Controllability Analysis of A Two Degree of Freedom Nonlinear Attitude Control System

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Abstract

We study a physically simple two degree of freedom attitude control system that has a single control input. The physical assumptions are described. Equations of motion are derived and expressed in a nonlinear control form. We demonstrate that the system is inherently nonlinear. Conditions for small time local controllability of the state and of the configuration are presented.

1 Introduction

An air spindle is supported by an air bearing so that it can rotate without friction about its vertical axis. The air spindle supports a rigidly attached platform that rotates in a horizontal plane. A control torque is exerted on the spindle by a motor. An unactuated mass particle is constrained to move without friction along a straight track that is rigidly mounted on the horizontal platform.

This air spindle system is similar to a physical testbed that exists in the Attitude Dynamics and Control Laboratory at the University of Michigan. The air spindle in the laboratory testbed is not actuated, but several masses that move in straight tracks rigidly mounted on the platform are actuated. This experimental testbed has been described in [1] and nonlinear control results have been developed and experimentally validated in [5]. The air spindle system studied in this paper has different actuation assumptions and leads to different, but equally interesting, nonlinear control problems.

2 Equations of Motion

In this section, we derive the equations of motion for the air spindle with an unactuated mass constrained to move along a straight track. We make the following assumptions: the platform is perfectly leveled so that gravity has no influence; external disturbances, e.g. friction and aerodynamics, can be ignored; the unactuated mass is modeled as a point mass.

We introduce the following notation:

$$\begin{array}{rcl} I & = & \text{the inertia of the platform;} \\ \theta & = & \text{the attitude angle of the platform;} \\ m & = & \text{the mass that moves without friction on a horizontal track fixed to the platform;} \\ z & = & \text{the relative position of the mass with respect to its track that is fixed to the platform;} \\ l & = & \text{the length of the normal from the vertical axis of the spindle to the track.} \end{array}$$

We choose $z = 0$ to correspond to the position of the mass in the track when its distance to the vertical axis of the spindle is minimum; this minimum distance is l.

We choose an inertial frame $OXYZ$, where the vertical axis OZ is aligned with the rotating axis of the platform and OXY is in the horizontal plane. Let $(\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3)$ denote unit vector in a platform fixed coordinate frame for the track, where $(\mathsf{e}_1, \mathsf{e}_2)$ lies in the OXY plane, and e_1 is aligned with the track axis; e_2 is perpendicular to e_1, and e_3 is aligned with the vertical axis. Let $\Omega = \dot{\theta}\mathsf{e}_3$ denote the angular velocity vector of the platform, and let $\rho = z\mathsf{e}_1 + l\mathsf{e}_2$ denote the position vector of the mass in the platform fixed coordinate frame. Using the identities $\dot{\mathsf{e}_1} = \Omega \times \mathsf{e}_1 = \dot{\theta}\mathsf{e}_2$ and $\dot{\mathsf{e}_2} = \Omega \times \mathsf{e}_2 = -\dot{\theta}\mathsf{e}_1$, we have $\dot{\rho} = (\dot{z} - l\dot{\theta})\mathsf{e}_1 + z\dot{\theta}\mathsf{e}_2$. Hence,

$$|\dot{\rho}|^2 = (\dot{z} - l\dot{\theta})^2 + (z\dot{\theta})^2 = (z^2 + l^2)\dot{\theta} + \dot{z}^2 - 2\dot{\theta}l\dot{z}.$$

The Lagrangian is equal to the kinetic energy given by

$$L(\dot{\theta}, z, \dot{z}) \quad = \quad \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m|\dot{\rho}|^2 = \frac{1}{2}[I + m(z^2 + l^2)]\dot{\theta}^2 + \frac{1}{2}m[\dot{z}^2 - 2l\dot{z}\dot{\theta}].$$

Equivalently,

$$L(\dot{\theta}, z, \dot{z}) = \frac{1}{2} \begin{pmatrix} \dot{\theta} & \dot{z} \end{pmatrix} M(z) \begin{pmatrix} \dot{\theta} \\ \dot{z} \end{pmatrix}$$

where $M(z)$ is the inertia matrix given by

$$M(z) = \begin{bmatrix} I + m(z^2 + l^2) & -ml \\ -ml & m \end{bmatrix}.$$

Moreover, $M(z)$ can be viewed as a Riemannian metric, which defines the kinetic energy on the tangent space $T(S^1 \times R^1)$. Note that M is independent of θ; this reflects the fact that the air spindle is symmetric to platform rotation.

