N} = \mathcal{G}_C$ as:

$$\Sigma_{\mathcal{M},\mathcal{N}}\colon \mathcal{M} \times \mathcal{N} \to \mathcal{W} \quad : \quad (m, n) \mapsto m + n \tag{2.7}$$

where note that if $[P, C]$ is well posed then $\Sigma_{\mathcal{M},\mathcal{N}}$ is invertible and $\Sigma_{\mathcal{M},\mathcal{N}}^{-1} = H_{P,C}$. Finally we define two nonlinear parallel projection operators:

$$\Pi_{\mathcal{M}//\mathcal{N}}\colon \mathcal{W} \to \mathcal{W} \quad : \quad \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}$$

$$\Pi_{\mathcal{N}//\mathcal{M}}\colon \mathcal{W} \to \mathcal{W} \quad : \quad \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ y_2 \end{pmatrix}$$

A useful property of these parallel projections is that stability (resp. gf. stability) of one parallel projection implies stability (resp. gf. stability) of the other.

The (nonlinear) gap metric is defined w.r.t. a normed vector space $\mathcal{F}$ as follows:

$$\begin{aligned} \vec{\delta}_{\mathcal{F}}(P, P_1) &= \begin{cases} \inf_{\Phi\in\mathcal{O}} \|(\Phi - I)|_{\mathcal{G}_P}\|_{\mathcal{F}} & \text{if } \mathcal{O} \neq \emptyset. \\ \infty & \text{if } \mathcal{O} = \emptyset. \end{cases} \\ \delta_{\mathcal{F}}(P, P_1) &= \max\{\vec{\delta}_{\mathcal{F}}(P, P_1), \vec{\delta}_{\mathcal{F}}(P_1, P)\}. \end{aligned} \tag{2.8}$$

where

$$\mathcal{O} = \{\Phi\colon \mathcal{M} \to \mathcal{M}_1 \ : \ \Phi \text{ is causal, bijective and } \Phi(0) = 0\} \tag{2.9}$$

and $\mathcal{M} = \mathcal{G}_P$, $\mathcal{M}_1 = \mathcal{G}_{P_1}$.

[1]In [3], the gain function is defined by $\gamma(r) = \sup_{\|x\|\le r,\ \tau>0} \|T_\tau Fx\|$, where T_τ denotes the truncation operator $T_\tau f(t) = f(t)$ for $t \in [0, \tau]$ and 0 otherwise. However, it can be shown that the two definitions coincide if F is causal.

For the case of finite dimensional, strictly proper linear time invariant systems the standard gap metric is defined by

$$\begin{aligned}\vec{\delta}_0(P,P_1) &= \sup_{0\neq m_1\in\mathcal{M}_1}\inf_{0\neq m\in\mathcal{M}}\frac{\|m_1-m\|_2}{\|m\|_2}\\ \delta_0(P,P_1) &= \max\{\vec{\delta}_0(P,P_1),\vec{\delta}_0(P_1,P)\}.\end{aligned} \tag{2.10}$$

The nonlinear gap provides a generalisation of the linear gap in the sense that if $\delta_0(P,P_1)<1$, then

$$\delta_0(P,P_1)=\delta_{L^2[0,\infty)}(P,P_1). \tag{2.11}$$

As our main interest in this paper is with $L^2=L^2[0,\infty)$ results, we define $\delta(P,P_1):=\delta_{L^2[0,\infty)}(P,P_1)$.

3 The First Order L^2 result

We first develop the result for the case of a first order linear system perturbed by L^2 disturbances. The main result is as follows:

Theorem 3.1. *Let $\mathcal{U}=\mathcal{Y}=L^2$, and let $P^*(\theta,y_1^0)\colon\mathcal{U}\to\mathcal{Y}_e$ be defined:*

$$P^*(\theta,y_1^0)(u_1)\ :\ \dot{y}_1=\theta y_1+u_1,\ y_1(0)=y_1^0\in\mathbb{R},\ \theta\in\mathbb{R}. \tag{3.12}$$

Then there exists a controller $C^*\colon\mathcal{Y}\to\mathcal{U}$ and a continuous function $\mu\colon\mathbb{R}^3\to(0,\infty)$ such that if $P_1^*\colon\mathcal{U}\to\mathcal{Y}_e$ satisfies the following inequality:

$$\delta(P^*(\theta,y_1^0),P_1^*)\le\mu(\|(u_0,y_0)^T\|_{L^2},|\theta|,|y_1^0|), \tag{3.13}$$

then $H_{P_1^*,C^*}(u_0,y_0)$ is bounded in L^2.

