Interconnection structures in physical systems: a mathematical formulation

Goran Golo, Orest V. Iftime, Abraham J. van der Schaft
Cornelis J. Drebbel Institute for Mechatronics
Department of Applied Mathematics, University of Twente
P.O. Box 217, 7500 AE Enschede, The Netherlands

Abstract

The power-conserving structure of a physical system is known as interconnection structure. This paper presents a mathematical formulation of the interconnection structure in Hilbert spaces. Some properties of interconnection structures are pointed out and their three natural representations are treated. The developed theory is illustrated on two examples: electrical circuit and one-dimensional transmission line.

1 Introduction

Most of the current modelling and simulation approaches to (complex) physical systems are based on some sort of *network representation*. The physical system under consideration is seen as the interconnection of a number of subsystems possibly from different domains (mechanical, electrical, and so on). This way of modelling has several advantages. One of them is that the knowledge about subsystems can be stored in libraries, and is reusable for later occasions. Also, due to this modularity, the modelling process can be performed in an *iterative* way, gradually refining the model -if necessary- by adding other subsystems. Further, the approach is suited to general control design where the overall behaviour of the system is sought to be improved by the addition of other subsystems or controlling devices.

In this paper we concentrate on the mathematical description of *power-conserving part* of a network representation of a physical system called *interconnection structure*. The relevance of interconnection structures in analysis of network models is enormous. It is used for the structural analysis of networks models [1, 2] and for the derivation of simulation model [3]. The proper treatment of interconnection structure is essential for the spatial discretisation of a class of physical systems described by partial differential equations [4].

Our starting assumption is that an interconnection structure is a Dirac structure[1]. This approach was initiated in [9, 10]. In these papers the authors show the relevance of Dirac structures in descriptions of *LC*-circuits with dependent storage elements [9] and how Dirac structures can be used in the description of kinematic structures of mechanisms [10]. These ideal are further elaborated in [11, 12, 13, 14]. The concept of Dirac structures (slightly

[1] The notation of Dirac strictures was introduced by Courant and Weinstein [5] and furthermore investigate by Courant in [6] as a generalisation of Poisson and (pre)-symplectic structures. Dorfman [7, 8] developed an algebraic theory of Dirac structures in the context of the study of completely integrable systems of partial differential equations.

modified) is essential for the description of distributed parameter systems with nonzero energy flow, as shown in [15]. This idea is extended in [16].

In this paper we concentrate on Dirac structures defined on real Hilbert spaces. The reasons why we study Dirac structures on Hilbert spaces is twofold. The Hilbert spaces are generally enough to cover a large class of physical systems and they offer enough tools for analysing such systems. The paper is organised as follows. In Section 2 definition of Dirac structure is recalled and the notation used through the paper is introduced. Dirac structure on Hilbert spaces are introduced in Section 3. Representations of Dirac structures (kernel, image and scattering) are discussed in Section 4. The developed theory is illustrated on two examples (electrical circuit and one-dimensional transmission line) in Section 5.

2 Preliminaries

Let $\mathcal{F}$, $\mathcal{E}$ be real vector spaces whose elements are labelled by f, e, respectively. We call $\mathcal{F}$ the space of *flows*, and $\mathcal{E}$ the space of *efforts*. A pair $p = (f, e)$ is called *port* and the set of all possible values of the port p is the real vector space $\mathcal{P} = \mathcal{F} \times \mathcal{E}$. We assume that the variables f, e are conjugate, i.e. there exists a *scalar product*

$$\langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}.$$

The scalar product satisfies the following two conditions:

(i) it is a linear function of each argument,

(ii) it is non-degenerate.

Using the scalar product one can define a symmetric *bilinear form* $\ll, \gg : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$

$$\ll \left(f^1, e^1\right), \left(f^2, e^2\right) \gg := \left\langle e^1 | f^2 \right\rangle + \left\langle e^2 | f^1 \right\rangle, \ \forall \left(f^1, e^1\right), \left(f^2, e^2\right) \in \mathcal{P}.$$

Observe that a scalar product and the corresponding bilinear form are related as

$$\langle e | f \rangle = \frac{1}{2} \ll p, p \gg, \forall p = (f, e) \in \mathcal{P}.$$

First, a *Telegen structure* on $\mathcal{P}$ is defined.

Definition 2.1 (Telegen structure).
Let $\mathcal{Z}$ be a subspace of the vector space $\mathcal{P} = \mathcal{F} \times \mathcal{E}$. We say that $\mathcal{Z}$ is a Telegen structure on $\mathcal{P}$ if

$$\langle e | f \rangle = 0$$

for every $(f, e) \in \mathcal{Z}$.

The term "$\mathcal{Z}$ is a subspace of $\mathcal{P}$" means that $\mathcal{Z}$ is a vector space in its own right under the operations obtained by restricting the operations of $\mathcal{P}$ to $\mathcal{Z}$.

The orthogonal complement of $\mathcal{Z}$ in $\mathcal{P}$, denoted with $\mathcal{Z}^{\perp}$ is defined by

$$\mathcal{Z}^{\perp} = \{p \in \mathcal{P} : \ll p, \tilde{p} \gg = 0, \forall \tilde{p} \in \mathcal{Z}\}.$$

Proposition 2.1 (Telegen structures).
Let $\mathcal{Z}$ be a subspace of $\mathcal{P}$. Then $\mathcal{Z}$ is a Telegen structure on $\mathcal{P}$ if and only if

$$\mathcal{Z} \subseteq \mathcal{Z}^{\perp}.$$

Proof:
Necessity: Suppose that $\mathcal{Z}$ is a Telegen structure on $\mathcal{P}$, let $p^1 = (f^1, e^1)$, $p^2 = (f^2, e^2)$ be any two elements of $\mathcal{Z}$. It means that $\langle e^1 | f^1 \rangle = 0$ and that $\langle e^2 | f^2 \rangle = 0$. Since $\mathcal{Z}$ is a subspace of $\mathcal{P}$, then $p^1 + p^2$ also belongs to $\mathcal{Z}$. Hence

$$\begin{aligned}
&\langle e^1 + e^2 | f^1 + f^2 \rangle = 0 \\
&\Rightarrow \langle e^1 | f^1 \rangle + \langle e^1 | f^2 \rangle + \langle e^2 | f^1 \rangle + \langle e^2 | f^2 \rangle = 0 \\
&\Rightarrow \langle e^1 | f^2 \rangle + \langle e^2 | f^1 \rangle = 0.
\end{aligned}$$

