On a unitary model for two-time parameter stationary processes

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Abstract

The basic problem in prediction theory is to estimate a certain, desired, behavior of the phenomenon under study (in our case a stationary process) using the information already obtained about it. In order to accomplish this we should separate completely the deterministic part from the part corrupted by noises. This is done via a Wold-type decomposition. In the case of a two-time parameter stationary process we characterize its different parts in some certain Wold-type decompositions in terms of a unitary model given by Berger, Coburn and Lebow for the commuting isometric pair associated to the process.

KEYWORDS: stationary process, complete correlated action, white noise, moving average, deterministic, purely non-deterministic, prediction, operator model.

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1 Introduction

In the study of stationary processes an essential role is played by various factorization theorems by analytic functions. As proved, for example, by Wiener and Masani ([17], [18]) or Helson and Lowdenslager ([7], [8]), in the finite multivariate case it is necessary to use factorization theorems by matrix-valued analytic functions. It becomes natural to suppose that factorization theorems by operator-valued analytic functions have an important contribution in the infinite multivariate prediction problems. Another result, of geometric type, namely the Wold decompostion theorem in its various (operator) variants played an important role in the development of the prediction theory. We should also mention that the Sz.-Nagy-Foiaş functional model ([10]) implicitely provides specific properties to prediction theory methods, properties that allowed to find a suitable operator background for infinite variate prediction problems.

This fruitful direction, emphasized by powerful tools provided by the operator theory, is also followed by our paper. More precisely, after the necessary preliminaries, to any discrete two-time parameter stationary process f we associate an isometric pair on the "past and present" Hilbert space (isometric pair which is completely non-unitary if f is purely non-deterministic). The so-called "unitary model" ([1],[2]) associated to this bi-isometry characterize important particular classes of stationary processes as we can observe in the last section of the paper.

We should finally mention that the paper continues and completes the work of the author and his collaborators on the structure of two-time parameter stationary processes presented in [5], [6], [11] or [12].

2 Complete Correlated Actions

The prediction problems for stationary processes consist in finding information about the process in the next moment by using all the knowledge about it up to the present. In order to accomplish this, we must perform a number of specific experiments on the process. In the model proposed by Suciu and Valuşescu ([15],[16]), the authors consider the action of operators from $\mathcal{L}(\mathcal{E})$ ($\mathcal{E}$ is a Hilbert space) on a right $\mathcal{L}(\mathcal{E})$-module H. The experiment results are measured in a system intimately linked by the metrics of the space, in the context of a complete correlated action.

We have to introduce some new notions:

Definition 2.1. A *correlation of the* $\mathcal{L}(\mathcal{E})$*-action on* H is just an "inner product" on H,

$$H \times H \ni (h, g) \mapsto \Gamma(h, g) \in \mathcal{L}(\mathcal{E}),$$

that is

- $\Gamma(h, h) \geq 0,\ h \in H$; $\Gamma(h, h) = 0$ iff $h = 0$;
- $\Gamma(h, g)^* = \Gamma(g, h),\ h, g \in H$;
- $\Gamma(h, g_1 + g_2) = \Gamma(h, g_1) + \Gamma(h, g_2),\ h, g_1, g_2 \in H$;
- $\Gamma(h, gA) = \Gamma(h, g)A,\ h, g \in H,\ A \in \mathcal{L}(\mathcal{E})$.

The triple $\{\mathcal{E}, H, \Gamma\}$ is said to be the *correlated action of* $\mathcal{L}(\mathcal{E})$ *on* H. H is the *state space* of the correlated action, while $\mathcal{E}$ is its *parameter space*.

Example 2.1. The *operator model* $\{\mathcal{E}, \mathcal{L}(\mathcal{E}, \mathcal{K}), \Gamma_m\}$ is a correlated action of $\mathcal{L}(\mathcal{E})$ on $\mathcal{L}(\mathcal{E}, \mathcal{K})$ ($\mathcal{K}$ is a Hilbert space) having $\mathcal{L}(\mathcal{E}, \mathcal{K})$ as the state space, $\mathcal{E}$ as the parameter space and Γ_m,

$$\Gamma_m(S, T) = S^*T,\ S, T \in \mathcal{L}(\mathcal{E}, \mathcal{K})$$

as the correlation.

Theorem 2.1 ([16]). *Any correlated action (c.a.)* $\{\mathcal{E}, H, \Gamma\}$ *can be embedded in an operator model modulo a unique* $\mathcal{L}(\mathcal{E})$*-module map*

$$H \ni h \mapsto V_h \in \mathcal{L}(\mathcal{E}, \mathcal{K}) \tag{2.1}$$

which verifies

(i) $\Gamma(h, g) = V_h^* V_g,\ h, g \in H$;

(ii) $\mathcal{K} = \bigvee_{h \in H} V_h \mathcal{E}$.

