Boundary Observability in the Quasi-Static Thermoelastic Contact Problem

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Abstract

We study the observability properties of a nonlinear parabolic system that models the temperature evolution of a thermoelastic rod that may come into contact with a rigid obstacle. Basically the system dynamics are described by a one-dimensional nonlocal partial differential equation of parabolic type with a nonlinear and nonlocal boundary condition. For a specified nonlocal observation operator, we show the system to be observable and get the estimates for the boundary temperature and the current state of the system. The technique of deriving the boundary estimate is essentially based on using Carleman-type estimates. Finally, for sufficiently smooth solutions, we show that the observability result is equivalent to the boundary observability of the system.

1 The dynamical system

Let $\Omega_T = (0,1) \times (0,T)$ for $T > 0$. Consider a system described by the equations

$$(1+a^2)\theta_t - \theta_{xx} = a\frac{d}{dt}\max\left\{a\int_0^1 \theta(\xi,t)d\xi - g,\ 0\right\}, \qquad in\ \Omega_T, \tag{1.1}$$

$$\theta(0,t) = 0, \qquad in\ (0,T), \tag{1.2}$$

$$-\theta_x(1,t) = k\left(g - a\int_0^1 \theta(\xi,t)d\xi\right)\theta(1,t), \qquad in\ (0,T), \tag{1.3}$$

$$\theta(x,0) = \theta_0(x), \qquad in\ (0,1). \tag{1.4}$$

Here $0 < a < 1$, $k(s)$, $s = g - a\int_0^1 \theta(\xi,t)d\xi \in R$, is a nonnegative function. Equations (1.1)–(1.4) model the *temperature variation* in a thermoelastic rod which is situated between two walls that are kept at different temperatures. One end of the rod is fixed to a wall, while the other end is free to expand or contract. The expansion of the rod resulting from the evolution of the temperature and the stresses is limited by the existence of the other wall. The contact of the free end with the wall results in the nonlocal parabolic equation with a nonlinear and nonlocal boundary condition [1].

Following [1], we define a strong solution of (1.1)–(1.4) as a function in $W_2^{2,1}(\Omega_T)$, which satisfies (1.1)–(1.4), and use the following result.

Theorem 1.1. *Given $\theta_0 \in H^1(0,1)$ with $\theta_0(0) = 0$ and $k(\cdot) \in C^1(R)$, there exists a strong solution to* (1.1)–(1.4).

In [1], a weak solution is also defined and an existence theorem in proved for $\theta_0 \in L^2(0,1)$, $k \in C(R)$, $k \geq 0$, where k satisfies the condition $k(s) \leq \alpha|s| + \beta$, $s \in R$, $\alpha, \beta > 0$. Based on this result, we take $L^2(\Omega_T)$ as a "generalized state space" of the system (1.1)–(1.4).

We assume that the conditions ensuring the existence of a strong solution hold, but that θ_0 is unknown. Measurements $w(t)$ and $z(t)$ of the process (1.1)–(1.4) are assumed to be available:

$$w(t) = \theta_x(0,t), \quad z(t) = \int_0^1 \theta(\xi, t)\, d\xi, \ t \in (0,T). \tag{1.5}$$

Since $\theta \in W^{2,1}(\Omega_T)$, we get $w(\cdot) \in H^{1/4}(0,T)$, $z(\cdot) \in H^1(0,T)$.

In this paper, we investigate if it is possible to estimate the current state $\theta(\cdot, T) \in L^2(0,1)$ and the boundary condition $\theta(1,t)$, $t \in (0,T)$ from the measurements (1.5). To give a formal description of the properties focused on in this paper, we refer to [6] and [2]. We say that the system (1.1)–(1.5) is **weakly observable**, if for any two solutions $\theta^{(1)}(x,t)$ and $\theta^{(2)}(x,t)$ with $w^{(1)}(\cdot) = w^{(2)}(\cdot)$, $z^{(1)}(\cdot) = z^{(2)}(\cdot)$, it follows that $\theta^{(1)}(x,t) = \theta^{(2)}(x,t)$ for almost all $(x,t) \in \Omega_T$. For the solutions in $W^{2,1}(\Omega_T)$, the weak observability property also implies that $\theta^{(1)}(1,\cdot) = \theta^{(2)}(1,\cdot)$.