Therefore p^1 is orthogonal to p^2 with respect to $\ll, \gg$ and thus $\mathcal{Z} \subseteq \mathcal{Z}^{\perp}$.
Sufficiency: Take any element $p = (f, e)$ that belongs to $\mathcal{Z}$. Since $\mathcal{Z}$ is a subspace such that $\mathcal{Z} \subseteq \mathcal{Z}^{\perp}$ then

$$\langle e | f \rangle = \frac{1}{2} \ll p, p \gg = 0.$$

Therefore $\mathcal{Z}$ is a Telegen structure on $\mathcal{P}$. ■
From now on we concentrate on a special class of Telegen structures called *Dirac structures*.

Definition 2.2 (Dirac structures).
Let $\mathcal{D}$ be a subset on $\mathcal{P}$. We say that $\mathcal{D}$ is a Dirac structure on $\mathcal{P}$ if $\mathcal{D} = \mathcal{D}^{\perp}$.

Remark 2.1.
It is obvious that every Dirac structure is a subspace of $\mathcal{P}$. Thus every Dirac structure is a Telegen structure by virtue of Proposition 2.1.

3 Dirac structures on real Hilbert spaces

In this section we study Dirac structures on real Hilbert spaces (in the sequel the word real is omitted). First we provide a basic set-up for the definition of Dirac structures on Hilbert spaces and then some properties of the Dirac structures are investigated.

Suppose that $\mathcal{F}$, $\mathcal{E}$ are Hilbert spaces whose inner products are $\langle \cdot, \cdot \rangle_{\mathcal{F}}$, $\langle \cdot, \cdot \rangle_{\mathcal{E}}$, respectively. We make throughout the following assumption.

Assumption 3.1 (Relation between the flow space and the effort space).
$\mathcal{F}$, $\mathcal{E}$ are isometrically isomorphic.

Assumption 3.1 entails the existence of a bijective isometry $r_{\mathcal{F},\mathcal{E}} : \mathcal{F} \to \mathcal{E}$. That is, $r_{\mathcal{F},\mathcal{E}}$ is an invertible linear transformation satisfying

$$\left\langle r_{\mathcal{F},\mathcal{E}} f^1, r_{\mathcal{F},\mathcal{E}} f^2 \right\rangle_{\mathcal{E}} = \left\langle f^1, f^2 \right\rangle_{\mathcal{F}}, \ \forall f^1, f^2 \in \mathcal{F}.$$

Let $r_{\mathcal{E},\mathcal{F}} : \mathcal{E} \to \mathcal{F}$ be the inverse of $r_{\mathcal{F},\mathcal{E}}$. Substitution of $r_{\mathcal{F},\mathcal{E}} f^1$ by e^1 and $r_{\mathcal{F},\mathcal{E}} f^2$ by e^2 gives

$$\left\langle r_{\mathcal{E},\mathcal{F}} e^1, r_{\mathcal{E},\mathcal{F}} e^2 \right\rangle_{\mathcal{F}} = \left\langle e^1, e^2 \right\rangle_{\mathcal{E}}, \ \forall e^1, e^2 \in \mathcal{E}.$$

We can now introduce a non-degenerate map $\langle \cdot | \cdot \rangle : \mathcal{F} \times \mathcal{E} \to \mathbb{R}$ as follows

$$\langle e | f \rangle := \langle f, r_{\mathcal{E},\mathcal{F}} e \rangle_{\mathcal{F}}, \ \forall f \in \mathcal{F}, \ \forall e \in \mathcal{E}. \tag{3.1a}$$

Since $r_{\mathcal{F},\mathcal{E}}$ is an isometry, we have that this pairing may be also represented as

$$\langle e | f \rangle = \langle e, r_{\mathcal{F},\mathcal{E}} f \rangle_{\mathcal{E}}, \forall f \in \mathcal{F}, \ \forall e \in \mathcal{E}. \tag{3.1b}$$

On the other hand, the space of port values $\mathcal{P} = \mathcal{F} \times \mathcal{E}$ is a Hilbert space whose inner product is given by

$$\left\langle p^1, p^2 \right\rangle_{\mathcal{P}} := \left\langle f^1, f^2 \right\rangle_{\mathcal{F}} + \left\langle e^1, e^2 \right\rangle_{\mathcal{E}}, \ \forall p^1, p^2 \in \mathcal{P},$$

where $p^1 = (f^1, e^1)$ and $p^2 = (f^2, e^2)$. The bilinear form is related to the inner product as

$$\ll p^1, p^2 \gg = \langle p^1, R p^2 \rangle_{\mathcal{P}}, \ \forall p^1, p^2 \in \mathcal{P},$$

where the linear operator $R : \mathcal{P} \to \mathcal{P}$ is defined by

$$R = \begin{bmatrix} 0 & r_{\mathcal{E},\mathcal{F}} \\ r_{\mathcal{F},\mathcal{E}} & 0 \end{bmatrix}.$$

Since $r_{\mathcal{F},\mathcal{E}}$ is an isometry, R is a bounded linear operator satisfying $RR = I_{\mathcal{P}}$ ($I_{\mathcal{P}}$ is the identity operator on $\mathcal{P}$). It is clear that R is also an invertible operator and its inverse is $R^{-1} = R$. Now, we concentrate on Telegen structures defined on Hilbert spaces. The closure of a subset $\mathcal{Z}$ with respect to the induced norm is denoted by $\mathrm{cl}(\mathcal{Z})$.

Proposition 3.1 (Closure of Telegen structure).
Let $\mathcal{Z}$ be a Telegen structure on $\mathcal{P}$. Then $\mathrm{cl}(\mathcal{Z})$ is a Telegen structure.