We further assume that the platform is controlled by an external torque about the spindle axis, but there is no direct control applied to the mass particle. This implies that the mass particle is unactuated. With this assumption, the equations of motion are

$$(J + mz^2)\ddot{\theta} \quad = \quad -2mz\dot{\theta}\dot{z} + ml\ddot{z} + u, \tag{2.1}$$

$$m\ddot{z} \quad = \quad mz\dot{\theta}^2 + ml\ddot{\theta}, \tag{2.2}$$

where $J = I + ml^2$ and u is the control torque on the platform.

3 Controllability Properties

In this section, we study controllability properties at equilibrium for the controlled air spindle with one unactuated mass. We provide two different controllability analyses that lead to consistent results. We first study state controllability using Lie bracket tools; we provide conditions for small time local controllability using results of [6, 7, 4]. Then we study configuration controllability using symmetric product tools; we provide conditions for small time local configuration controllability and equilibrium controllability using results of [2, 3]. The implications of the two results are slightly different, but in the particular problem studied in this paper, the obtained conditions are identical.

3.1 State Controllability Analysis

Since the platform rotation is actuated and $I + mz^2 > 0$ for all z, we can simplify the equations using feedback. Introducing the following control transformation defined by

$$u = 2mz\dot{z}\dot{\theta} - mlz\dot{\theta}^2 + (I + mz^2)v,$$

where v the transformed control, the equations become

$$\ddot{\theta} = v, \tag{3.3}$$

$$\ddot{z} = z\dot{\theta}^2 + lv. \tag{3.4}$$

The above equations can be written in the standard nonlinear form as

$$\dot{x} = f(x) + g(x)v, \tag{3.5}$$

where x is the state given by

$$x = \begin{bmatrix} \theta \\ \dot{\theta} \\ z \\ \dot{z} \end{bmatrix}, \quad \text{and} \quad f(x) = \begin{bmatrix} \dot{\theta} \\ 0 \\ \dot{z} \\ z\dot{\theta}^2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ l \end{bmatrix}.$$

If $v = 0$, then the equilibrium manifold is given by $\{(\theta, \dot{\theta}, z, \dot{z}) | \dot{\theta} = 0,\ \dot{z} = 0\}$. That is, the system is in equilibrium at any fixed platform attitude and any fixed mass particle location on the track. It is clear that the system is not linearly controllable at any equilibrium when $\dot{\theta} = \dot{z} = 0$; a nonlinear controllability analysis is necessary.

We first present a negative result on the air spindle controllability using a theorem of Sussmann [6].

Proposition 1. If $z \neq 0$, then the air spindle defined by eqn.(3.5) is *not* small time locally controllable at any equilibrium.

Proof: The following Lie brackets can be computed:

$$ad_f g(x) = [f, g](x) = \begin{bmatrix} 1 \\ 0 \\ l \\ 2z\dot{\theta} \end{bmatrix}, \quad [g, [f, g]](x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2z \end{bmatrix}, \quad ad_f^2 g(x) = [f, [f, g]](x) = \begin{bmatrix} 0 \\ 0 \\ 2z\dot{\theta} \\ \dot{\theta}(l\dot{\theta} - 2\dot{z}) \end{bmatrix}.$$

We need the following technical result:

Claim: $ad_f^i g(x), i \geq 2$ have the form

$$ad_f^i g(x) = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} h_1(z, \dot{z}, \dot{\theta}) \\ \dot{\theta} h_2(z, \dot{z}, \dot{\theta}) \end{bmatrix},$$

where h_1 and h_2 are two polynomial functions of $(z, \dot{z}, \dot{\theta})$.

We prove this by induction. When $i = 2$, it is obvious that $ad_f^2 g(x)$ is in the claimed form. Now suppose this claim holds for $ad_f^i g$ with $i = 2, \cdots, k$ for $k \geq 3$. Consider the Lie bracket $ad_f^{k+1} g$. Since

$$ad_f^{k+1} g(x) = [f, ad_f^k g](x) = \frac{\partial ad_f^k g}{\partial x} f(x) - \frac{\partial f}{\partial x} ad_f^k g(x),$$

where $ad_f^k g(x) = \big(0, \ \ 0, \ \ \dot{\theta} h_1^k(z, \dot{z}, \dot{\theta}), \ \ \dot{\theta} h_2^k(z, \dot{z}, \dot{\theta})\big)^T$ by induction hypothesis, we have