Furthermore the controller C^* is explicitly constructed as follows:

$$\begin{aligned}C^*(y_2)\ :\ u_2 &= -\hat{\theta}y_2-y_2\\ \hat{\theta}(t) &= \|\sqrt{\alpha}y_2\|_{L^2[0,t]}^{\frac{1}{2}}=\left(\int_0^t\alpha y_2^2\,dt\right)^{\frac{1}{4}}.\end{aligned} \tag{3.14}$$

Note that the above adaptive law (equation 3.14) is similar to the standard parametric adaptive law:

$$\dot{\hat{\theta}}=\alpha y_2^2,\qquad \hat{\theta}(0)=0 \tag{3.15}$$

which can be equivalently written as:

$$\hat{\theta}(t)=\|\sqrt{\alpha}y_2\|_{L^2[0,t]}^2=\int_0^t\alpha y_2^2\,dt. \tag{3.16}$$

It is also worthwhile to observe that the adaptive law 3.14 can be written in the equivalent differential form:

$$\dot{\hat{\theta}} = \alpha \frac{1}{4\hat{\theta}^3} y_2^2, \quad \hat{\theta}(0) = 0 \tag{3.17}$$

where the singularity at $\hat{\theta} = 0$ (eg. when $t = 0$) is non-essential.

The claim of Theorem 3.1 can be written in a more classical way, for we can say that for any $\|(u_0, y_0)\| \leq d$, there exists a non-zero robust stability margin $b_{P^*(\theta, y_1^0), C^*}$:

$$0 < b_{P^*(\theta, y_1^0), C^*} = \inf_{0 \leq d' \leq d} \mu(d', |\theta|, |y_1^0|), \tag{3.18}$$

in particular if θ and y_1^0 are only known to within some bounds $|\theta| \leq \Theta$, $|y_1^0| \leq \gamma$, we can guarantee:

$$0 < b_{P^*(\theta, y_1^0), C^*} = \inf_{0 \leq d' \leq d} \inf_{|t| \leq \Theta} \inf_{|y| \leq \gamma} \mu(d', t, y). \tag{3.19}$$

4 Properties of the Adaptive Controller

4.1 Well posedness

Let $\mathcal{U} = \mathcal{Y} = L^2$, and consider the controller C^* defined by:

$$\begin{aligned} C^*(y_2) \ : \ u_2 &= -\hat{\theta} y_2 - y_2 \\ \dot{\hat{\theta}} &= \alpha \frac{1}{4(\hat{\theta})^3} y_2^2, \quad \hat{\theta}(0) = 0 \end{aligned} \tag{4.20}$$

Proposition 4.1. *The feedback interconnection $[P^*(\theta, y_1^0), C^*]$ is well posed.*

Proof. We only demonstrate that $u_0, y_0 \in L^2$ implies $u_1, y_1 \in L^2$, since the formal check of existence and uniqueness of solutions is routine, and the corresponding properties for u_2, y_2 follow from the parallel projection properties. So, let $0 \leq t^* \leq \infty$ be defined:

$$t^* = \inf\{t \geq 0 \ : \ \hat{\theta}(t) = \theta\}. \tag{4.21}$$

if the infimum exists, and $t^* = \infty$ otherwise. Then:

$$\begin{aligned} \|y_1\|_{L^2[0,t^*]} &\leq \|y_2\|_{L^2[0,t^*]} + \|y_0\|_{L^2[0,t^*]} \leq \frac{1}{\sqrt{\alpha}} \hat{\theta}(t^*)^2 + \|y_0\|_{L^2[0,t^*]} \\ &= \frac{1}{\sqrt{\alpha}} \theta^2 + \|y_0\|_{L^2[0,t^*]} \leq \frac{1}{\sqrt{\alpha}} \theta^2 + \|y_0\|_{L^2[0,\infty)}. \end{aligned} \tag{4.22}$$