Proof:
Take a sequence $\{p_k\} = \{(f_k, e_k)\}$ of elements in $\mathcal{Z}$ that converge to $p = (f, e)$. We have that

$$\langle e | f \rangle = \frac{1}{2} \ll p, p \gg = \frac{1}{2} \langle p, Rp \rangle_{\mathcal{P}}.$$

From the continuity of $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ we obtain

$$\frac{1}{2}\langle p, Rp\rangle = \frac{1}{2}\left\langle \lim_{k\to\infty} p_k, R\lim_{k\to\infty} p_k\right\rangle = \frac{1}{2}\lim_{k\to\infty} \ll p_k, p_k \gg = 0.$$

This implies $\langle e|f\rangle = 0$. Therefore, $\mathrm{cl}(\mathcal{Z})$ is a Telegen structure on $\mathcal{P}$. ■

Let $\mathcal{Z}^{\mathrm{c}}$ denotes the orthogonal complement of $\mathcal{Z}$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{P}}$, i.e.,

$$\mathcal{Z}^{\mathrm{c}} = \{p \in \mathcal{P} : \langle p, \tilde{p}\rangle_{\mathcal{P}} = 0, \forall \tilde{p} \in \mathcal{Z}\}.$$

A before, we denote by $\mathcal{Z}^{\perp}$ the subset defined by

$$\mathcal{Z}^{\perp} = \{p \in \mathcal{P} :\ll p, \tilde{p} \gg = 0, \forall \tilde{p} \in \mathcal{Z}\}.$$

Proposition 3.2 (Relation between $\mathcal{Z}^{\mathrm{c}}$ and $\mathcal{Z}^{\perp}$).
Let $\mathcal{Z}$ be a subspace on $\mathcal{P}$. Then

$$\mathcal{Z}^{\perp} = R\mathcal{Z}^{\mathrm{c}}.$$

Proof:
This is a direct consequence of relation between $\ll\cdot,\cdot\gg$ and $\langle\cdot,\cdot\rangle_{\mathcal{P}}$:

$$\ll \mathcal{Z}, \mathcal{Z}^{\perp} \gg = 0 \Rightarrow \langle \mathcal{Z}, R\mathcal{Z}^{\perp}\rangle = 0 \Rightarrow R\mathcal{Z}^{\perp} \subseteq \mathcal{Z}^{\mathrm{c}},$$
$$\langle \mathcal{Z}, \mathcal{Z}^{\mathrm{c}}\rangle = 0 \Rightarrow \ll \mathcal{Z}, R\mathcal{Z}^{\mathrm{c}} \gg = 0 \Rightarrow R\mathcal{Z}^{\mathrm{c}} \subseteq \mathcal{Z}^{\perp} \Rightarrow \mathcal{Z}^{\mathrm{c}} \subseteq R\mathcal{Z}^{\perp}.$$

Consequently, $\mathcal{Z}^{\mathrm{c}} = R\mathcal{Z}^{\perp}$ and thus $\mathcal{Z}^{\perp} = R\mathcal{Z}^{\mathrm{c}}$. ■

Remark 3.1.
Since $\mathcal{Z}^{\mathrm{c}}$ is a closed subspace and R is a bounded operator then $\mathcal{Z}^{\perp}$ is a closed subspace.

Let the symbol $\boxplus$ stand for the orthogonal direct sum with respect to $\langle\cdot,\cdot\rangle_{\mathcal{P}}$. Thus, $\mathcal{Z}_1 \boxplus \mathcal{Z}_2 = \mathcal{P}$ means that $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \mathcal{P}$ and that $\mathcal{Z}_1$ is orthogonal to $\mathcal{Z}_2$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{P}}$.

Proposition 3.3 (Dirac structures).
Let $\mathcal{D}$ be a subspace on $\mathcal{P}$. The following statements are equivalent:

(i) $\mathcal{D}$ *is a Dirac structure on* $\mathcal{P}$.

(ii) $\mathcal{D}^{\mathrm{c}} = R\mathcal{D}$.

(iii) $\mathcal{D} \boxplus R\mathcal{D} = \mathcal{P}$.

(iv) $\mathcal{D}^{\mathrm{c}} \boxplus R\mathcal{D}^{\mathrm{c}} = \mathcal{P}$ *and* $\mathcal{D}$ *is a closed subspace of* $\mathcal{P}$.

Proof:
(**i**) $\Rightarrow$ (**ii**) Suppose that $\mathcal{D}$ is a Dirac structure. Then $\mathcal{D}$ is a subspace of $\mathcal{P}$ (Remark 2.1) and $\mathcal{D}^{\perp} = R\mathcal{D}^{\mathrm{c}}$ (Proposition 3.2). Since $\mathcal{D} = \mathcal{D}^{\perp}$ then the condition (*ii*) is satisfied.
(**ii**) $\Rightarrow$ (**iii**) Suppose that $\mathcal{D}^{\mathrm{c}} = R\mathcal{D}$. This implies that $\mathcal{D}$ is a closed subspace. Therefore $\mathcal{D} \boxplus \mathcal{D}^{\mathrm{c}} = \mathcal{P}$, which means that condition (*iii*) is satisfied.
(**iii**) $\Rightarrow$ (**iv**) Suppose that the condition (*iii*) is satisfied. Then $\mathcal{D}$ is a closed subspace and $\mathcal{D}^{\mathrm{c}} = R\mathcal{D}$. Consequently, $\mathcal{D} = R\mathcal{D}^{\mathrm{c}}$. Hence

$$\mathcal{D} \boxplus R\mathcal{D} = \mathcal{P} \Rightarrow R\mathcal{D}^{\mathrm{c}} \boxplus RR\mathcal{D}^{\mathrm{c}} = \mathcal{P} \Rightarrow \mathcal{D}^{\mathrm{c}} \boxplus R\mathcal{D}^{\mathrm{c}} = \mathcal{P}.$$

