SOLUTION OF SECOND ORDER LINEARIZATION

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Abstract

For a nonlinear system with a control input, a generalized form of the homological equation can be formulated to reduce the system to its normal form. In this paper, it has been shown that for a linearly controllable system with an appropriate choice of state feedback, the generalized homological equation can be solved to give an explicit solution, of a reduced order, to the problem of second order linearization.

1 Introduction

Linearization of nonlinear dynamic system was originally investigated by Poincare[1, 2] in terms of homological equations. Krener et. al.[3] have considered a nonlinear system with a control input and showed that a generalized form of the homological equation can be formulated in this case.Devanathan[4]has shown that, for a linearly controllable system with an appropriate choice of state feedback, the system matrix can be made non-resonant.This concept is further exploited in this paper in the case of second order linearization. First, the normalizing transformation is explicitly solved for in an unique way. This is followed by the solution of the parameters of the input transformation using a system of equations which can be reduced to $(\frac{n(n-1)}{2})$ equations in n variables whose rank is $(n-1)$.This is in contrast to the result of [3] which derives a system of $[(\frac{n^2(n+1)}{2})+n^2]$ equations in $(\frac{n^2(n+1)}{2})+\frac{n(n+1)}{2}+n)$ variables with a rank of $(\frac{n^2(n+1)}{2})+\frac{n(n+1)}{2}+n-1)$. Moreover, the solution derived in this paper is in an explicit form while that of [3] is not.

2 Background

Krener et.al. [3] have extended the Poincare's result on the normalizing transformation to include the control input in a straightforward way. Consider the equation

$$\dot{x} = Ax + Gu + f_2(x) + f_3(x) + \cdots + g_1(x)u + g_2(x)u + \cdots \tag{2.1}$$

where the eigenvalues of A are non-resonant, u corresponds to a scalar input and $f_m(x)$ $(g_{m-1}(x))$ corresponds to a vector-valued polynomial containing terms of the form $x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}$, $q_i \geq 0, i = 1, 2, \cdots, n$ are integers and $\sum_{i=1}^{n} q_i = m(\sum_{i=1}^{n} q_i = m-1), m \geq 2$. Using a change of variable as in

$$x = y + h(y) \tag{2.2}$$

consider a change of input as in

$$v = (1 + \beta_1(x))u + \alpha_2(x)\,, \;\; 1 + \beta_1(x) \neq 0 \tag{2.3}$$

where $\beta_1(x)$ and $\alpha_2(x)$ correspond to terms linear and of second order in x respectively. Following Devanathan [4], it can be shown that the second order term "$f_2(x)$" and the "$g_1(x)u$" term can be removed provided the following generalized homological equations are satisfied.

$$\frac{\partial h_2(y)}{\partial y}(Ay) - Ah_2(y) + G\alpha_2(y) = f_2(y), \tag{2.4}$$

$$\frac{\partial h_2(y)}{\partial y}(Gu) + G\beta_1(y)u = g_1(y)u, \;\; \forall\, u, \tag{2.5}$$

3 State Feedback for Non-resonance

In this paper, we consider a class of systems of the form

$$\dot{\xi} = f(\xi) + g(\xi)\zeta \tag{3.1}$$

where ξ is an n-tuple vector and $f(\xi)$and $g(\xi)$ are vector fields.For simplicity, we assume a scalar input ζ. We require that the system (3.1) be linearly controllable [5],i.e., the pair (F, G) is controllable where $F = \frac{\partial f}{\partial \xi}(0)$ and $G = g(0)$, at the assumed equilibrium point at the origin. One can consider, without loss of generality, F and G to be in Brunovsky form [6]. The power series expansion of (3.1) about the origin can then be written as

$$\dot{x} = Fx + G\phi + O_1(x)^{(2)} + \gamma_1(x, \phi)^{(1)} \tag{3.2}$$

where superscript (2) corresponds to terms in x of degree greater than one, superscript (1) corresponds to terms in x of degree greater than or equal to one and x and ϕ are the transformed state and input variables respectively. To put (3.2) in the form of (2.1), where matrix A is non-resonant,we introduce state feedback as in