Now we bound $y_1(t^*)$. Define $V \colon \mathbb{R} \to \mathbb{R}_+$ by:

$$V(y_1) = \frac{1}{2} y_1^2. \tag{4.23}$$

Now,

$$\begin{aligned}
\dot V &= y_1 \dot y_1 \\
&= y_1(\theta y_1 + u_1) \\
&= y_1(\theta y_1 + u_0 - u_2) \\
&= y_1(\theta y_1 + u_0 + \hat\theta y_2 + y_2) \\
&= y_1(\theta y_1 + u_0 + \hat\theta(y_0 - y_1) + y_0 - y_1) \\
&= (\theta - \hat\theta - 1) y_1^2 + u_0 y_1 + y_0 y_1 + \hat\theta y_0 y_1 \qquad (4.24)
\end{aligned}$$

and applying Young's inequality ($ab - \frac{1}{4}b^2 \le a^2$) twice we obtain:

$$\begin{aligned}
\dot V &\le -\frac{1}{2} y_1^2 + (1+\hat\theta)^2 y_0^2 + u_0^2 + (\theta - \hat\theta) y_1^2 \\
&\le -\frac{1}{2} y_1^2 + 3(1+\hat\theta^2) y_0^2 + u_0^2 + (|\theta| + |\hat\theta|) y_1^2 \qquad (4.25)
\end{aligned}$$

Observing that $\hat\theta$ is non-negative and increasing, then by integrating, we obtain:

$$\begin{aligned}
V(y_1(t^*)) - V(y_1(0)) &= \int_0^{t^*} \dot V \, dt \\
&\le -\frac{1}{2} \|y_1\|^2_{L^2[0,t^*]} + 3(1+\hat\theta^2(t^*)) \|y_0\|^2_{L^2[0,t^*]} \\
&\quad + \|u_0\|^2_{L^2[0,t^*]} + (|\theta| + |\hat\theta(t^*)|) \|y_1\|^2_{L^2[0,t^*]} \\
&\le -\frac{1}{2} \|y_1\|^2_{L^2[0,t^*]} + 3(1+\theta^2) \|y_0\|^2_{L^2[0,t^*]} + \|u_0\|^2_{L^2[0,t^*]} + 2|\theta| \|y_1\|^2_{L^2[0,t^*]}, \qquad (4.26)
\end{aligned}$$

which implies:

$$\begin{aligned}
y_1^2(t^*) &\le 2V(0) - \frac{1}{2} \|y_1\|^2_{L^2[0,t^*]} + 6(1+|\theta|^2) \|y_0\|^2_{L^2[0,t^*]} + 2\|u_0\|^2_{L^2[0,t^*]} + 4|\theta| \|y_1\|^2_{L^2[0,t^*]}, \\
&\le (y_1^0)^2 + 6(1+|\theta|^2) \|y_0\|^2_{L^2[0,t^*]} + 2\|u_0\|^2_{L^2[0,t^*]} + 4|\theta| \left(\frac{1}{\sqrt{\alpha}} \theta^2 + \|y_0\|_{L^2[0,t^*]}\right)^2 \\
&\le (y_1^0)^2 + 6(1+|\theta|^2) \|y_0\|^2_{L^2[0,\infty)} + 2\|u_0\|^2_{L^2[0,\infty)} + 4|\theta| \left(\frac{1}{\sqrt{\alpha}} \theta^2 + \|y_0\|_{L^2[0,\infty)}\right)^2 . \qquad (4.27)
\end{aligned}$$

We now consider the L^2 estimates on $[t^*, \infty)$. Since $\hat\theta$ is increasing it follows that $\theta - \hat\theta \le 0$ for $t \ge t^*$, hence we can establish an inequality of the form:

$$\begin{aligned}
\dot V &\le -y_1^2 + u_0 y_1 + y_0 y_1 + \hat\theta y_0 y_1 \\
&\le -\frac{1}{2} y_1^2 + 3((1+\hat\theta^2)) y_0^2 + u_0^2. \qquad (4.28)
\end{aligned}$$

Integrating on $[t^*, t)$, we obtain:

$$V(y_1(t)) - V(y_1(t^*)) = \int_{t^*}^{t} \dot V \, dt \le -\frac{1}{2} \|y_1\|^2_{L^2[t^*,t)} + 3(1+\hat\theta^2(t)) \|y_0\|^2_{L^2[t^*,t)} + \|u_0\|^2_{L^2[t^*,t)} \quad (4.29)$$

which implies that $\forall t \geq t^*$,

$$\|y_1\|^2_{L^2[t^*,t)} \leq y_1^2(t^*) + 6(1+\hat{\theta}^2(t))\|y_0\|^2_{L^2[t^*,t)} + 2\|u_0\|^2_{L^2[t^*,t)}, \tag{4.30}$$