(**iv**) $\Rightarrow$ (**i**) Suppose that condition (*iv*) is satisfied. Closedness of $\mathcal{D}$ implies $\mathcal{D}^{\mathrm{cc}} = \mathcal{D}$. On the other hand $\mathcal{D}^{\mathrm{c}} \boxplus R\mathcal{D}^{\mathrm{c}} = \mathcal{P}$ implies $\mathcal{D}^{\mathrm{cc}} = R\mathcal{D}^{\mathrm{c}}$. Therefore, $\mathcal{D}^{\mathrm{c}} = R\mathcal{D}$. Furthermore, Proposition 3.2 implies that $\mathcal{D}^{\mathrm{c}} = R\mathcal{D}^{\perp}$. Therefore $\mathcal{D} = \mathcal{D}^{\perp}$. ■

Remark 3.2.
Condition (iv) implies $\mathcal{D}^{\mathrm{c}}$ is a Dirac structure on $\mathcal{P}$. Conversely, if $\mathcal{D}^{\mathrm{c}}$ is a Dirac structure then $\mathrm{cl}(\mathcal{D})$ *is a Dirac structure. But note that not necessarily $\mathcal{D}$ is a Dirac structure since $\mathcal{D}$ need not be closed.*

The following proposition gives a necessary and sufficient condition for a Telegen structure to be a Dirac structure.

Proposition 3.4 (Relation between Dirac structures and Telegen structures).
Let $\mathcal{Z}$ be a closed subspace of $\mathcal{P}$. Then $\mathcal{Z}$ is a Dirac structure if and only if $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ are Telegen structures on $\mathcal{P}$.

Proof:
Necessity: Suppose that $\mathcal{Z}$ is a Dirac structure. Then $\mathcal{Z}$ is a Telegen structure (Remark 2.1). Since $\mathcal{Z}^{\perp} = \mathcal{Z}$, then $\mathcal{Z}^{\perp}$ is a Telegen structure too.
Sufficiency: Suppose that $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ are Telegen structures. Then Proposition 2.1 implies that

$$\mathcal{Z} \subseteq \mathcal{Z}^{\perp} \subseteq \mathcal{Z}^{\perp\perp}. \tag{3.2}$$

Both $\mathcal{Z}$ and $\mathcal{Z}^{\perp}$ are closed subspaces of $\mathcal{P}$ (see Remark 3.1). Proposition 3.2 implies

$$\left.\begin{array}{c} \mathcal{Z} \boxplus R\mathcal{Z}^{\perp} = \mathcal{P} \Rightarrow \mathcal{Z}^{\perp} \boxplus R\mathcal{Z} = \mathcal{P} \\ \mathcal{Z}^{\perp} \boxplus R\mathcal{Z}^{\perp\perp} = \mathcal{P} \end{array}\right\} \Rightarrow \mathcal{Z} = \mathcal{Z}^{\perp\perp}. \tag{3.3}$$

Now the relations (3.2) and (3.3) imply that $\mathcal{Z}^{\perp\perp} = \mathcal{Z}^{\perp} = \mathcal{Z}$. Therefore $\mathcal{Z}$ is a Dirac structure on $\mathcal{P}$. ■

4 Representation of Dirac structures

In this section three representations of Dirac structures are considered. Kernel and image representations of a Dirac structure are discussed in subsection 4.1, and scattering representation is discussed in Subsection 4.2.

4.1 Kernel and image representation

Since a Dirac structure $\mathcal{D}$ is a closed subspace, then there exists a linear transformation $T : \mathcal{P} \to \mathcal{L}$ with a dense domain such that [17]

$$\mathcal{D} = \ker(T).$$

This representation is called a *kernel representation.*

Let $T^* : \mathcal{L} \to \mathcal{L}$ stand for the adjoint of the linear transformation T, i.e. T^* is the unique solution of the equation

$$\langle Tp, l \rangle_{\mathcal{L}} = \langle p, T^* l \rangle_{\mathcal{P}} \, ,$$

where $\langle \cdot, \cdot \rangle_{\mathcal{L}}$ is the inner product on $\mathcal{L}$. Then (see [18], pp. 357, Theorem 5.22.6)

$$\mathcal{D}^{\mathrm{c}} = \mathrm{cl}(\mathrm{im}(T^*))$$

If $\mathcal{D}$ is a Dirac structure , then proposition 3.3 implies that

$$\mathcal{D} = \mathrm{cl}(\mathrm{im}(RT^*)),$$

This representation is called *image representation* of the Dirac structure $\mathcal{D}$.

Proposition 4.1 (Kernel and image representation of Dirac structures).
Consider a linear transformation $T : \mathcal{P} \to \mathcal{L}$ whose domain is a dense subspace of $\mathcal{P}$. $\mathcal{D} = \ker(T)$ is a Dirac structure on $\mathcal{P}$ if and only if $\ker(T)$ and $\mathrm{im}(RT^*)$ are Telegen structures on $\mathcal{P}$.

Proof:
The subspace $\mathcal{D} = \ker(T)$ is a closed subspace and it represents a Telegen structure. If $\mathrm{im}(RT^*)$ is a Telegen structure then also its closure $\mathrm{cl}(\mathrm{im}(RT^*))$, which represents $\mathcal{D}^{\perp}$, is a Telegen structure (see Proposition 3.1). The conditions of Proposition 3.4 are satisfied and $\mathcal{D}$ is a Dirac structure. ∎

4.2 Scattering representation

In this subsection we consider scattering representation of Dirac structures (see for the finite dimensional case [13]). The scattering variables v and w belongs to a Hilbert space $\mathcal{G}$ which is isometrically isomorphic to $\mathcal{F}$ and $\mathcal{E}$. The scattering variables are defined as

$$v = \frac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f \right) , \tag{4.1a}$$

$$w = \frac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f \right). \tag{4.1b}$$

Here, $r_{\mathcal{E},\mathcal{G}} : \mathcal{E} \to \mathcal{F}$, $r_{\mathcal{F},\mathcal{G}} : \mathcal{F} \to \mathcal{F}$ are bijective isometries. The scalar paring $\langle e|f \rangle$ expressed in the scattering variables has the following form

$$\langle e|f \rangle = \frac{1}{2} \langle w, w \rangle_{\mathcal{G}} - \frac{1}{2} \langle v, v \rangle_{\mathcal{G}}. \tag{4.2}$$

Proposition 4.2. *If $\mathcal{D}$ is a Dirac structure on $\mathcal{P}$ then the following statements hold*

(i) For every v in $\mathcal{G}$ there exists a unique $(f, e) \in \mathcal{D}$ such that (4.1a) is satisfied.

