Controllabilities and Stabilities of switched Systems (with applications to the Quantum Systems)

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Abstract

We study various stabilities and controllabilities of linear switched systems , including those appearing in the quantum computations context . A number of new results and connections is presented , most of them with proofs .

1 Main Definitions and Problems

Let us consider a set $S = \{A_\gamma : \gamma \in K\}, A_\gamma : C^n \rightarrow C^n$. I.e., S is a set of complex $n \times n$ matrices , K is an index set.
Recall that linear discrete inclusion $LDI(S)$ is a set of discrete dynamical systems

$$x_{i+1} = A_{R(i)} x_i; i \geq 0, R : N \rightarrow K. \tag{1}$$

Correspondingly , linear continuous inclusion $LCI(S)$ is a set of continuous time dynamical systems

$$\dot{X(t)} = A_{R(t)} X(t), \tag{2}$$

where $R(.)$ is a piece-wise continuous from the right switching rule .
$LDI(S)(LCI(S))$ is called Absolutely Asymptoticaly Stable (AAS) if all trajectories in (1) ((2)) converge to zero . $LDI(S)(LCI(S))$ is called Switching Asymptoticaly Stable (SAS) if there exists at least one switching rule such that the corresponding time variant system is Asymptoticaly Stable .
It was shown in [1] , [2] that if the set S is bounded then the convergence is in fact uniform , moreover $LDI(S)$ is (AAS) iff there exists a norm $\|.\|_d$ on C^n such that the induced norms $\|A_\gamma\| \leq a < 1$ for all $\gamma \in K$ ([1]) ;
$LCI(S)$ is (AAS) iff there exists a norm $\|.\|_c$ on C^n such that the induced norms $\|exp(A_\gamma t) \leq exp(at)\|$ for all $\gamma \in K, t \geq 0$ and some $a < 0$ (see , for example , [2]) .
I will mostly consider in this paper the case when it is known in advance that there exists a norm $\|.\|_*$ on C^n such that the induced norms $\|A_\gamma\| \leq 1$ for all $\gamma \in K$ in the discrete case and $\|exp(A_\gamma t) \leq 1\|$ for all $\gamma \in K, t \geq 0$ in the continuous case .
Let us call such norms as a priori. In the discrete case , if a priori norm is "good" , say polytope or Hilbert , then , at least , there are finite algorithms to check (AAS) property . But even for l_1 norm this desision problem is NP-HARD [2] . One of the main results of this paper is that in the continuous case if a priori norm is polytope then $LCI(S)$ is (AAS) iff each $A_\gamma, \gamma \in K$ is Hurwitz . I will give also a generalization of this result for so called Lindblad's operators introduced first in the Quantum Mechanics context.
I will give a very powerful sufficient condition for Hilbert a priori norms in the continuous case . A surprising feature of this result is that the proof is "infinite dimensional" though the result

itself is finite dimensional .

Definition 1.1: Let us denote linear space (over complex numbers) of $N \times N$ complex matrices as $M(N)$. A positive semidefinite matrix $\rho_{A,B} : C^N \otimes C^N \rightarrow C^N \otimes C^N$ is called bipartite unnormalized density matrix
(**BUDM**) , if $tr(\rho_{A,B}) = 1$ then this $\rho_{A,B}$ is called bipartite density matrix .
It is convinient to represent bipartite $\rho_{A,B} = \rho(i_1, i_2, j_1, j_2)$ as the following block matrix :

$$\rho_{A,B} = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,N} \\ A_{2,1} & A_{2,2} & \dots & A_{2,N} \\ \dots & \dots & \dots & \dots \\ A_{N,1} & A_{N,2} & \dots & A_{N,N} \end{pmatrix}, \tag{3}$$

where $A_{i_1,j_1} =: \{\rho(i_1, i_2, j_1, j_2) : 1 \leq i_2, j_2 \leq N\}, 1 \leq i_1, j_1 \leq N$.
A (**BUDM**) ρ called **separable** if