Now let us estimate $\hat{\theta}$ $\forall t \geq t^*$. From the definition of the adaptive law 3.14, we have:

$$\hat{\theta}^4(t^*) = \int_0^{t^*} \alpha y_2^2 \, dt, \quad \hat{\theta}^4(t) = \int_0^t \alpha y_2^2 \, dt \tag{4.31}$$

so,

$$\begin{aligned} \hat{\theta}^4(t) - \hat{\theta}^4(t^*) = \hat{\theta}^4(t) - \theta^4 &= \alpha\|y_2\|^2_{L^2[t^*,t)} \\ &\leq \alpha(\|y_1\|_{L^2[t^*,t)} + \|y_0\|_{L^2[t^*,t)})^2 \\ &\leq 3\alpha(\|y_1\|^2_{L^2[t^*,t)} + \|y_0\|^2_{L^2[t^*,t)}), \end{aligned} \tag{4.32}$$

and in particular by the inequality $(1+a)^{\frac{1}{2}} \leq 1 + \frac{a}{2}$ we obtain:

$$\begin{aligned} \hat{\theta}^2(t) &\leq \left(\theta^4 + 3\alpha(\|y_1\|^2_{L^2[t^*,t)} + \|y_0\|^2_{L^2[t^*,t)})\right)^{\frac{1}{2}}, \\ &\leq \sqrt{3\alpha}\|y_1\|_{L^2[t^*,t)}\left(1 + \frac{\theta^4 + 3\alpha\|y_0\|^2_{L^2[t^*,t)}}{6\alpha\|y_1\|^2_{L^2[t^*,t)}}\right). \end{aligned} \tag{4.33}$$

Substituting inequality 4.33 into inequality 4.30,

$$\begin{aligned} \|y_1\|^2_{L^2[t^*,t)} &\leq y_1^2(t^*) + 6\left(1 + \sqrt{3\alpha}\|y_1\|_{L^2[t^*,t)}\left(1 + \frac{\theta^4 + 3\alpha\|y_0\|^2_{L^2[t^*,t)}}{6\alpha\|y_1\|^2_{L^2[t^*,t)}}\right)\right)\|y_0\|^2_{L^2[t^*,t)} \\ &\quad +2\|u_0\|^2_{L^2[t^*,t)}, \end{aligned} \tag{4.34}$$

Rearranging and letting $t \to \infty$:

$$\begin{aligned} \|y_1\|^3_{L^2[t^*,\infty)} &\leq (6\sqrt{3\alpha}\|y_0\|_{L^2[t^*,\infty)})\|y_1\|^2_{L^2[t^*,\infty)} \\ &\quad + \left(y_1^2(t^*) + 6\|y_0\|^2_{L^2[t^*,\infty)} + 2\|u_0\|^2_{L^2[t^*,\infty)}\right)\|y_1\|_{L^2[t^*,\infty)} \\ &\quad + \sqrt{\frac{3}{\alpha}}\left(\theta^4 + 3\alpha\|y_0\|^2_{L^2[t^*,\infty)}\right)\|y_0\|^2_{L^2[0,t^*)} \\ &\leq (6\sqrt{3\alpha}\|y_0\|_{L^2[0,\infty)})\|y_1\|^2_{L^2[t^*,\infty)} \\ &\quad + \left(y_1^2(t^*) + 6\|y_0\|^2_{L^2[0,\infty)} + 2\|u_0\|^2_{L^2[0,\infty)}\right)\|y_1\|_{L^2[t^*,\infty)} \\ &\quad + \sqrt{\frac{3}{\alpha}}\left(\theta^4 + 3\alpha\|y_0\|^2_{L^2[0,\infty)}\right)\|y_0\|^2_{L^2[0,\infty)} \end{aligned} \tag{4.35}$$

Since the r.h.s of inequality 4.35 is quadratic in $\|y_1\|_{L^2[t^*,\infty)}$ with positive coefficients, it follows that $\|y_1\|_{L^2[t^*,\infty)}$ is bounded as a function of $|y_1(t^*)|$, $|\theta|$, $\|y_0\|_{L^2[0,\infty)}$, $\|u_0\|_{L^2[0,\infty)}$. Furthermore, the cubic inequality 4.35 can be solved explicitly to give this bound (see later).