(ii) For every w in $\mathcal{G}$ there exists a unique $(f, e) \in \mathcal{D}$ such that (4.1b) is satisfied.

Proof:
We prove the first statement. The proof consists of two parts: existence and uniqueness.
Existence: Define $\tilde{p} = (\tilde{f}, \tilde{e}) = (0, \sqrt{2}\, r_{\mathcal{G},\mathcal{E}} v)$ where $r_{\mathcal{G},\mathcal{E}} : \mathcal{G} \to \mathcal{E}$ is the inverse of $r_{\mathcal{E},\mathcal{G}}$. It is clear that

$$v = \frac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} \tilde{e} - r_{\mathcal{F},\mathcal{G}} \tilde{f} \right).$$

$\mathcal{D}$ is a Dirac structure, and thus $\tilde{p}$ can be decomposed in a unique way (see Proposition 3.3 (*iii*)) as

$$\tilde{p} = p^1 + R p^2,\ \ p^1 = (f^1, e^1), p^2 = (f^2, e^2) \in \mathcal{D}.$$

The components of p^1, p^2 satisfy the following conditions

$$\begin{aligned} f^1 + r_{\mathcal{E},\mathcal{F}} e^2 &= 0 \\ e^1 + r_{\mathcal{F},\mathcal{E}} f^2 &= \sqrt{2}\, r_{\mathcal{G},\mathcal{E}} v. \end{aligned} \tag{4.3}$$

Define $p \in \mathcal{D}$ as $p = (f, e) = p^1 - p^2$. Then

$$\begin{aligned} \tfrac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f \right) = \tfrac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e^1 - r_{\mathcal{E},\mathcal{G}} e^2 - r_{\mathcal{F},\mathcal{G}} f^1 + r_{\mathcal{F},\mathcal{G}} f^2 \right) = \\ -\tfrac{r_{\mathcal{F},\mathcal{G}}}{\sqrt{2}} \left(f^1 + r_{\mathcal{E},\mathcal{F}} e^2 \right) + \tfrac{r_{\mathcal{E},\mathcal{G}}}{\sqrt{2}} \left(e^1 + r_{\mathcal{E},\mathcal{F}} f^2 \right) \underset{(4.3)}{=} v. \end{aligned}$$

Hence $(f, e) \in \mathcal{D}$ satisfies (4.1a).
Uniqueness: Suppose that there are two elements $p^1 = (f^1, e^1)$, $p^2 = (f^2, e^2)$, both in $\mathcal{D}$, such that (4.1a) is satisfied, i.e.

$$v = \frac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e^1 - r_{\mathcal{F},\mathcal{G}} f^1 \right),$$

$$v = \frac{1}{\sqrt{2}} \left(r_{\mathcal{E},\mathcal{G}} e^2 - r_{\mathcal{F},\mathcal{G}} f^2 \right).$$

Equalising these two equations yields

$$0 = r_{\mathcal{E},\mathcal{G}}(e^1 - e^2) - r_{\mathcal{F},\mathcal{G}}(f^1 - f^2). \tag{4.4a}$$

The scalar product related to $p^1 - p^2 \in \mathcal{D}$ is zero. By using the identity (4.2) one finds that

$$0 = \langle e^1 - e^2 | f^1 - f^2 \rangle = \frac{1}{2}\| r_{\mathcal{E},\mathcal{G}}(e^1 - e^2) + r_{\mathcal{F},\mathcal{G}}(f^1 - f^2)\|^2_{\mathcal{G}} - \frac{1}{2}\| r_{\mathcal{E},\mathcal{G}}(e^1 - e^2) - r_{\mathcal{F},\mathcal{G}}(f^1 - f^2)\|^2_{\mathcal{G}}.$$

Taking into account (4.4a), the last relation becomes

$$0 = r_{\mathcal{E},\mathcal{G}}(e^1 - e^2) + r_{\mathcal{F},\mathcal{G}}(f^1 - f^2). \tag{4.4b}$$

The equations (4.4a), (4.4b) imply $p^1 = p^2$. ■

Consider a Dirac structure $\mathcal{D}$ defined on $\mathcal{P}$. Define the map $O : \mathcal{F} \to \mathcal{E}$ in the following way. For any $v \in \mathcal{F}$ find $p = (f, e) \in \mathcal{D}$ such that the relation (4.1a) is satisfied. Then calculate w by using (4.1b). Now the map O is defined by

$$w = Ov.$$

The following can be concluded about the map O:

- The map O is well defined. Indeed, for any v there exists a unique $(f, e) \in \mathcal{D}$ (Proposition 4.2 statement (i)) such that (4.1a) is satisfied. This also means that w is unique.
- The map O is a linear transformation.
- If $w = Ov$ then $\langle w, w\rangle_{\mathcal{G}} = \langle v, v\rangle_{\mathcal{G}}$. This is a direct consequence of (4.2).
- The map O is injective. Indeed, $0 = Ov$ and the previous remark imply $v = 0$.
- The map O is surjective. For any $w \in \mathcal{E}$ there exists a unique pair (f, e) (Proposition 4.2 statement (i)) such that (4.1b) is satisfied. By (4.1a) this defines v such that $Ov = w$.
- The last three remarks imply that O is a unitary operator, that is $O^*O = OO^* = I_{\mathcal{G}}$.

These remarks are summarised as follows.

Proposition 4.3 (Scattering representation of Dirac structures).
If $\mathcal{D}$ is a Dirac structure then there exists a unitary linear operator O such that

$$\mathcal{D} = \left\{ (f, e) : r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f = O \left(r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f \right) \right\}. \tag{4.5}$$

This representation of Dirac structure is called a scattering representation. The converse result is also proved.