$$\rho = \rho_{(X,Y)} =: \sum_{1 \leq i \leq K} x_i x_i^\dagger \otimes y_i y_i^\dagger, \tag{4}$$

and **entangled** otherwise .
If vectors $x_i, y_i; 1 \leq i \leq K$ in (6) are real then ρ is called **real separable** .
A linear operator $T : M(N) \rightarrow M(N)$ called positive if $T(X) \succeq 0$ for all $X \succeq 0$, and dominant positive if $T(X) \succeq \alpha X$ for all $X \succeq 0$ and some $\alpha > 0$. A positive operator T is called completely positive if

$$T(X) = \sum_{1 \leq i \leq N^2} A_i X A_i^\dagger; A_i, X \in M(N) \tag{5}$$

Choi's representation of linear operator $T : M(N) \rightarrow M(N)$ is a block matrix $CH(T)_{i,j} =: T(e_i e_j^\dagger)$. Dual to T respect to the inner product $< X, Y >= tr(XY^\dagger)$ is denoted as T^*. Very usefull and easy Choi's result [6] states that T is completely positive iff $CH(T)$ is (**BUDM**) . Using this natural (linear) correspondence between completely positive operators and (**BUDM**) , we will freely "transfer" properties of (**BUDM**) to completely positive operators . For example , a linear operator T is called separable iff $CH(T)$ is separable , i.e.

$$T(Z) = T_{(X,Y)}(Z) = \sum_{1 \leq i \leq K} x_i y_i^\dagger Z y_i x_i^\dagger \tag{6}$$

Or equivalently , for some $A_i, B_i \succeq 0$

$$T(Z) = \sum_{1 \leq i \leq K} A_i tr(Z B_i) \tag{7}$$

A positive operator T is called stochastic (substochastic) if $I = T^*(I)$ $(I \succeq T^*(I))$ ∎

I will study in this paper quantum linear discrete inclusions , i.e. $LDI(S)$ where S is a set of positive substochastic operators . These linear discrete inclusions are motivated by the models of quantum computation where some "information" leaks to the environment . From the

other hand , quantum linear discrete inclusions generalize $LDI(R)$, where R is either a set of standard substochastic matrices or a set of matrices which are nonstrict contractions respect to some Hilbert norm .
Other generalizations of $LDI(R)$, with R being a set of matrices which are nonstrict contractions respect to some Hilbert norm , which I will study are discrete and continuous time controlled switched systems :

$$x(n+1) = A_{t(n)}x(n) + B_{t(n)}u(n)$$

, where all the matrices involved act on some linear finite-dimensional space X.
Here $t : N \to 1, 2, .., k$ is a switching rule. There are two natural notions of controllability in this framework:
1. Absolute, i.e. corresponding time-variant systems are controllable for all switching rules (corresponds to the (AAS) property).
2. Switching controllability, i.e. there exists at least one switching rule which produces a controllable time-variant system (corresponds to the (SAS) property).
Similarly one defines these notions for the continuous time case and piece-wise constant switching rules.
Notice that the discrete time case make sence for systems over finite fields.

2 General framework : monotone automatons

Let us now consider a partially ordered set $\mathbf{X}$ with unique maximum element $\mathbf{1}$ and unique minimal element $\mathbf{0}$. We will call such set Nether if it does contain infinite strictly decreasing chains and N-Nether if does not contain strictly decreasing chains of length N. For a given finite set of maps

$$G = \{f_i : \mathbf{X} \to \mathbf{X}; \mathbf{1} \leq \mathbf{i} \leq \mathbf{k}\}$$