$$\phi = -Kx + u \tag{3.3}$$

where

$$K = [k_n, k_{n-1}, \cdots, k_2, k_1]^t \tag{3.4}$$

(3.2) then becomes

$$\dot{x} = Ax + G\,u + O(x)^{(2)} + \gamma(x, u)^{(1)} \tag{3.5}$$

where

$$A = F - GK \tag{3.6}$$

Remark 3.1 *We can choose the eigenvalues of (3.6), without loss of generality, to be real and distinct ([4]). Since, matrix A in (3.6) is in phase-variable canonical form, using a change of coordinate involving the Vandermonde matrix [7], (3.5) can be put in the form of (2.1) where matrix A is diagonal and* $G = [1, 1, \cdots, 1]^t$. *The solution of the generalized homological equations (2.4) and (2.5), in this case, results in the second order linearization.*♣

4 Equations for Second Order Linearization

Let

$$h_2(y) = [h_{21}(y), h_{22}(y), \cdots, h_{2n}(y)]^t \tag{4.1}$$

$$h_{2i}(y) = \sum_Q h_{2iQ} Y^Q, i = 1, 2, \cdots, n \tag{4.2}$$

$$Q = [q_1, q_2, \cdots, q_n] \tag{4.3}$$

$$\sum_{j=1}^{n} q_j = 2, \; 2 \geq \; q_j \geq 0, j = 1, 2, \cdots, n \tag{4.4}$$

$$Y^Q = y_1^{q_1} y_2^{q_2} \cdots y_n^{q_n} \tag{4.5}$$

$$\alpha_2(y) = \sum_Q \alpha_{2Q} Y^Q \tag{4.6}$$

$$f_{2i}(y) = \sum_Q f_{2iQ} Y^Q, i = 1, 2, \cdots, n \tag{4.7}$$

$$f_2 = [f_{21}, f_{22}, \cdots, \; f_{2n}]^t \tag{4.8}$$

Further, given Q as in (4.3),let

$$Q_j = [q_1, q_2, \cdots, q_{j-1}, (q_j - 1), q_{j+1}, \cdots, q_n], \; q_j = 2, j \in (1, 2, \cdots, n) \tag{4.9}$$

and for $j, k \in (1, 2, \cdots, n)$.

$$Q_j^k = \begin{cases} [q_1, q_2, \cdots, q_{k-1}, (q_k + 1), q_{k+1}, \cdots, q_{j-1}, (q_j - 1), q_{j+1}, \cdots, q_n], & j > k \\ [q_1, q_2, \cdots, q_{j-1}, (q_j - 1), q_{j+1}, \cdots, q_{k-1}, (q_k + 1), q_{k+1}, \cdots, q_n], & k > j \\ Q, & k = j, \end{cases} \tag{4.10}$$

$q_j = 2, q_k = 0, j, k \in (1, 2, \cdots, n)$

Let

$$\beta_1(y) = \sum_{Q_j} b_{Q_j} Y^{Q_j} \tag{4.11}$$

$$g_1 = [g_{11}, g_{12}, \cdots, , g_{1n}]^t \tag{4.12}$$

$$g_{1i} = \sum_{Q_j} g_{1iQ_j} Y^{Q_j}, i = 1, 2, \cdots, n \tag{4.13}$$

Then, combining Equations (2.4) and (2.5), one can get for $q_j = 2, q_k = 0, j = 1, 2, \cdots, n; k \in (1, 2, \cdots, n); i = 1, 2, \cdots, n$

$$q_j(< Q, \Lambda > -\lambda_i)^{-1}(\alpha_{2Q}) + \sum_{k \neq j} [(q_k + 1)(< Q_j^k, \Lambda > -\lambda_i)^{-1}(\alpha_{2Q_j^k})] - b_{Q_j} = d_{ij} \tag{4.14}$$

where d_{ij} is a known quantity given by

$$d_{ij} = q_j(< Q, \Lambda > -\lambda_i)^{-1}(f_{2iQ}) + \sum_{k \neq j} [(q_k + 1)(< Q_j^k, \Lambda > -\lambda_i)^{-1}(f_{2iQ_j^k})] - g_{1iQj} \tag{4.15}$$

The unknown parameters α_{2Q}, $\alpha_{2Q_j^k}$ and b_{Q_j} in the input transformation are to be solved from the equation (4.14). Using the simplified notations as follows:

$$\alpha_{2Q_j^k} = \alpha_{jk}; \quad \alpha_{2Q} = \alpha_{jj} \ and \ b_{Q_j} = b_j; q_j = 2, q_k = 0, \quad j, k \in (1, 2, \cdots, n) \ ; i = 1, 2, \cdots, n \tag{4.16}$$