Proposition 4.4 (Scattering representation of Dirac structures).
$\mathcal{D}$ represented by (4.5), where O is a unitary operator, is a Dirac structure.

Proof:
First, $\mathcal{D}$ is expressed as the kernel of the linear transformation $T : \mathcal{P} \to \mathcal{G}$ defined as

$$T = \left[\ (I_\mathcal{G} - O)\, r_{\mathcal{E},\mathcal{G}} \quad (I_\mathcal{G} + O)\, r_{\mathcal{F},\mathcal{G}} \ \right].$$

The linear transformation T is bounded and thus $\mathcal{D}$ is closed. Since O is a unitary operator then

$$\|r_{\mathcal{E},\mathcal{G}} e + r_{\mathcal{F},\mathcal{G}} f\|_\mathcal{G}^2 - \|r_{\mathcal{E},\mathcal{G}} e - r_{\mathcal{F},\mathcal{G}} f\|_\mathcal{G}^2 = 0,$$

and consequently $\langle e|f \rangle = 0$. Thus $\mathcal{D}$ is a Telegen structure. The image of T is the whole space $\mathcal{L}$ (choose $e = r_{\mathcal{F},\mathcal{E}} f$) and thus it is closed. Consequently, $\mathcal{D}^\perp$ may be represented as

$$\mathcal{D}^\perp = \mathrm{im}(RT^*) = \mathrm{im} \begin{bmatrix} r_{\mathcal{G},\mathcal{F}}\,(I_\mathcal{G} - O^*) \\ r_{\mathcal{G},\mathcal{E}}\,(I_\mathcal{G} + O^*) \end{bmatrix}.$$

For any element of $\mathcal{D}^\perp$, $p = (f, e) = RT^* g$,

$$\begin{aligned} &\langle e|f \rangle = \langle r_{\mathcal{F},\mathcal{E}} f, f \rangle_\mathcal{F} = \langle r_{\mathcal{G},\mathcal{F}}\,(I_\mathcal{G} - O^*)\, g, r_{\mathcal{G},\mathcal{F}}\,(I_\mathcal{G} - O^*)\, g \rangle_\mathcal{F} = \\ &\langle r_{\mathcal{G},\mathcal{F}}\,(I_\mathcal{G} + O^*)\, g, rGF\,(I_\mathcal{G} - O^*)\, g \rangle_\mathcal{F} = \langle g, (I_\mathcal{G} - OO^*)\, g \rangle_\mathcal{G} = 0. \end{aligned}$$

Therefore, $\mathcal{D}^\perp$ is a Telegen structure. The conditions of the proposition (3.4) are satisfied and $\mathcal{D}$ is a Dirac structure. ■

5 Examples

The developed theory is illustrated on two examples: electrical circuit (Subsection 5.1) and one-dimensional transmission line (Subsection 5.2).

5.1 Electrical circuit

Consider an electrical circuit having no ports, n_b branches, n_n nodes and n_s separate parts. The voltage of the ith branch is denoted by e_i and the current through the ith branch is denoted by f_i. The positive direction of the current is in accordance with the positive polarity of the voltage.

Kirchhoff's current law places $n_n - n_s$ constraints on the currents, so that only $n_b - n_n + n_s$ currents may be specified independently. It is expressed as

$$Ff = 0,$$

where $f^\mathrm{T} = [f_1, \ldots, f_{n_b}]$, $F \in \{-1, 0, 1\}^{(n_n - n_s) \times n_b}$ and $\mathrm{rank}(F) = n_n - n_s$. Similarly, Kirchhoff's voltage law places $n_b - n_n + n_s$ constraints on the voltages and it can be expressed as

$$Ee = 0,$$

where $e^{\mathrm{T}} = [e_1, \ldots, e_{n_b}]$, $E \in \{-1, 0, 1\}^{(n_b - n_n + n_s) \times n_b}$ and $\mathrm{rank}(F) = n_b - n_n + n_s$. Also the matrices F, E are related as follows

$$FE^{\mathrm{T}} = 0. \tag{5.1}$$

Let $\mathcal{F} = \mathbb{R}^{n_b}$ be the space of the currents f and let $\mathcal{E} = \mathbb{R}^{n_b}$ be the space of the voltages e. The subspace of admissible currents and voltages imposed by the Kirchhoff's laws is given by

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} : T \begin{bmatrix} f \\ e \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} f \\ e \end{bmatrix} = 0 \right\}. \tag{5.2}$$

The scalar product has the following form

$$\langle e | f \rangle = e^{\mathrm{T}} f.$$

Tellegen's theorem (see e.g. [19]) implies that

$$e^{\mathrm{T}} f = 0, \ \forall (f, e) \in \mathcal{D}.$$

Therefore $\mathcal{D}$ is a Telegen structure. Matrix representation of the operator R is

$$R = \begin{bmatrix} 0 & I_{n_b} \\ I_{n_b} & 0 \end{bmatrix}$$

where I_{n_b} is n_b-dimensional identity matrix. The matrix representation of the adjoint of T is $T^* = T^{\mathrm{T}}$ and an element of $\mathrm{im}(RT^*)$ is given by

$$\begin{bmatrix} f \\ e \end{bmatrix} = RT^{\mathrm{T}} l = \begin{bmatrix} E^{\mathrm{T}} & 0 \\ 0 & F^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}.$$

Therefore,

$$e^{\mathrm{T}} f = l_2^{\mathrm{T}} F E^{\mathrm{T}} l_1 \underset{(5.1)}{=} 0.$$

Now it is clear that $\mathrm{im}(RT^*)$ is a Telegen structure and thus $\mathcal{D}$ is a Dirac structure by virtue of Proposition 3.4. In other words, interconnection structure of an electrical circuit is represented by Kirchhoff's laws.