$$ad_f^{k+1} g(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \star & \dot{\theta}\frac{\partial h_1^k}{\partial z} & \dot{\theta}\frac{\partial h_1^k}{\partial \dot{z}} \\ 0 & \star & \dot{\theta}\frac{\partial h_2^k}{\partial z} & \dot{\theta}\frac{\partial h_2^k}{\partial \dot{z}} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ 0 \\ \dot{z} \\ z\dot{\theta}^2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2z\dot{\theta} & \dot{\theta}^2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} h_1^k \\ \dot{\theta} h_2^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} h_1^{k+1} \\ \dot{\theta} h_2^{k+1} \end{bmatrix},$$

where

$$h_1^{k+1}(z, \dot{z}, \dot{\theta}) = \dot{z}\frac{\partial h_1^k}{\partial \dot{z}} + z\dot{\theta}^2 \frac{\partial h_1^k}{\partial \dot{z}} - h_2^k, \quad h_2^{k+1}(z, \dot{z}, \dot{\theta}) = \dot{z}\frac{\partial h_2^k}{\partial z} + z\dot{\theta}^2 \frac{\partial h_2^k}{\partial \dot{z}} - \dot{\theta}^2 h_1^k.$$

This proves the claim.

Using this result, we see that at an equilibrium x_e where $\dot{z} = \dot{\theta} = 0$,

$$[g, [f, g]](x_e) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2z \end{bmatrix}, \quad f(x_e) = 0, \ \ g(x_e) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ l \end{bmatrix}, \quad ad_f g(x_e) = \begin{bmatrix} 1 \\ 0 \\ l \\ 0 \end{bmatrix}, \quad ad_f^i g(x_e) = 0, \quad i \geq 2.$$

Therefore $[g,[f,g]](x_e) \notin \text{span}\{f, g, ad_f^i g, \forall i \in Z^+\}(x_e)$ when $z \neq 0$. By Sussmann's theorem, eqn.(3.5) is *not* small time locally controllable at any equilibrium with $z \neq 0$. $\square$

A positive controllability result is presented as follows.

Proposition 2. If the offset $l \neq 0$, then eqn.(3.5) is small time locally controllable at any equilibrium with $z = 0$.

Proof: This result follows from Sussmann's sufficient conditions for small time local controllability [7]. We first show local accessibility at equilibrium. The Lie bracket computations show that

$$h(x) = \big[[f,g],[g,[f,g]]\big](x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2l \end{bmatrix}, \quad [h,f](x) = \begin{bmatrix} 0 \\ 0 \\ -2l \\ 0 \end{bmatrix}.$$

Therefore, $\big\{g,[f,g],h,[h,f]\big\}$ span $\mathbb{R}^4$ at any equilibrium if $l \neq 0$. Therefore, eqn.(3.5) is locally accessible at any equilibrium.

Next, we check the good and bad Lie bracket relations for small time local controllability. We first notice that all the above spanning brackets are *good* in the sense of Sussmann, and that the highest degree of these Lie brackets is 6 for $[h,f]$. Therefore, we only need to check the bad brackets with degree lower than 6. These bad brackets can be classified into two groups according to the number of times that f appears: bad brackets containing f once; bad brackets containing f three times. We analyze these bad brackets subsequently.

Up to a sign, the bad brackets in the first group are f, $ad_g^2 f = -[g,[f,g]]$, and $ad_g^4 f$. It is clear that $f = 0$ at equilibrium and that $[g,[f,g]]$ is zero when $z = 0$. Moreover, $ad_g^4 f$ is identically zero.

Up to a sign, there are four bad brackets in the second group: $[ad_f^3 g, g]$, $[[ad_f^2 g, g], f]$, $[ad_f^2 g, [f,g]]$, and $[[ad_g^2 f, f], f]$. Computations show that all of these Lie brackets are zero at any equilibrium with $z = 0$.

Thus, we claim that all bad brackets are linear combinations of lower degree good brackets. Consequently, eqn.(3.5) is small time locally controllable at any equilibrium with $z = 0$, if the offset $l \neq 0$. $\square$

3.2 Configuration Controllability Analysis

First, we present a formula that simplifies symmetric product computations. Let X and Z be two vector fields on $Q = S^1 \times \mathbb{R}$. Using the definitions of covariant derivative and symmetric product, it is easy to show the symmetric product between X and Z can be expressed as

$$\big\langle X : Z \big\rangle = M^{-1}(z) \big\langle X; Z \big\rangle,$$

where

$$\big\langle X; Z \big\rangle = \frac{\partial [M(z)X]}{\partial q} Z + \frac{\partial [M(z)Z]}{\partial q} X - DM(X, Z), \qquad q = (\theta, z).$$