$\mathcal{K}$ is said to be the *measuring space* of the c.a. If the embedding (2.1) is onto the c.a. is said to be *complete* (c.c.a.).

3 Discrete Two-Time Parameter Stationary Processes

A *discrete two-time parameter stationary process* is just a sequence $f = \{f_n\}_{n\in\mathbb{Z}^2} \subset H$ such that

$$\Gamma(f_n, f_m) = \Gamma(f_{n-m}, f_{(0,0)}),\ n, m \in \mathbb{Z}^2.$$

We can mention some **important spaces** attached to f:

- the *measuring space* $\mathcal{K}^f = \bigvee_{n\in\mathbb{Z}^2} V_{f_n}\mathcal{E}$;
- the *past and present* up to the moment $n \in \mathbb{Z}^2$, $\mathcal{K}^f_n = \bigvee_{k\leq n} V_{f_k}\mathcal{E}$;
- the *past* up to the moment $n \in \mathbb{Z}^2$, $\mathcal{K}^{f,0}_n = \bigvee_{k\leq n, k\neq n} V_{f_k}\mathcal{E}$;
- the *distant past* $\mathcal{K}^f_{-\infty} = \bigcap_{n\in\mathbb{Z}^2} \mathcal{K}^f_n$.

Some **remarkable processes** are defined below:
f is said to be

- *white noise* if $\Gamma(f_n, f_m) = 0$, for $n \neq m$;
- a *moving average of a white noise* g if f contains g (i.e. $\Gamma(f_n, g_m)$ depends only on the difference $n - m$, that is f and g are cross-correlated, $\Gamma(f_n, g_m) = 0$ for $n - m \notin -\mathbb{Z}^2_+$, $V_{g_{(0,0)}}\mathcal{E} \subset \mathcal{K}^f_{(0,0)}$ and $\operatorname{Re}\Gamma(f_n - g_n, g_n) \geq 0$, for all n) and $\mathcal{K}^f_{(0,0)} = \mathcal{K}^g_{(0,0)}$;
- the *predictable part of* f is the process $\tilde{f} = \{\tilde{f}_n\}_{n\in\mathbb{Z}^2}$ given by

$$V_{\tilde{f}_n} = P_{\mathcal{K}^{f,0}_n} V_{f_n},\ n \in \mathbb{Z}^2;$$

- *deterministic* if $f = \tilde{f}$;
- *purely non-deterministic* if $\mathcal{K}^f_{-\infty} = \{0\}$;
- *i-deterministic* if

$$\mathcal{K}^f_{(0,0)} = \mathcal{K}^i_{-\infty} := \bigcap_{n\leq 0} \mathcal{K}^f_{n(2-i,i-1)},\ i = 1, 2;$$

- *i-purely non-deterministic* if $\mathcal{K}^i_{-\infty} = \{0\},\ i = 1, 2.$

The following theorem allows us to distinguish another important classes of stationary processes:

Theorem 3.1 ([4]). *There is a four-fold decomposition of f,*

$$f = f^d + f^s + f^m + f^e$$

into Γ-orthogonal processes cross-correlated with f and each to another such that
(a) f^d is a deterministic process;
(b) f^s, f^m, f^e are purely non-deterministic and:
(i) f^s is a moving average of a maximal white noise contained in f;

(ii) *there is a wandering subspace* $\mathcal{L}$ *for the bi-shift operator associated to* $f:\ U_f = (U_f^{(1)}, U_f^{(2)})$, $U_f^{(i)} V_{f_n} = V_{f_{n+(2-i,i-1)}}$, $i = 1, 2$, *orthogonal to* $\mathcal{K}^f_{(0,0)}$ *and such that*

– $V_{f_n^m} \in \bigoplus_{k\in\mathbb{Z}^2} U_f^k \mathcal{L},\ n \in \mathbb{Z}^2$;

– $V_{f_n^m} \in \bigoplus_{k\notin\mathbb{Z}_+^2} U_f^k \mathcal{L},\ n \in -\mathbb{Z}_+^2$;

(iii) f^e *cannot be decomposed into an orthogonal sum of processes of the form* (i) *or* (ii).

Remark 3.1. As observed in the theorem above, every discrete two-time parameter stationary process f can be uniquely decomposed into the orthogonal sum between a deterministic process f^d and a purely non-deterministic one, namely $f^s + f^m + f^e$. By the prediction theory point of view we are only interested into the study of the purely non-deterministic part as we shall do in the following.