5.2 One-dimensional transmission line

Consider a one dimensional transmission line whose length is S. The telegraphers equations are given by

$$\begin{aligned} f_q(z) &= -\tfrac{\partial f_\phi(z)}{\partial z}, \\ e_\phi(z) &= -\tfrac{\partial e_q(z)}{\partial z}, \end{aligned} \tag{5.3a}$$

and the boundary conditions are given by

$$\begin{aligned} f_{\phi}\left(0\right) &= -f_{\mathrm{BL}}, \\ e_{q}\left(0\right) &= e_{\mathrm{BL}}, \\ f_{\phi}\left(S\right) &= f_{\mathrm{BR}}, \\ e_{q}\left(S\right) &= e_{\mathrm{BR}}. \end{aligned} \tag{5.3b}$$

Here (f_q, e_q) are the power variables of the electric port, (f_ϕ, e_ϕ) are the power variables of the magnetic port, $(f_{\mathrm{BL}}, e_{\mathrm{BL}})$ are the power variables of an end of the transmission line and $(f_{\mathrm{BR}}, e_{\mathrm{BR}})$ are the power variables of the other end of the transmission line.

The space of flow variables is defined as

$$\mathcal{F} = L_2^2(0, S) \times \mathbb{R}^2$$

and its element is denoted by

$$f = (f_q, f_\phi, f_{\mathrm{BL}}, f_{\mathrm{BR}}).$$

The space of effort variables is defined as

$$\mathcal{E} = L_2^2(0, S) \times \mathbb{R}^2$$

and its element is denoted by

$$e = (e_q, e_\phi, e_{\mathrm{BL}}, e_{\mathrm{BR}}).$$

Here $L_2(0, S)$ is a the space of square integrable function on $[0, S]$. The scalar product has the following form

$$\langle e|f\rangle = \int_0^S f_q\left(z\right) e_q\left(z\right) \mathrm{d}z + \int_0^S f_\phi\left(z\right) e_\phi\left(z\right) \mathrm{d}z + e_{\mathrm{BL}} f_{\mathrm{BL}} + e_{\mathrm{BR}} f_{\mathrm{BR}}.$$

The first term on the right side represents the power exchanged with electrical port, the second term on the left side represents the power exchanged with the magnetic port and the last two terms represent the powers exchanged with the external ports. Since $\mathcal{F} = \mathcal{E}$ then

$$r_{\mathcal{F},\mathcal{E}} = r_{\mathcal{E},\mathcal{F}} = I_{\mathcal{F}}.$$

where $I_{\mathcal{F}}$ is the identity operator in $\mathcal{F}$. It is obvious that $\mathcal{F}$ represent a Hilbert space whose inner product is defined by

$$\langle f^1, f^2 \rangle = \ll f^1, f^2 \gg .$$

The operator R is defined by

$$R = \begin{bmatrix} 0 & I_{\mathcal{E}} \\ I_{\mathcal{F}} & 0 \end{bmatrix}.$$

The space of admissible flows f and efforts e imposed by (5.3) is given by

$$\mathcal{D} = \{p = (f, e) \in \mathcal{F} \times \mathcal{E} : Tp = 0\},$$

where $T : \mathcal{F} \times \mathcal{F} \to \mathcal{L} = L_2^2(0, S) \times \mathbb{R}^4$

$$Tp = \begin{bmatrix} f_q(z) + \frac{\partial f_\phi(z)}{\partial z} \\ e_\phi(z) + \frac{\partial e_q(z)}{\partial z} \\ e_{\mathrm{BL}} - e_q(0) \\ f_{\mathrm{BL}} + f_\phi(0) \\ e_{\mathrm{BR}} - e_q(S) \\ f_{\mathrm{BR}} - f_\phi(S) \end{bmatrix}.$$

The domain of the operator T is

$$\mathcal{K}(T) = \mathrm{L}_2^2(0, S) \times \mathbb{R}^2 \times \mathcal{K}^2(\frac{\partial}{\partial z}) \times \mathbb{R}^2$$

where

$$\mathcal{K}(\frac{\partial}{\partial z}) = \{e \in \mathrm{L}_2(0, 1) : e(z) \text{ absolutely continuous and } \frac{\partial e(z)}{\partial z} \in \mathrm{L}_2(0, 1)\}.$$

The subspace $\mathcal{K}(\frac{\partial}{\partial z})$ is a dense subspace on $L_2(0, S)$ (see see [17], pp. 145, exercise 2.7). Thus $\mathcal{K}(T)$ is a dense subspace. First we prove that $\mathcal{D}$ is a Telegen structure. Indeed, if $(f, e) \in \mathcal{F} \times \mathcal{E}$ then

$$\begin{aligned} \langle e|f\rangle &\underset{(5.3a)}{=} -\int_0^S \frac{\partial f_\phi(z)}{\partial z} e_q(z)\,\mathrm{d}z - \int_0^S \frac{\partial f_q(z)}{\partial z} e_\phi(z)\,\mathrm{d}z + e_{\mathrm{BL}} f_{\mathrm{BL}} + e_{\mathrm{BR}} f_{\mathrm{BR}} \\ &= -\int_0^S \frac{\partial (f_\phi(z)\, e_q(z))}{\partial z}\mathrm{d}z + e_{\mathrm{BL}} f_{\mathrm{BL}} + e_{\mathrm{BR}} f_{\mathrm{BR}} \\ &= -f_\phi(S)\, e_q(S) + f_\phi(0)\, e_q(0) + e_{\mathrm{BL}} f_{\mathrm{BL}} + e_{\mathrm{BR}} f_{\mathrm{BR}} \\ &\underset{(5.3b)}{=} 0. \end{aligned}$$

The adjoint of the linear transformation T is given by

$$T^* l = \begin{bmatrix} l_q \\ l_\phi \\ l_{f_{\mathrm{BL}}} \\ l_{f_{\mathrm{BR}}} \\ -\frac{\partial l_\phi}{\partial z} \\ -\frac{\partial l_q}{\partial z} \\ l_{e_{\mathrm{BL}}} \\ l_{e_{\mathrm{BR}}} \end{bmatrix}. \tag{5.4}$$