If the system is weakly observable, our next concern is whether the a posteriori estimates

$$\|\theta(\cdot,T)\|^2_{L^2(0,1)} \leq \gamma \left(\|w(\cdot)\|^2_{L^2(0,T)} + \|\dot{z}\|^2_{H^1(0,T)} \right). \tag{1.6}$$

$$\int_{0+\varepsilon}^{T} \theta^2(1,\cdot) dt \leq \eta \left(\|w(\cdot)\|^2_{L^2(0,T)} + \|\dot{z}\|^2_{H^1(0,T)} \right) \tag{1.7}$$

may be derived for the solutions to (1.1)–(1.4) where the constants γ and η do not depend on a particular solution, but where η may depend on ε. Linear parabolic systems, whose solutions satisfy an estimate like (1.6), are said to be **strongly observable** [6]. We will use the same notion to say that the nonlinear system (1.1)–(1.5) is strongly observable if its solutions satisfy (1.6). Similarly, the system (1.1)–(1.5) will be said to be **boundary identifiable** if (1.7) holds.

In this paper, we derive and prove the theorems establishing the weak and the strong observability for the system (1.1)–(1.5), and examine the special cases, for which the estimate (1.7) holds.

2 Observability theorems

Theorem 2.1. *The system* (1.1)–(1.5) *is weakly observable.*

Proof. Let there be two solutions $\theta^{(1)}$ and $\theta^{(2)}$ such that for a.e. $t \in (0,T)$,

$$\theta_x^{(1)}(0,t) = \theta_x^{(2)}(0,t),$$
$$\int_0^1 \theta^{(1)}(\xi,t)d\xi = \int_0^1 \theta^{(2)}(\xi,t)d\xi.$$

From equations (1.1)–(1.5), it follows that the difference $\theta(x,t) = \theta^{(1)}(x,t) - \theta^{(2)}(x,t)$ has to yield the conditions

$$\begin{aligned}
(1+a^2)\theta_t - \theta_{xx} = 0, \qquad & in\ \Omega_T,\\
\theta(0,t) = 0,\ \theta_x(0,t) = 0 \qquad & in\ (0,T),\\
\theta_x(1,t) = K(t)\theta(1,t), \qquad & in\ (0,T),
\end{aligned}$$

where $K(t) = k(g - a\int_0^1 \theta^{(1)}(\xi,t)d\xi) = k(g - a\int_0^1 \theta^{(2)}(\xi,t)d\xi)$. Applying known results on boundary observability of linear parabolic equations [5, 7], we conclude that $\theta^{(1)}(x,t) = \theta^{(2)}(x,t)$ for all t. The theorem 2.1 is proved.

Theorem 2.2. *The system* (1.1)–(1.5) *is strongly observable.*

Proof. We integrate equation (1.1) over the interval $(0,1)$ and see that

$$(1+a^2)\int_0^1 \theta_t(\xi,t)d\xi = \theta_x(1,t) - \theta_x(0,t) + a\frac{d}{dt}\max\left\{a\int_0^1 \theta(\xi,t)d\xi - g,\ 0\right\}$$

from which we deduce

$$\theta_x(1,t) = (1+a^2)\dot{z}(t) + w(t) - a\frac{d}{dt}\max\{az(t) - g,\ 0\} \tag{2.1}$$

and the estimate $\|\theta_x(1,\cdot)\|_{L^2(0,T)} \le (1+a^2)\|\dot{z}\|_{L^2(0,T)} + \|w\|_{L^2(0,T)}$.

We now represent θ in the form $\theta(x,t) = \theta^{(1)}(x,t) + \theta^{(2)}(x,t)$ where $\theta^{(1)}(x,t)$ yields

$$\begin{aligned}
(1+a^2)\theta_t^{(1)} = \theta_{xx}^{(1)}, \qquad & in\ \Omega_T,\\
\theta^{(1)}(0,t) = 0, \qquad & in\ (0,T),\\
\theta_x^{(1)}(1,t) = 0, \qquad & in\ (0,T),\\
\theta^{(1)}(x,0) = \theta_0(x), \qquad & in\ (0,1),
\end{aligned}$$

and $\theta^{(2)}(x,t)$ satisfies the equations

$$\begin{aligned}
(1+a^2)\theta_t^{(2)} = \theta_{xx}^{(2)} + a\frac{d}{dt}\max\{az(t) - g,\ 0\}, \qquad & in\ \Omega_T,\\
\theta^{(2)}(0,t) = 0, \qquad & in\ (0,T),\\
\theta_x^{(2)}(1,t) = v(t), \qquad & in\ (0,T),\\
\theta^{(2)}(x,0) = 0, \qquad & in\ (0,1),
\end{aligned}$$

where $v(t)$ is the expression on the right-hand side of (2.1).