Since we have bounded $\|y_1\|_{L^2[0,t^*)}$ in terms of $|\theta|$, $\|y_0\|_{L^2[0,\infty)}$, and $|y_1(t^*)|$ in terms of $|y_1^0|$, $|\theta|$, $\|y_0\|_{L^2[0,\infty)}$, $\|u_0\|_{L^2[0,\infty)}$ it follows that we have bounded $\|y_1\|_{L^2[0,\infty)}$ in terms of:

$$|y_1^0|,\ |\theta|,\ \|y_0\|_{L^2[0,\infty)},\ \|u_0\|_{L^2[0,\infty)}, \tag{4.36}$$

as required. A similar bound for $\|u_1\|_{L^2[0,\infty)}$ can now also be found, since:

$$\begin{aligned}
\|u_1\|_{L^2[0,\infty)} &= \|u_0 - u_2\|_{L^2[0,\infty)} \\
&\leq \|u_0\|_{L^2[0,\infty)} + \| - \hat{\theta}(y_0 - y_1) - (y_0 - y_1)\|_{L^2[0,\infty)} \\
&\leq \|u_0\|_{L^2[0,\infty)} + \|\hat{\theta}\|_{L^\infty[0,\infty)}(\|y_0\|_{L^2[0,\infty)} + \|y_1\|_{L^2[0,\infty)}) + \|y_0\|_{L^2[0,\infty)} + \|y_1\|_{L^2[0,\infty)} \\
&\leq \|u_0\|_{L^2[0,\infty)} + \alpha^{\frac{1}{4}}(\|y_0\|_{L^2[0,\infty)} + \|y_1\|_{L^2[0,\infty)})^{\frac{3}{2}} + \|y_0\|_{L^2[0,\infty)} + \|y_1\|_{L^2[0,\infty)}. \qquad (4.37)
\end{aligned}$$

Hence it follows that $\|u_1\|_{L^2[0,\infty)}$ is bounded as a function of

$$|y_1^0|,\ |\theta|,\ \|y_0\|_{L^2[0,\infty)},\ \|u_0\|_{L^2[0,\infty)}. \tag{4.38}$$

This completes the proof. □

4.2 Incorporation of initial conditions and other parameterisations

The first observation is that for $\theta > 0$, $H_{P^*(\theta,0),C^*}$ is not stable in the sense of equation 2.3, simply because

$$(u_0, y_0) \approx 0 \quad \not\Rightarrow \quad H_{P^*(\theta,0),C^*}(u_0, y_0) \not\approx 0. \tag{4.39}$$

This is a generic problem for closed loops with non-zero responses to zero disturbances or non-continuous behaviour at this point. This arises in a variety of situations; some examples are:

- Adaptive controllers when applied to unstable plants.
- Memoryless feedback designs such as example 5 in [3], when applied to systems with non-zero initial conditions.

Secondly, since the controller C^* is itself highly nonlinear, it seems unlikely that an analysis based on linear gains will be applicable. Hence we take our notion of stability to be that of the existence of a (nonlinear) gain function. Finally note that the adaptive problem concerns the analysis of a controller on a parameterised set of nominal plants (ie. by the uncertain parameter θ, and also typically the initial condition y_1^0). However the standard gap framework applies to a single fixed nominal plant P. The approach taken in this paper is to view the uncertain parameters themselves as inputs to the plant. This has the effect of replacing a linear plant by a nonlinear plant with extra input channels, but has the important advantage of needing only to study a single nominal plant.

In general, suppose the nominal plant is parameterised by $p \in \Pi$ for some appropriate choice of Euclidean space Π. We then augment the $\mathcal{U}$ disturbance channel to:

$$\mathcal{U} := \tilde{\Pi} \times L^2 \tag{4.40}$$

where $\tilde{\Pi}$ denotes the set of constant maps $\mathbb{R}_+ \to \Pi$ ie.

$$\tilde{\Pi} = \{f \colon \mathbb{R}_+ \to \Pi \mid \exists p \in \Pi \text{ s.t. } f(t) = p \; \forall t \in \mathbb{R}_+\}. \tag{4.41}$$

Since Π and $\tilde{\Pi}$ are naturally isometrically isomorphic, henceforth we always implicitly make the natural identifications between Π and $\tilde{\Pi}$ and also write Π for $\tilde{\Pi}$.

The plant and controller equations will then be redefined appropriately with respect to the new domains and co-domains (see below). In particular the controller equations are chosen to assign 0 to the Π channel, to ensure the nonlinear projection properties of the parallel projection hold. The framework of [3] then applies directly. This idea allows us to consider system responses to non-zero initial conditions $\Pi = \mathbb{R}^n$, where n is the dimension of the state space of the plant P, and to parameter variations in the plant eg. $\Pi = \mathbb{R}^p$ where p is the dimension of the parameter space.