The domain of T^* is

$$\begin{aligned}\mathcal{G}(T^*) = \quad & \big\{(l_\phi, l_q, l_{e_{\mathrm{BL}}}, l_{e_{\mathrm{BR}}}, l_{f_{\mathrm{BL}}}, l_{f_{\mathrm{BR}}}) \in \mathrm{L}_2(0,S) \times \mathrm{L}_2(0,S) \times \mathbb{R}^4 : \\ & l_q, l_\phi \text{ are absolutely continuous functions, } \frac{\partial l_q}{\partial z}, \frac{\partial l_\phi}{\partial z} \in L_2(0,S), \\ & l_q(0) = l_{f_{\mathrm{BL}}}, l_q(S) = l_{f_{\mathrm{BR}}}, l_\phi(0) = -l_{e_{\mathrm{BL}}}, l_\phi(S) = l_{e_{\mathrm{BR}}}\big\}.\end{aligned} \tag{5.5}$$

Now, we prove that im(RT^*) is a Telegen structure. Indeed,

$$\begin{aligned}\langle e|f\rangle \underset{(5.4)}{=} & -\int_0^S \frac{\partial l_\phi(z)}{\partial z} l_q(z) - \int_0^S \frac{\partial l_q(z)}{\partial z} l_\phi(z) + l_{f_{\mathrm{BL}}} l_{e_{\mathrm{BL}}} + l_{f_{\mathrm{BR}}} l_{e_{\mathrm{BR}}} \\ = & -\int_0^S \frac{\partial (l_\phi(z) l_q(z))}{\partial z} + l_{f_{\mathrm{BL}}} l_{e_{\mathrm{BL}}} + l_{f_{\mathrm{BR}}} l_{e_{\mathrm{BR}}} \\ = & \; l_\phi(0) l_q(0) - l_\phi(S) l_q(S) + l_{f_{\mathrm{BL}}} l_{e_{\mathrm{BL}}} + l_{f_{\mathrm{BR}}} l_{e_{\mathrm{BR}}} \\ \underset{(5.5)}{=} & \; 0.\end{aligned}$$

Therefore $\mathcal{D}$ is a Dirac structure by virtue of Proposition 3.4. Thus Equation (5.3) represents the interconnection structure of one-dimensional transmission line.

6 Summary and recommendations for further research

In this paper, the interconnection structures (Dirac structures) on Hilbert spaces have been defined. Some properties of Dirac structure are pointed out and it has been proved that any Dirac structure can be associated with kernel, image and scattering representations.

Further research will be focused on compositionally properties of Dirac structures. It has been proved in [20] that the composition of any two finite dimensional Dirac structures yields a Dirac structure again. Some preliminary results show that this is not necessarily true for Dirac structures defined on infinite dimensional Hilbert spaces.

Acknowledgement: We would like to thank H. Zwart, associate professor from University of Twente, for his valuable comments on this paper.

References

[1] D.C. Karnopp, D.L. Margolis, and R.C. Rosenberg. *System Dynamics: Modelling and Simulation of Mechatronic System.* John Wiley and Sons, Inc. New York, 2000.

[2] G. Golo, P.C. Breedveld, B.M. Maschke, and A.J. van der Schaft. Geometric formulation of generalized bond graphs models part i: Generalized junction structure. Technical report, Faculty of Mathematical Sciences, 2000.

[3] G. Golo, P.C. Breedveld, B.M. Maschke, and A.J. van der Schaft. A novel systematic approach for derivation of simulation models from bond graphs. submitted to the Journal of Franklin Insitute, 2002.

[4] G. Golo, V. Talasila, and A.J. van der Schaft. Hamiltonian discretizations of the telegrapher's equations. to be submitted, 2002.

[5] T.J. Courant and A. Weinstein. Beyond poisson structures. In *in Seminaire sudrhodanien de geometrie 8, Travaux en Cours 27*. Hermann, Paris, 1988.

[6] T.J. Courant. Dirac manifolds. *Trans. Amer. math. Soc.*, 319:631–661, 1990.

[7] I. Dorfman. Dirac structures of integrable evolution equations. *Phys. Lett. A*, 125:240–246, 1987.

[8] I. Dorfman. *Dirac structures and Integrability of Nonlinear Evolution Equations*. John Wiley, Chichester, 1993.

[9] P.E. Crouch. Handwritten notes, 94.

[10] A.J. van der Schaft and B.M. Maschke. The hamiltonian formulation of energy conserving physical systems with external ports. *Archiv fur Elektronik und Ubertragungstechnik*, 49:362–371, 1995.

[11] B.M. Maschke and A.J. van der Schaft. *Modelling and Control of Mechanical Systems*, chapter Interconnected Mechanical Systems, part II: the Dynamics of Spatial Mechanical Networks, pages 17–30. Imperial College Press, 1997.

[12] A.J. van der Schaft and B.M. Maschke. *Modelling and Control of Mechanical Systems*, chapter Interconnected Mechanical Systems, part I: geometry of interconnection and implicit Hamiltonian systems, pages 1–15. Imperial College Press, 1997.

[13] A.J. van der Schaft. *L_2-Gain and Passivity Techniques in Nonlinear Control*. Springer-Verlag, 2000.

[14] S. Stramigioli. *Modeling and IPC Control of Interactive Mechanical Systems: a coordinate free approach*. Springer Verlag, 2001.

[15] B.M. Maschke and A.J. van der Schaft. Port controlled hamiltonian representation of distributed parameter systems. *Proc. IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, Princeton University, Editors N.E. Leonard, R. Ortega*, pages 28–38, 2000.

[16] A.J. van der Schaft and B.M. Maschke. Hamiltonian formulation of distributed-parameter systems with boundary energy flow. *Journal of Geometry and Physics*, page to be published, 2002.

[17] T. Kato. *Perturbation Theory for Linear Operators*. Springer-Verlag, Berlin, 1980.

[18] A.W. Naylor and G.R. Sell. *Linear Operator Theory in Engineering and Sciences*. Holt, Rinehart and Winston, INC. New York, 1971.

[19] B.D.H. Tellegen. A general network theorem, with applications. *Philips Res. Rept.*, 7:168–175, 1952.

[20] A.J. van der Schaft. *The Mathematics of Systems and Control: from Intelegent Control to Behavioral Systems*, chapter Interconnection and Geometry, pages 201–217. University of Groningen, 1999.