To estimate $\theta^{(2)}(\cdot, T)$ and $\theta^{(2)}(1, \cdot)$, we use results for linear parabolic equations [8] and conclude that

$$\|\theta^{(2)}(0,\cdot)\|_{L^2(0,T)} \leq \gamma_1 \left(\|v\|_{L^2(0,T)} + a^2\|\dot{z}\|_{L^2(0,T)}\right),$$
$$\|\theta^{(2)}(\cdot,T)\| \leq \gamma_2 \left(\|v\|_{L^2(0,T)} + a^2\|\dot{z}\|_{L^2(0,T)}\right)$$

for constants γ_1, γ_2 that are independent of v and z. From observability properties for linear parabolic systems [5, 7] we deduce that

$$\|\theta^{(1)}(\cdot,T)\|_{L^2(0,1)} \leq \gamma_3\|\theta_x^{(1)}(0,\cdot)\|_{L^2(0,T)} \leq \gamma_3 \left(\|\theta_x^{(2)}(0,\cdot)\|_{L^2(0,T)} + \|w\|_{L^2(0,T)}\right).$$

Hence,

$$\|\theta(\cdot,T)\|_{L^2(0,1)} \leq \|\theta^{(1)}(\cdot,T)\|_{L^2(0,1)} + \|\theta^{(2)}(\cdot,T)\|_{L^2(0,1)} \leq C_1\|\dot{z}\|_{L^2(0,T)} + C_2\|w\|_{L^2(0,T)}$$

for some C_1 and C_2. Thus, Theorem 2.2 is proved.

3 Boundary identifiability

We now turn to the study of the boundary identifiability property for the system (1.1)–(1.5). We note that if the system was linear, the boundary identifiability property would follow directly from the estimates obtained in the theorem above. Since the system under consideration is nonlinear, this question is open. In what follows we examine some cases, for which it is possible to get an estimate for the boundary conditions $\theta(1,t)$ based on the measurements (1.5).

Our first result comes directly from the estimate (2.1) and the condition (1.3).

Theorem 3.1. *Let the function $k(s)$ be continuously differentiable and $k(s) \geq \underline{k} > 0$ for all $s \in R$. Then the system* (1.1)–(1.5) *is boundary identifiable.*

Under the conditions of this theorem, we derive from (2.1) that $\|\theta(1,\cdot)\|_{L^2(0,T)} \leq \underline{k}^{-1}((1 + a^2)\|\dot{z}\|_{L^2(0,T)} + \|w\|_{L^2(0,T)})$, and the theorem is proved. However, the condition $k(s) \geq \underline{k} > 0$ is very restrictive and does not cover many typical forms of $k(s)$, for example, the case $k'(s) \leq 0$ and $|k(s)| \leq \alpha|s| + \beta$ treated in [1, 3, 4]. The next two theorems present the results related to the cases, where the constant η in (1.7) only works for special classes of measurements.

Theorem 3.2. *Let $k(s)$ be continuously differentiable and $k'(s) \leq 0$ for all s. Then there exists a constant $\eta > 0$, for which the estimate*

$$\int_0^T e^{-\lambda t^{-2}} \theta^2(1,t)dt \leq \eta(\|\dot{z}\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)}) \tag{3.1}$$

holds for any solution $\theta(x,t)$ to the problem (1.1)–(1.4) *with $\theta_0 \in H^1(0,1)$ and* $\operatorname{ess\,inf}_{t\in(0,T)} \dot{z}(t) \geq 0$.