Equation (4.14) can be put in the form

$$d_j = C_j \, \alpha_j - L \, b_j \tag{4.17}$$

where, for $j = 1, 2, \cdots, n$,

$$d_j = [d_{1j}, d_{2j}, \cdots, d_{nj}]^t \tag{4.18}$$

$$\alpha_j = [\alpha_{1j}, \alpha_{2j}, \cdots, \alpha_{nj}]^t \tag{4.19}$$

$$L = [1, 1, 1, \cdots, 1]^t \tag{4.20}$$

(4.21)

and C_j is an $n \times n$ matrix whose (i, k)-th element is given by

$$C_j(i, k) = \begin{cases} \frac{1}{\lambda_j + \lambda_k - \lambda_i}, & j \neq k \\ (\frac{2}{2\lambda_j - \lambda_i}), & j = k \end{cases} \tag{4.22}$$

$i, k = 1, 2, \cdots, n; \ j = 1, 2, \cdots, n$ The following two results are stated without proof.

Theorem 4.1 *For each $j = 1, 2, \cdots, n$, the $n \times n$ matrix C_j is nonsingular.* ♣

Representing C_j in terms of its columns as

$$C_j = [c_{j1}, c_{j2}, \cdots, c_{NJ}] \tag{4.23}$$

and noting that
$c_{jk} = c_{kj}, \alpha_{jk} = \alpha_{kj}, k, j \in (1, 2, \cdots, n)$
Equation (4.17) can now be put in the form

$$d = [D \, B] \begin{bmatrix} \alpha \\ b \end{bmatrix} \tag{4.24}$$

where d is a n^2-tuple vector given as

$$d = [d_1^t, d_2^t, \cdots, d_n^t]^t \tag{4.25}$$

where $d_j, j \in (1, 2, \cdots, n)$ is an n-tuple vector given as in (4.18). α is a $\frac{n(n+1)}{2}$ -tuple vector whose elements are $\alpha_{jk}, j = 1, 2, \cdots, n, k = j, j+1, \cdots, n$. b is n-tuple vector given as

$$b = [b_1, b_2, \cdots, b_n]^t \tag{4.26}$$

D is a (n^2 x $\frac{n(n+1)}{2}$) matrix which can be explicitly expressed in terms of the elements $c_{jk}, j, k \in (1, 2, \cdots, n)$. B is a $n^2 \times n$ matrix put in block diagonal form as

$$B = dig[-L, -L, \cdots, -L] \tag{4.27}$$

Theorem 4.2 *The rank of the $n^2 \times (\frac{n(n+1)}{2} + n)$ matrix [D B] is $(\frac{n(n+1)}{2} + n - 1)$. Moreover, equation (4.24) further reduces to a system of $\frac{n(n-1)}{2}$ linear equations in b_j, $j = 1, 2, \cdots, n$ with rank $(n-1)$ whose solution corresponds to the solution of second order linearization.*♣

5 Example

We illustrate the complete procedure through an example. Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0.5x_1^2 \\ -x_1 x_3 \\ 0.5x_2^2 \end{bmatrix} + \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} u + O(x)^{(3)} + O'(x)^{(2)} u \tag{5.1}$$

where

$$x = [x_1, x_2, x_3]^t, \tag{5.2}$$

and superscript (3)(superscript (2)) corresponds to terms in x of degree $\geq 3(2)$. The state feedback $u = v - Kx$ applied to the above system where $K = [2, \frac{29}{6}, \frac{23}{6}]^t$ results in the eigenvalues $-1, \frac{-4}{3}, \frac{-3}{2}$ leading to the system