4.3 The closed loop is gf-stable

Now we return to the concrete example. Define the signal spaces as follows:

$$\begin{aligned} u_0 = (\theta, y_1^0, u_0^*)^T \;&\in\; \mathcal{U} := \mathbb{R}^2 \times L^2 \\ y_0 \;&\in\; \mathcal{Y} := L^2, \end{aligned} \tag{4.42}$$

where the $\mathcal{U}$ norm is taken to be

$$\|(\theta, y_1^0, u_0^*)^T\| = |\theta| + |y_1^0| + \|u_0^*\|_{L^2}. \tag{4.43}$$

We define the plant as:

$$\begin{aligned} P \;&:\; \mathbb{R}^2 \times L^2 \to L^{2,e} \\ P(\theta, y_1^0, u_1^*) = y_1 \quad : \quad \dot{y}_1 \;&=\; \theta y_1 + u_1^*, \quad y_1(0) = y_1^0, \end{aligned} \tag{4.44}$$

where note that P is *not* a linear operator. The controller is defined formally as:

$$\begin{aligned} C \;&:\; L^2 \to \mathbb{R}^2 \times L^2 \\ C(y_2) = (0, 0, u_2^*) \quad : \quad u_2^* \;&=\; -\hat{\theta} y_2 - y_2, \\ \dot{\hat{\theta}} \;&=\; \alpha \frac{1}{4\hat{\theta}^3} y_2^2, \quad \hat{\theta}(0) = 0 \end{aligned} \tag{4.45}$$

Note that

$$P(\theta, y_1^0, u_1^*) = P^*(\theta, y_1^0)(u_1^*), \tag{4.46}$$

and

$$C(y_2) = (0, 0, C^*(y_2)). \tag{4.47}$$

We now come to the key result:

Proposition 4.2. *$H_{P,C}$ is gain function stable, furthermore, a bound γ on the $\Pi_{\mathcal{M}//\mathcal{N}}$ gain function $\gamma^*\colon \mathbb{R} \to [0,\infty)$ can be taken to be continuous with $\gamma(0) = 0$ and γ non-negative on $(0,\infty)$.*

Proof. The gain function γ^* is defined:

$$\gamma^*(r) = \sup\{\|\Pi_{\mathcal{M}//\mathcal{N}} x\| \ : \ \|x\| \leq r\} \tag{4.48}$$

where

$$x = (u_0, y_0) = (\theta, y_1^0, u_0^*, y_0)^T \in \mathcal{W} = \mathbb{R}^2 \times L^2 \times L^2, \tag{4.49}$$

so,

$$\gamma^*(r) = \sup\{\|\Pi_{\mathcal{M}//\mathcal{N}}(\theta, y_1^0, u_0^*, y_0)\| : u_0^*, y_0 \in L^2,\ \theta, y_1^0 \in \mathbb{R},\ \theta^2 + (y_1^0)^2 + \|u_0^*\|_{L^2}^2 + \|y_0\|_{L^2}^2 \leq r^2\}. \tag{4.50}$$

To establish gf-stability, we consider the bounds 4.27, 4.35 of Proposition 4.1 to obtain:

$$\begin{aligned}\|y_1\|^3_{L^2[t^*,\infty)} \ \leq\ & (6\sqrt{3\alpha}\|y_0\|_{L^2[0,\infty)})\|y_1\|^2_{L^2[t^*,\infty)} + \\ & \Big((y_1^0)^2 + 6(1+|\theta|^2)\|y_0\|^2_{L^2[0,\infty)} + 2\|u_0\|^2_{L^2[0,\infty)} \\ & +4|\theta|(\frac{1}{\sqrt{\alpha}}\theta^2 + \|y_0\|_{L^2[0,\infty)})^2 + 6\|y_0\|^2_{L^2[0,\infty)} + 2\|u_0\|^2_{L^2[0,\infty)}\Big)\|y_1\|_{L^2[t^*,\infty)} \\ & +\sqrt{\frac{3}{\alpha}}\left(\theta^4 + 3\alpha\|y_0\|^2_{L^2[0,\infty)}\right)\|y_0\|^2_{L^2[0,\infty)}\end{aligned} \tag{4.51}$$

which yields the cubic inequality:

$$\begin{aligned}&\|y_1\|^3_{L^2[t^*,\infty)} - 6\sqrt{3\alpha}r\|y_1\|^2_{L^2[t^*,\infty)} - \\ &\quad \left(\frac{4}{\alpha}r^5 + (\frac{8}{\sqrt{\alpha}} + 6)r^4 + 4r^3 + 17r^2\right)\|y_1\|_{L^2[t^*,\infty)} - \sqrt{\frac{3}{\alpha}}(r^6 + 3\alpha r^4) \leq 0.\end{aligned} \tag{4.52}$$