Proof. We will use the method of Carleman's estimates for proving the theorem. Following the general idea [5], we introduce the operator $\mathcal{P} : W_2^{2,1}(\Omega_T) \mapsto L^2(\Omega_T)$, acting as follows: $\mathcal{P}y = (1+a^2)y_t - y_{xx}$. Next, we construct the function $\phi(x,t) = ((x-2)^2 - \sigma)\psi(t)$ where the constant σ is such that $(x-2)^2 - \sigma < 0$ for all $x \in [0,1]$. The function $\psi(t)$ belongs to $C^1(0,T]$, is positive, nonincreasing in $(0,T]$, and is such that

$$\psi(t) = \begin{cases} \dfrac{1}{t^2}, & t \in (0,t_0] \cup [T-t_0, T], \\ \dfrac{4-\sigma}{T^2(1-\sigma)}, & t \in [t_1, t_2] \end{cases}$$

for some $0 < t_0 < t_1 < t_2 \leq T$. For any $\rho > 0$, we define the operator $\mathcal{L}_\rho y = e^{\rho\phi(x,t)}\mathcal{P}e^{-\rho\phi(x,t)}y$ acting on an appropriate set of functions $\{y(x,t)\}$, $(x,t) \in \Omega_T$. We derive the following expression for this operator

$$\begin{aligned} \mathcal{L}_\rho y &= e^{\rho\phi(x,t)}\left((1+a^2)\partial_t - \partial_{xx}\right)e^{-\rho\phi(x,t)}y \\ &= -(1+a^2)\rho\phi_t y + (1+a^2)y_t - \rho^2(\phi_x)^2 y + (\phi_{xx})^2\rho y + 2\rho\phi_x y_x - y_{xx}. \end{aligned}$$

The differential equation (1.1) may be written as

$$\mathcal{P}u = p(t) \equiv \frac{a}{2}\frac{d}{dt}\left(a\int_0^1 \theta(\xi,t)d\xi - g + \left|a\int_0^1 \theta(\xi,t)d\xi - g\right|\right), \tag{3.2}$$

and, for the function $y(x,t) = e^{\rho\phi(x,t)}\theta(x,t)$, we obtain the equality $\mathcal{L}_\rho y = e^{\rho\phi(x,t)}p(t)$. Squaring both sides of this equation, we derive the inequality

$$\begin{aligned} &2\langle -\rho^2(\phi_x)^2 y - y_{xx}, (1+a^2)y_t + 2\rho\phi_x y_x\rangle \\ &\leq 3\left(\|(1+a^2)\rho\phi_t y\|^2_{L_2(\Omega_T)} + \|e^{\rho\phi(x,t)}p\|^2_{L_2(\Omega_T)} + \|\rho\phi^2_{xx}y\|^2_{L_2(\Omega_T)}\right) \end{aligned} \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega_T)$.

Integrating by parts, and keeping in mind that $y(x,0) = 0$, $y_x(x,0) = 0$, and $y(0,t) = 0$, $y_t(0,t) = 0$, we deduce the following expressions for the terms on the left-hand side of this inequality

$$\begin{aligned} -2\int_{\Omega_T} \rho^2(\phi_x)^2 y(1+a^2)y_t\, dx\, dt &= -\rho^2(1+a^2)\int_{\Omega_T}\phi_x^2(y^2)_t\, dxdt \\ &= -4\rho^2(1+a^2)T^{-4}\int_0^1 (x-2)^2y^2(x,T)\, dx \\ &\quad -8\rho^2(1+a^2)\int_{\Omega_T}(x-2)^2\psi(t)\psi'(t)y^2(x,t)\, dxdt, \end{aligned} \tag{3.4}$$

$$\begin{aligned} -4\rho^3\int_{\Omega_T}(\phi_x)^3 y y_x\, dxdt &= -16\rho^3\int_{\Omega_T}\psi^3(t)(x-2)^3(y^2)_x\, dxdt \\ &= 16\rho^3\int_0^T \psi^3(t)y^2(1,t)\, dt + 48\rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)\, dx\, dt, \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
-4\int_{\Omega_T} y_{xx}\rho\phi_x y_x \; dxdt &= 4\rho\int_0^T y_x^2(1,t)\psi(t)\; dt + 4\rho\int_{\Omega_T} y_x^2(x,t)\psi(t)\; dxdt \\
&-8\rho\int_0^T y_x^2(0,t)\psi(t)dt,
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
-2(1+a^2)\int_{\Omega_T} y_{xx}y_t \; dxdt &= -2(1+a^2)\int_0^T y_t y_x \,|_0^1 \; dt + (1+a^2)\int_{\Omega_T} ((y_x)^2)_t \; dxdt \\
&= -2(1+a^2)\int_0^T y_t(1,t)y_x(1,t)\; dt + (1+a^2)\int_0^1 y_x^2(x,T)\; dx \\
&= 2(1+a^2)\int_0^T e^{2\rho(1-\sigma)\psi(t)}\left[\rho(1-\sigma)\psi'(t)u(1,t) + u_t(1,t)\right] \\
&\times\left[2\rho\psi(t) + K(t)\right]u(1,t)dt + (1+a^2)\int_0^1 y_x^2(x,T)\; dx \\
&= (1+a^2)e^{2\rho(1-\sigma)\psi(T)}(2\rho\psi(T) + K(T))u^2(1,T) \\
&-(1+a^2)\int_0^T e^{2\rho(1-\sigma)\psi(t)}(2\rho\psi' + K'(t))u^2(1,t)dt + (1+a^2)\int_0^1 y_x^2(x,T)\; dx,
\end{aligned}
\tag{3.7}
$$