we define a language in alphabet $\{1, ..., k\}$ as $L_G = \{\omega : f_\omega(\mathbf{0}) < \mathbf{1}\}$.
Here if the word $\omega = \omega_1 ... \omega_m, \omega_i \in \{1, ..., k\}$ then $f_\omega =: f_{\omega_1} ... f_{\omega_m}$.
We will call a tuple $\{X, G\}$ a monotone automaton if all maps in G are monotone (respect to the order in $\mathbf{X}$ and $f_i(\mathbf{1}) = \mathbf{1}, \mathbf{1} \leq \mathbf{i} \leq \mathbf{k}$. An equivalent constructure corresponds to the case when $\mathbf{1}$ and $\mathbf{0}$ are switched . Let consider a finite alphabet $\{1, ..., k\}$. For a given sequence of words $\omega^i : i \geq 0$ define the following recursion on words:

$$r(0) = \omega^0, r(n+1) = r(n)\omega^{n+1}r(n).$$

We will call words which can be obtained in n steps of such recursion as n-pumped words. Here products of words is defined in a standard way as their concatenation. A word ω^1 is a subword of a word ω^2 if $\omega^2 = \alpha\omega^1\beta$ for some words α and β. The following result was proved in [5] and reproved in [3]. There is really easy proof of it, based on van der Waerden result about existence of infinite arithmetic progressions in one of "lumps" of a finite partition of a set of nonnegative integers .

Lemma 2.1: *1. Any infinite word in alphabet* $\{1, ..., k\}$ *contains infinitely-pumped subword.*
2. There exists (very rapidly growing) function $F(N, k)$ *such that any word of length* $F(N, k)$ *contains* N*-pumped word.*

In fact the second statement easily follows from the first one via a compactness argument.

The following lemma is a direct corollary of Lemma 2.1.

Lemma 2.2: *Suppose that* f_i *are monotonic for all* $1 \leq i \leq k$ *and the language* L_G *is infinite (contains infinite number of words). Then the following statements hold.*
1. If $\mathbf{X}$ *is Nether and the language* L_G *is infinite (contains infinite number of words) then there exists a nontrivial* $\mathbf{x} \in \mathbf{X}$ *(i.e.* $\mathbf{0} < \mathbf{x} < \mathbf{1}$*) and a word* ω *such that* $f_\omega(\mathbf{x}) = \mathbf{x}$*.*
2. If $\mathbf{X}$ *is* N*-Nether and the language* L_G *contains a word of length* $F(N, k)$ *then there exists a nontrivial* $\mathbf{x} \in \mathbf{X}$ *(i.e.* $\mathbf{0} < \mathbf{x} < \mathbf{1}$*) and a word* ω *such that* $f_\omega(\mathbf{x}) = \mathbf{x}$*.*

3 Absolute asymptotic stability of quantum linear discrete inclusions

Let us consider a quantum linear discrete inclusion $LDI(S)$, where $S = \{T_1, ..., T_k\}$ is a finite set of positive substochastic operators , $T_i : M(N) \rightarrow M(N)$. We want to associate with $LDI(S)$ a monotone automaton $\{X, G\}$ with X being partially ordered N-Nether set . In this case X is a partially ordered set of linear subspaces of C^N and the corresponding monotone maps are defined as follows :

$$f_i(Z) = Ker(T_i^*(Pr(Z)) - I); 1 \leq i \leq k, \tag{8}$$

where $Pr(Z)$ denotes an orthogonal projector on the linear subspace Z. The next proposition is easy to prove.

Proposition 3.1: *Define a language* $L_S = \{\omega : f_\omega(C^N) \neq \{0\}\}$*. Then* $LDI(S)$ *is(AAS) iff* L *is finite.*

Corollary 3.2: *There is an algorithm to check whether given given quantum linear discrete inclusion* $LDI(S)$ *is(AAS), where* $S = \{T_1, ..., T_k\}$ *is a finite set of positive substochastic operators ,* $T_i : M(N) \rightarrow M(N)$ *.*

Quantum linear discrete inclusions have a natural a priori norm which is so called trace norm , i.e. $\|X\|_{tr} =: tr((XX^*)^{\frac{1}{2}})$. The language L_S above is a particular example of languages introduced in [1] : for a finite set of finite square matrices $S = \{A_1, ..., A_k\}$ with a priori norm $\|.\|$ define a language $L_{S,\|.\|} = \{\omega : \|A_\omega\| = 1\}$. If a apriori norm $\|.\|$ is polytope then $L_{S,\|.\|}$ is a regular language with an upper bound on the number of states of corresponding finite automaton depending only on a apriori norm $\|.\|$ (not depending on the size of S) [1]. One of corollaries of this result is the following proposition .