$$\dot{z} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & \frac{-4}{3} & 0 \\ 0 & 0 & \frac{-3}{2} \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} v +$$
$$\begin{bmatrix} 18z_1^2 + \frac{4108}{3} z_2^2 + \frac{2421}{2} z_3^2 - \frac{946}{3} z_1 z_2 - 2587 z_2 z_3 + \frac{609}{2} z_1 z_3 \\ 27z_1^2 + 1523 z_2^2 + \frac{2565}{2} z_3^2 - 410 z_1 z_2 - \frac{5619}{2} z_2 z_3 + 387 z_1 z_3 \\ 30z_1^2 + \frac{4648}{3} z_2^2 + 1284 z_3^2 - 436 z_1 z_2 - \frac{8506}{3} z_2 z_3 + 408 z_1 z_3 \end{bmatrix}$$
$$- \begin{bmatrix} 12z_1 + 56z_2 + 45z_3 \\ 11z_1 + 57z_2 + 45z_3 \\ 12z_1 + 56z_2 + 44z_3 \end{bmatrix} v + O''(z)^{(3)} + O'''(z)^{(2)} v \tag{5.3}$$

Formulating the equation(4.17), j=1,2,3

$$\begin{bmatrix} 2 & \frac{3}{4} & \frac{2}{3} & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & \frac{6}{7} & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & \frac{6}{5} & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{4} & 0 & \frac{6}{5} & \frac{6}{11} & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{2}{3} & 0 & 0 & 1 & 0 \\ 0 & \frac{6}{5} & 0 & \frac{12}{7} & \frac{3}{4} & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & \frac{6}{11} & 1 & 0 & 0 & 1 \\ 0 & 0 & \frac{6}{7} & 0 & \frac{2}{3} & \frac{6}{5} & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{3}{4} & \frac{4}{3} & 0 & 0 & 1 \end{bmatrix} \delta = \begin{bmatrix} \frac{-17}{2} \\ \frac{-65}{7} \\ \frac{-36}{5} \\ \frac{5677}{110} \\ \frac{117}{2} \\ \frac{623}{10} \\ \frac{-937}{22} \\ \frac{-331}{7} \\ \frac{-101}{2} \end{bmatrix} \tag{5.4}$$

where

$$\delta = [\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{22}, \alpha_{23}, \alpha_{33}, b_1, b_2, b_3]^t \tag{5.5}$$

The matrix on the OHS of equation (5.4) has rank 8 only.The parameters $\alpha_{jk}, j, k \in (1, 2, 3)$ can be uniquely expressed in terms of $b_j, j = 1, 2, 3$.Hence, for reasons of consistency of the values of $b_j, j = 1, 2, 3$, we have,

$$\begin{bmatrix} 3 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 34 \\ 36 \\ -40 \end{bmatrix} \tag{5.6}$$

In the above, one of the $b_j, j \in (1, 2, 3)$ is arbitrary. Putting $b_3 = 0, b_1 = 18$ and $b_2 = -20$, the normalizing and the input transformations are computed respectively as

$$z = h_2(y) + y \tag{5.7}$$

where

$$h_2(y) = \begin{bmatrix} -15y_1^2 + 352y_1y_2 - 351y_1y_3 - 1215y_2^2 + 2154y_2y3 - 924y3^2 \\ -36y_1^2 + 564y_1y_2 - 522y_1y_3 - 1634y_2^2 + 2781y_2y3 - 1152y3^2 \\ -54y_1^2 + 708y_1y_2 - 630y_1y_3 - 1890y_2^2 + 3148y_2y3 - 1890y3^2 \end{bmatrix} \tag{5.8}$$

$$\begin{aligned} w = (1 + 18z_1 - 20z_2)v + 3z_1^2 + 154z_1z_2 - 222z_1z_3 \\ -\frac{1967}{3}z_2^2 + 1362z_2z_3 - \frac{1275}{2}z_3^2 \end{aligned} \tag{5.9}$$

The original system (5.1) then reduces to

$$\dot{y} = dig(-1, -\frac{4}{3}, -\frac{3}{2})y + [111]^t\, w + O_1^{(3)}(y) + O_1^{\prime(2)}(y, w) \tag{5.10}$$

where the second degree terms in y and the product terms of the form "(linear term in y)w" are completely removed.

6 Conclusions

The problem of transforming a dynamic system into its normal form is an old one. In the reduction to the normal form, the role played by the resonance of the system matrix is well-known. In the context of linearly controllable system, however, the control input can be used to provide state feedback so as to ensure the non-resonance of the system matrix. This, in turn, can lead to the explicit solution of second order linearization as shown in this paper.The future work is to extend the linearization technique (for a linearly controllable system) to an arbitrary order.

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