where we have denoted $K(t) = k(az(t) - g)$. Substituting these formulas in (3.3), we get

$$
\begin{aligned}
&\rho^3\int_0^T \psi^3(t)y^2(1,t)\; dt + \rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)\; dx\; dt + (1+a^2)\int_0^1 y_x^2(x,T)\; dx \\
&\quad +\Psi_1 + \Psi_2 + \Psi_3 \le 3a^4\|\dot z\|^2_{L^2(0,T)} + 8\rho\int_0^T y_x^2(0,t)\psi(t)dt
\end{aligned}
\tag{3.8}
$$

where

$$
\Psi_1 = 46\rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)dxdt - 4\rho^2(1+a^2)T^{-4}\int_0^1 (x-2)^2y^2(x,T)dx,
$$

$$
\Psi_2 = 15\rho^3\int_0^T \psi^3(t)y^2(1,t)\; dt - (1+a^2)\int_0^T e^{2\rho(1-\sigma)\psi(t)}K'(t)u^2(1,t)dt,
$$

$$
\Psi_3 = \rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)dxdt - 3\rho^2\int_{\Omega_T}((1+a^2)^2\phi_t^2 + \phi_{xx}^2)y^2(x,t)dxdt.
$$

It is clear that $\Psi_3 > 0$ if ρ is sufficiently large. Further, $K'(t) = k'(az(t) - g)\dot z(t) \le 0$ and, hence, $\Psi_2 > 0$. Now we examine the sign of Ψ_1. To do this we will need the following lemma.

Lemma 3.1. *There exists a constant $C > 0$ such that the inequality*

$$C \int_0^1 \theta^2(x,T)dx \le \int_{\Omega_T} \theta^2(x,t)\ dxdt + \int_0^T p^2(t)\ dt \tag{3.9}$$

holds for any solution $u(x,t)$ of (1.1)–(1.4).

Proof of Lemma 3.1. Let $\theta(x,t)$ yield the system (1.1)–(1.4). Taking $\beta > 0$, we consider the function $v(x,t) = e^{\frac{-\beta}{t^2}}\theta(x,t),\ (x,t) \in \Omega_T$. Integrating by parts we obtain

$$\begin{aligned}
\int_0^1 v^2(x,T)\ dx &= 2\int_{\Omega_T} vv_t\ dxdt = 2\int_{\Omega_T} e^{\frac{-2\beta}{t^2}}(\theta_t + 2\beta\theta t^{-3})\theta\ dxdt \\
&= 2\int_{\Omega_T} e^{\frac{-2\beta}{t^2}}((1+a^2)^{-1}\theta_{xx} + 2\beta\theta t^{-3} + (1+a^2)^{-1}p(t))\theta\ dxdt \\
&= 4\beta\int_{\Omega_T} e^{\frac{-2\beta}{t^2}}t^{-3}\theta^2\ dxdt + 2(1+a^2)^{-1}\int_0^T e^{\frac{-2\beta}{t^2}}\theta(1,t)\theta_x(1,t)\ dt \\
&-2(1+a^2)^{-1}\int_{\Omega_T} e^{\frac{-2\beta}{t^2}}\theta_x^2\ dxdt + 2(1+a^2)^{-1}\int_{\Omega_T} e^{\frac{-2\beta}{t^2}}p(t)\theta(t)dt.
\end{aligned}$$

Hence,

$$e^{-\frac{\beta}{T^2}}\int_0^1 \theta^2(x,T)dx \le \int_{\Omega_T} e^{\frac{-2\beta}{t^2}}4\beta t^{-3}\theta^2\ dxdt + \int_{\Omega_T}\theta^2 dxdt + \int_0^T p^2(t)dt. \tag{3.10}$$

Having the inequality $e^{-\frac{2\beta}{t^2}}t^{-3} \le e^{-3/2}\left(\frac{3}{4\beta}\right)^{3/2} \equiv e_0$, we can estimate the first term on the right hand side of (3.10), which results in (3.9) with $C = e^{-2\beta/T^2}(e_0+1)^{-1}$. Lemma 3.1 is proved.