Here $DM(X, Z)$ is given by

$$DM(X, Z) = \Big(0, \quad X^T \frac{\partial M(z)}{\partial z} Z \Big)^T.$$

Now we use this simplified formula to compute certain symmetric products for the air spindle problem. For the control vector field Y, we have $M(z)Y = e_1$, where $e_1 = (1, 0)^T$. For the second order symmetric product, we have

$$\big\langle Y; Y \big\rangle(z) = -DM(Y, Y) = -\begin{bmatrix} 0 \\ e_1^T M^{-1}(z) \frac{\partial M(z)}{\partial z} M^{-1}(z) e_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-2mz}{(I+mz^2)^2} \end{bmatrix}.$$

It is easy to see that $\big\langle Y : Y \big\rangle$ evaluates as zero only if $z = 0$. Similarly, we can compute the third order symmetric product $\big\langle Y : \big\langle Y : Y \big\rangle \big\rangle$ as

$$\big\langle Y; \big\langle Y : Y \big\rangle \big\rangle(z) = \begin{bmatrix} 0 \\ \frac{2ml(-I+3mz^2)}{(I+mz^2)^4} \end{bmatrix}.$$

When evaluated as $z = 0$, it becomes

$$\big\langle Y; \big\langle Y : Y \big\rangle \big\rangle_{z=0} = \begin{bmatrix} 0 \\ \frac{-2ml}{I^3} \end{bmatrix}.$$

Therefore, $\Big\{ Y, \big\langle Y : \big\langle Y : Y \big\rangle \big\rangle \Big\}$ span the tangent space of Q at $z = 0$. This means that the air spindle is locally configuration accessible at $z = 0$. Furthermore, notice that these two symmetric products are good, and the bad symmetric product $\big\langle Y : Y \big\rangle$ is zero when evaluated at $z = 0$; thus the good-bad symmetric product relation is satisfied. Hence, we conclude that the air spindle with one unactuated mass is small time locally configuration controllable at $z = 0$. This also implies that the system is equilibrium controllable at $z = 0$.

We summarize the above analysis in the following proposition.

Proposition 3. If the offset $l \neq 0$, then eqn.(3.5) is small time locally configuration controllable and equilibrium controllable at any equilibrium with $z = 0$.

It is clear that if $z \neq 0$, the sufficient conditions for configuration controllability are not satisfied. These results agree with the state controllability analysis performed in the last section.

There is an important symmetry property for this system: it is clear that the dynamic equations are independent of the platform angle θ, therefore the air spindle system is symmetric with respect to platform rotation. Hence, arbitrary platform attitude can be reached at $z = 0$ if $l \neq 0$.

4 Conclusions

A physically simple two degree of freedom attitude control system has been introduced. Despite its physical simplicity, we have shown that its controllability properties are subtle. It is shown to be controllable only at a single mass particle position. This analysis raises many questions, some of which will be considered in the final version of the paper.

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References

[1] D. S. Bernstein, N. H. McClamroch, and A. M. Bloch, "Development of Air Spindle and Triaxial Bearing Testbeds for Spacecraft Dynamics and Control Experiments", Proceedings of 2001 American Control Conference, Arlington, VA, June 2001.

[2] A. D. Lewis and R. M. Murray, Controllability of simple mechanical control systems. SIAM Journal of Control and Optimization, 35(3):766-790, 1997.

[3] A. D. Lewis, Local configuration controllability for a class of mechanical systems with a single input, Proceedings of the 1997 European Control Conference, 1997.

[4] M. Reyhanoglu, A. van der Schaft, N. H. McClamroch and I. Kolmanovsky, Dynamics and Control of a Class of Underactuated Mechanical Systems, IEEE Transactions on Automatic Control, Vol.44(9), p.1663-1671, 1999.

[5] J. Shen, N. H. McClamroch, and D. S. Bernstein, "Air Spindle Attitude Control via Proof Mass Actuators", Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, December 2001.

[6] H. J. Sussmann, Lie Brackets and Local Controllability: a Sufficient Condition for Scalar-Input Systems, SIAM Journal of Control and Optimization, Vol. 21, No. 5, p.686-713, Sept. 1983.

[7] H. J. Sussmann, A General Theorem on Local Controllability, SIAM Journal of Control and Optimization, Vol. 25, No. 1, p.158-194, 1987.