At equality, the above equation has a positive root, since the cubic coefficient is positive and the other coefficients are negative. The minimal positive root is then clearly a bound on $\|y_1\|_{L^2[t^*,\infty)}$. Since roots of polynomial equations depend continuously on their coefficients, there is a continuous function $\lambda\colon [0,\infty) \to [0,\infty)$ for which

$$\|y_1\|_{L^2[t^*,\infty)} \leq \lambda(r).^2 \tag{4.53}$$

Since the polynomial 4.52 has all its roots at 0 when $r = 0$ we can take $\lambda(0) = 0$.
Inequality 4.22 implies:

$$\|y_1\|_{L^2[0,t^*]} \leq \frac{1}{\sqrt{\alpha}}r^2 + r. \tag{4.54}$$

Hence,

$$\|y_1\|_{L^2[0,\infty]} = \sqrt{\|y_1\|^2_{L^2[0,t^*]} + \|y_1\|^2_{L^2[t^*,\infty)}} \leq \sqrt{\frac{1}{\alpha}r^4 + \frac{2}{\sqrt{\alpha}}r^3 + r^2 + \lambda^2(r)} := \zeta(r). \tag{4.55}$$

[2] Note that as the inequality is cubic, this function could be computed explicitly.

Likewise, inequality 4.37 yields:

$$\|u_1\|_{L^2[0,\infty)} = \left(\theta^2 + (y_1^0)^2 + \|u_1^*\|^2_{L^2[0,\infty)}\right)^{\frac{1}{2}} \leq \left(r^2 + \left(2r + \alpha^{\frac{1}{4}}(r+\zeta(r))^{\frac{3}{2}} + \zeta(r)\right)^2\right)^{\frac{1}{2}}. \tag{4.56}$$

This establishes gain function stability with γ taken explicitly as:

$$\gamma(r) = \left(\frac{1}{\alpha}r^4 + \frac{2}{\sqrt{\alpha}}r^3 + 2r^2 + \lambda^2(r) + \left(2r + \alpha^{\frac{1}{4}}(r+\zeta(r))^{\frac{3}{2}} + \zeta(r)\right)^2\right)^{\frac{1}{2}}. \tag{4.57}$$

Now, it is easy to observe that γ is continuous, γ is non-negative on $(0,\infty)$ and $\gamma(0) = 0$, hence completing the proof. □

4.4 The idea behind the proof of the main result

The main result follows from the following theorem, which is a refinement of Theorem 6 of [3]. The only difference between the result below and Theorem 6 of [3] is that we allow Φ to be a map onto a subset of $\mathcal{M}_1$.

Theorem 4.1. *Let $H_{P,C}$ be gf-stable. If there exists a surjective mapping $\Phi\colon \mathcal{D} \to \mathcal{D}_1$, where $\mathcal{D} \subset \mathcal{M}$ and $\mathcal{D}_1 \subset \mathcal{M}_1$, and if there exists a function $\epsilon(\cdot) \in \mathcal{K}_\infty$ such that*

$$g(I-\Phi) \circ g(\Pi_{\mathcal{M}//\mathcal{N}})(\alpha) \leq (1+\epsilon)^{-1}(\alpha) \tag{4.58}$$

for all $\alpha \geq 0$, then

$$H_{P_1,C}\colon \Sigma_{\mathcal{D}_1,\mathcal{N}}(\mathcal{D}_1,\mathcal{N}) \to \mathcal{D}_1 \times \mathcal{N} \tag{4.59}$$

is gf-stable and

$$g(\Pi_{\mathcal{D}_1//\mathcal{N}})(\alpha) \leq g(\Phi) \circ g(\Pi_{\mathcal{M}//\mathcal{N}}) \circ (1+\epsilon^{-1})(\alpha). \tag{4.60}$$

Proof. The proof is essentially the same as Theorem 6 of [3]. □

We now state the critical result which replaces a gap constraint on the augmented (nonlinear) plant P with gap constraints on the original (linear) plant P^*.