Now we can estimate Ψ_1. We represent this term as the sum

$$\Psi_1 = \rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)dxdt - 4\rho^2(1+a^2)T^{-4}\int_0^1(x-2)^2y^2(x,T)dx + \Psi_{10}$$

where

$$\Psi_{10} = 45\rho^3\int_{\Omega_T}\psi^3(t)(x-2)^2y^2(x,t)dxdt - 4\rho^2(1+a^2)T^{-4}\int_0^1(x-2)^2y^2(x,T)dx.$$

We have

$$\begin{aligned}
\Psi_{10} &\ge 45\rho^3\int_{t_1}^{t_2}\int_0^1 \psi^3(t)e^{2\rho(1-\sigma)\psi(t)}\theta^2(x,t)dxdt - 16\rho^2(1+a^2)\int_0^1 y^2(x,T)dx \\
&\ge e^{2\rho(4-\sigma)T^{-2}}\left(45\rho^3\left[\frac{4-\sigma}{T^2(1-\sigma)}\right]^3\int_0^1\int_{t_1}^{t_2}\theta^2(x,t)dxdt - 16\rho^2(1+a^2)T^{-4}\int_0^1\theta^2(x,T)dx\right)
\end{aligned} \tag{3.11}$$

According to Lemma3.1, there exists a constant C_2, for which

$$C_2 \int_0^1 \theta^2(x,t_2)dx \leq \int_0^1 \int_{t_1}^{t_2} \theta^2(x,t)dxdt + \int_{t_1}^{t_2} p^2(t)dt$$

Taking $\rho > 0$, which satisfies the condition

$$46\rho^3 \left[\frac{4-\sigma}{T^2(1-\sigma)}\right]^3 C_2 > 16\rho^2(1+a^2)T^{-4} \equiv C_3,$$

we find that

$$\Psi_{10} \geq C_3 e^{2\rho(4-\sigma)T^{-2}} \left(\int_0^1 \theta^2(x,t_2)dx - C_3^{-1} e^{2\rho(\sigma-4)T^{-2}} \int_{t_1}^{t_2} p^2(t)dt - \int_0^1 \theta^2(x,T)dx\right) \tag{3.12}$$

We can get the estimate

$$\int_0^1 \theta^2(x,t_2)dx - \int_0^1 \theta^2(x,T)dx \geq -\frac{1}{2}\int_0^1 \int_{t_2}^T \theta^2(x,t)dxdt - \frac{1}{2}\int_{t_2}^T p^2(t)dt$$

by integrating by parts the equality

$$\int_0^1 \int_{t_2}^T \theta((1+a^2)\theta_t - \theta_{xx} - p(t))dt = 0.$$

Hence,

$$\Psi_{10} \geq C_3 e^{2\rho(4-\sigma)T^{-2}} \left(-\frac{1}{2}\int_0^1 \int_{t_2}^T \theta^2(x,t)dxdt - \frac{1}{2}\int_{t_2}^T f^2(t)dt - C_3^{-1} e^{2\rho(\sigma-4)T^{-2}} \int_{t_1}^{t_2} p^2(t)dt\right).$$

Then, if ρ is sufficiently large,

$$\Psi_1 > C_3 e^{2\rho(4-\sigma)T^{-2}} \left(-\frac{1}{2}\int_{t_2}^T p^2(t)dt - C_3^{-1} e^{2\rho(\sigma-4)T^{-2}} \int_{t_1}^{t_2} p^2(t)dt\right).$$

Using these estimates we deduce from (3.8) the inequality

$$\int_0^T e^{-\lambda t^{-2}} \theta^2(1,t)dt \leq \eta \left(\|\dot{z}\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)}\right) \tag{3.13}$$

with the appropriate constants η and λ. Remembering the notation (3.2), we deduce the needed inequality (3.1) and Theorem 3.2 is proved.

In much the same way as Theorem 3.2 was produced, we can derive the following result.