We can define **other remarkable processes**:
f is said to be

- *bi-shift process* if $f = f^s$;
- *modified bi-shift process* if $f = f^m$;
- *ultraevanescent process* if $f = f^e$.

The *prediction error operator* is defined as

$$f \mapsto \Delta[f] := \Gamma(f_0 - \tilde{f}_0, f_0 - \tilde{f}_0).$$

It is easy to see that f is deterministic iff $\Delta[f] = 0$ and $\Delta[f] = \Delta[f^s] + \Delta[f^m] + \Delta[f^e]$. In other words, the computation of the prediction error operator for a stationary process can be reduced into the computation of such operators for the three particular classes of purely non-deterministic stationary processes mentioned above (for which there are some certain computation formulas [4],[5]).

4 The Model

Berger, Coburn and Lebow developed a unitary model for completely non-unitary commuting isometric pairs:

Theorem 4.1 ([1]; to see also [2]). *Let* $V = (V_1, V_2)$ *be a completely non-unitary isometric commuting pair on a Hilbert space. Then there exist a unitary operator* U *and an orthogonal projection* P*, both acting on the same Hilbert space* $\mathcal{F}$*, such that* V *is unitarily equivalent with the isometric pair* $W = (W_1, W_2)$ *on* $H^2(\mathcal{F})$ *given by the inner functions*

$$W_1(z) = U[zP + (I - P)]$$
$$W_2(z) = [P + z(I - P)]U^*, \quad z \in \mathbb{T}.$$

The triple $\{\mathcal{F}, U, P\}$ is uniquely determined by V (up to a unitary equivalence) and is said to be the *unitary model* associated to V.

Given a purely non-deterministic discrete two-time parameter stationary process f in a c.c.a. $\{\mathcal{E}, H, \Gamma\}$, we can consider $\{\mathcal{F}, U, P\}$ as the unitary model associated to the isometric pair $U_f^*|_{\mathcal{K}^f_{(0,0)}}$ (which is completely non unitary since f is purely non-deterministic). It will be called the *unitary model* associated to f.

In the following some remarkable processes are characterized in terms of their unitary models. The proofs are similar to the ones given in [13],[14] and [3].

Theorem 4.2. *Let f be a discrete two-time parameter stationary process in a c.c.a. and $\{\mathcal{F}, U, P\}$ be the unitary model associated to its purely non-deterministic part. Then*

(i) *f is 1-purely non-deterministic iff $PU = PUP$ and $U|_{\ker P}$ is a shift;*

(ii) *f is 2-purely non-deterministic iff $PU = PUP$ and $U^*|_{\operatorname{ran} P}$ is a shift;*

(iii) *f is 1-deterministic and 2-purely non-deterministic iff $P = 0$;*

(iv) *f is 1-purely non-deterministic and 2-deterministic iff $P = I$;*

(v) *$f = f^{pd} + f^{dp}$ (the orthogonal sum between processes of the form (iii) and (iv)) iff $PU = UP$.*

As regarding the purely non-deterministic parts mentioned in Theorem 3.1 we obtain:

Theorem 4.3. *Let f be as above. Then*

(vi) *f is a moving average of a maximal white noise contained in f iff f is a bi-shift process iff $PU = PUP$ and U is a bilateral shift with defect space $U\operatorname{ran} P \cap \ker P$;*

(vii) *f is a modified bi-shift process iff $PU = PUP$ and U is a bilateral shift with defect space $U^*\operatorname{ran} P \cap \ker P$;*

$(viii)$ *$f = f^m + f^e$ iff $\mathcal{R} = \bigcap_{n\geq 0} \ker[(I-P)U^{*n+1}] \cap \ker(PU^n) = \{0\}$;*

(ix) *$f = f^s + f^e$ iff $\mathcal{L} = \bigcap_{n\geq 0} \ker[(I-P)U^{n+1}] \cap \ker(PU^{*n}) = \{0\}$;*

(x) *f is an ultraevanescent process iff $\mathcal{R} = \mathcal{L} = \{0\}$;*

Example 4.1. One of the well-known methods to build two-time parameter stationary processes f is to start from two one-time parameter stationary processes. More precisely, if f is purely non-deterministic, it holds:

f is the orthogonal sum between two one-time parameter discrete stationary processes (i.e. $f_{(m,n)} = x_m + y_n,\ (m,n) \in \mathbb{Z}^2$) iff $U = I$.

Corresponding results can be obtained also for the so-called harmonizable discrete two-time parameter processes, by using the dilation of such a process to a stationary one ([9]).

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