Theorem 4.2. *Let $\mathcal{U} = \mathcal{Y} = L^2$ and suppose $P^*(\theta, y_1^0)\colon \mathcal{U} \to \mathcal{Y}_e$ is a system, parameterised by $\theta \in \mathbb{R}^p$ and with initial condition $y_1^0 \in \mathbb{R}^n$. Let P, C be defined:*

$$\begin{array}{lrcl} C\colon \mathcal{Y} \to \mathbb{R}^{p+n} \times \mathcal{U}, & C(y_2) & = & (0,0,C^*(y_2)) \\ P\colon \mathbb{R}^{p+n} \times \mathcal{U} \to \mathcal{Y}_e, & P(\theta, y_1^0, u_1) & = & P^*(\theta, y_1^0)(u_1). \end{array} \tag{4.61}$$

Suppose $[P, C]$ is gf-stable. Then there exists a continuous function $\mu\colon \mathbb{R}^3 \to (0,\infty)$ such that if $P_1^*\colon \mathcal{U} \to \mathcal{Y}_e$ satisfies the following inequality:

$$\delta_{\mathcal{F}}(P^*(\theta, y_1^0), P_1^*) \leq \mu(\|(u_0, y_0)^T\|_{L^2}, |\theta|, |y_1^0|), \tag{4.62}$$

then $H_{P_1^*,C^*}(u_0, y_0)$ is bounded in L^2.

Proof. The proof can be found in [2] □

The main first order result, Theorem 3.1 now follows directly from Proposition 4.2 and Theorem 4.2.

5 Gap Robustness of an Adaptive Controller for Relative Degree one plants

In this section we extend the previous results from the first order plant to linear plants $P^*(a,b)\colon \mathcal{U} \to \mathcal{Y}$,

$$y_1 = P^*(a,b)(u_1) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} u_1, \tag{5.63}$$

which satisfy the classical assumptions of adaptive control namely:

1. The order of the plant (n) is known.
2. The relative degree ($\rho = n - m$) of the plant is one, $\rho = 1$.
3. The high frequency gain is unity. (ie. $b_0 = 1$).
4. The plant is minimum phase (ie. the polynomial $b_0 s^m + b_1 s^{m-1} + \cdots + b_m$ is Hurwitz).

The relative degree one result is then as follows:

Theorem 5.1. *Let $\mathcal{U} = \mathcal{Y} = L^2$, and let $P^*(a,b)\colon \mathcal{U} \to \mathcal{Y}_e$ be the plant defined by equation 5.63, which is assumed to be minimum phase, relative degree 1 and have unity high frequency gain. Then there exists a universal controller $C^*\colon \mathcal{Y} \to \mathcal{U}$ and a continuous function $\mu\colon \mathbb{R}^{1+n+m} \to (0,\infty)$ such that if $P_1^*\colon \mathcal{U} \to \mathcal{Y}_e$ satisfies the following inequality:*

$$\delta(P^*(a,b), P_1^*) \leq \mu(\|(u_0, y_0)^T\|_{L^2}, a, b), \tag{5.64}$$

then $H_{P_1^*,C^*}(u_0, y_0)$ is bounded.

Proof. See [2]. □

6 Conclusions

In this paper we have approached the classical problem of robustness of adaptive controllers to unmodelled dynamics within the framework of the nonlinear gap metric. The main idea of the approach is threefold:

- To augment the input signal space of parametrically uncertain linear plants with a channel representing the uncertain parameters.
- To synthesise the controller to ensure the existence of a certain closed loop gain function, and hence to give rise to a nonlinear gap margin.
- To relate the nonlinear gap margin to the (linear) gap margins on the individual plants.

For both first order plants and plants of relative degree one, adaptive controllers with gap robustness margins are constructed. Whilst the results have been presented in a qualitative manner, the proofs are fully constructive, and in principle can be used to explicitly compute the gap margins. The paper [2] contains many other related results.

Acknowledgement: I would like to thank M.C. Smith for his encouragement of this work. The research was supported in part by the EPSRC.

References

[1] A.K. El-Sakkary. The gap metric: Robustness of stabilization of feedback systems. *IEEE Trans. on Automatic Control*, 30(3):240–247, 1985.

[2] M. French. Adaptive control and robustness in the gap metric. 2002. In preparation.

[3] T.T. Georgiou and M.C. Smith. Robustness analysis of nonlinear feedback systems. *IEEE Trans. on Automatic Control*, 42(9):1200–1229, 1997.

[4] P.A. Ioannou and J. Sun. *Robust Adaptive Control*. Prentice Hall, 1st edition, 1996.

[5] C.E. Rohrs, L. Valavani, M. Athans, and G. Stein. Robustness of continuous-time adaptive control algorithms in the prescence of unmodelled dynamics. *IEEE Trans. on Automatic Control*, 30(9):881–889, 1985.

[6] G. Zames and A.K. El-Sakkary. Unstable systems and feedback: The gap metric. In *Proc. Allerton Conf.*, pages 380–385, October 1980.