Theorem 3.3. *Let $\dot{z}(\cdot) \in L^\infty(0,T)$ and*

$$\max\{\|z\|_{L^\infty(0,T)},\ \|\dot{z}(\cdot)\|_{L^\infty(0,T)}\} \leq R. \tag{3.14}$$

Then there exist constants $\eta = \eta(R)$ and $\lambda = \lambda(R)$ such that

$$\int_0^T e^{-\lambda t^{-2}} \theta^2(1,t)dt \le \eta \left(\|p\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)} \right) \tag{3.15}$$

holds for any solution $\theta(x,t)$ to the problem (1.1)–(1.4) *satisfying* (3.14) *and $\theta_0 \in H^1(0,1)$.*

Remark 3.1. *Under the conditions of Theorem 3.2, the inequality*

$$\|\theta(\cdot,T)\|_{L^2(0,1)} \le \gamma_1 \left(\|p\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)} \right)$$

holds, while the conditions of Theorem 3.3 ensure the estimate

$$\|\theta(\cdot,T)\|^2_{L^2(0,1)} \le \gamma_1(R) \left(\|p\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)} \right). \tag{3.16}$$

Since $\|p\|_{L^2(0,T)} \le \|\dot{z}\|_{L^2(0,T)}$, these two inequalities are stronger then the one proved in Theorem 2.2, but the conditions under which they hold are more restrictive.

We also note that (3.16) *generalizes the estimate known for parabolic equations [5] for the case of non-homogeneous equations with non-stationary boundary conditions.*

4 The boundary measurements

Suppose a solution $\theta(x,t)$ of (1.1)–(1.4) is a classic one and, hence, θ_t, θ_{xx} are continuous in $[0,1] \times [0,T]$. Since $\theta(0,t) \equiv 0$, we derive that

$$\theta_{xx}(0,t) = -\frac{a}{2}\frac{d}{dt}\left(az(t) - g + |az(t) - g| \right). \tag{4.1}$$

According to the notation (3.2), we can write $\theta_{xx}(0,t) = p(t)$ and write the estimates (3.1) and (3.16) as follows

$$\int_0^T e^{-\lambda t^{-2}} \theta^2(1,t)dt \le \eta(R) \left(\|\theta_{xx}(0,\cdot)\|^2_{L^2(0,T)} + \|w\|^2_{L^2(0,T)} \right),$$

$$\|\theta(\cdot,T)\|^2_{L^2(0,1)} \le \gamma_1 \left(\|\theta_{xx}(0,\cdot)\|^2_{L^2(0,T)} + \|w(\cdot)\|^2_{L^2(0,T)} \right).$$

This means that, when the solution to (1.1)–(1.4) is smooth enough, the boundary measurements $\theta_x(0,t)$ and $\theta_{xx}(0,t)$ provide information which is sufficient to reconstruct both $\theta(\cdot,T)$ and $\theta(1,t)$.

5 Conclusion

In this paper, we have derived observability results for the one-dimensional nonlinear parabolic PDE describing the temperature evolution of a thermoelastic rod which comes into contact with a rigid barrier. Future work will built on these results to study identification problem for thermoelastic contact problem.

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References

[1] K.T. Andrews, P. Shi, M. Shillor, S. Wright, Thermoelastic contact with Barber's heat exchange condition, *Applied Mathematics and Optimization*, 28 (1993), 11–48.

[2] H.T. Banks, K. Kunish, *Estimation techniques for distributed parameter systems*, Boston, Basel, Berlin, Birkhäuser, 1989.

[3] J.R. Barber, Stability of thermoelastic contact, *Proc. I Mech. Intern. Congress on Tribology*, London, 1987, 981–986.

[4] J.R. Barber, J. Dundurs, M. Comninou, Stability considerations in thermoelastic contact, *J. Appl. Mech.*, 47 (1980), 871-874.

[5] O. Y. Emanuilov, Boundary controllability of parabolic equations. (Russian) *Uspekhi Mat. Nauk*, 48 (1993), no. 3(291), 211–212; translation in Russian Math. Surveys 48 (1993), no. 3, 192–194

[6] A.B. Kurzhanskii, A.Y. Khapalov, An observation theory for distributed-parameter systems, *J. Math. System. Estim. Control*, 1 (1991), 389–440.

[7] G. Lebeau, L. Robbiano, Controle exact de l'equation de la chaleur, *Comm. Partial Differential Equations*, 20 (1995), no. 1-2, 335–356.

[8] J.L. Lions, E. Magenes, *Problemes aux limites non-homogenes at applications*. 2, Paris, Dunod. 1968.