Proposition 3.3: *Suppose that S is a compact set of finite square matrices having a polytope a apriori norm . Then $LDI(S)$ is(AAS) iff $LDI(D)$ is(AAS) for all finite subsets D of S .*

It was wrongly stated in [3] that the languages associated with Hilbert a apriori norms are regular . Here is a counter example :

Example 3.4: Consider the folowing three 2×2 matrices : $A_1 = xx^T, \|x\|_{l_2} = 1, A_2 = U(\alpha), A_3 = U(-\alpha)$. Here $U(\alpha)$ is a rotation on angle α. If α is π irrational then the language $L_{\{A_1,A_2,A_3\},l_2}$ is not regular . To see this notice that

$$L_{\{A_1,A_2,A_3\},l_2} \cap 1\{2,3\}^*1 = \{1\omega 1\} =: LW,$$

where ω are words in alphabet $\{2,3\}$ with equal number of 2 and 3 .
If $L_{\{A_1,A_2,A_3\},l_2}$ is regular then LW should be regular as an intersection of two regular languages . But it is easy to show that LW is not regular . ∎

Let A be $N \times N$ be column-substochastic matrix , i.e. nonnegative entry wise matrix which is a nonstrict contraction respect to l_1 . Associate with this matrix A the following positive substochastic operator :

$$S_A(X) = Diag(y_i, 1 \leq i \leq N), y_i = \sum_{1 \leq j \leq N} A(i,j)X(j,j).$$

If A is a contraction respect to l_2 then associate with it the following positive substochastic operator :

$$U_A(X) = AXA^*.$$

It is easy to see that S_A is a separable operator and U_A is separable iff $Rank(A) \leq 1$. Example 3.4 hints that nonregularity of languages associated with quantum linear discrete inclusions might be caused by the entanglement .

Proposition 3.5: *Suppose that S is a finite set of separable positive substochastic operators . Then the language L_S is regular . Moreover , there exists a polytope a apriori norm for $LDI(S_H)$, where S_H consists of restrictions of operators from S on linear space (over reals) of hermitian matrices .*

Example 3.6: Consider the folowing compact set of 2×2 matrices $S = \{U(\alpha)AU(-\alpha) : 0 \leq \alpha \leq 2\pi\}$. Then $LDI(S)$ is(AAS) iff $\|A\|_{l_2} < 1$. This is because up to a trivial scale there is only one rotationaly invariant norm on R^2. But if $A = xy^T$, where both x and y are norm one vectors and the angle $arccos(< x, y >)$ is π irrational, then $LDI(D)$ is(AAS) for all finite subsets D of S (see the last section). As $\|xy^T\|_{l_2} = 1$ thus $LDI(S)$ is not(AAS). This example shows that Proposition 3.3 does not hold even in separable case . ∎

The next result shows that for dominant positive operators the situation is rather simple.

Lemma 3.7: *Suppose that $T_i : M(N) \to M(N), 1 \leq i \leq N$ are dominant positive substochastic operators and spectral radius $\rho(T_i) < 1, 1 \leq i \leq N$. Then $\|T_1...T_N\|_{tr} < 1$.*

Proof: Suppose that $\|T_1...T_N\|_{tr} = 1$. Then there exists $X_1 \succeq 0, tr(X_1) = 1$ such that $tr(X_i) = 1, 1 \leq i \leq N+1$ where $X_{i+1} = T_i X_i$. As T_i are dominant positive substochastic operators thus $Im(X_i) \subset Im(X_{i+1})$. Since $\rho(T_i) < 1$ hence the above inclusions are strict . Therefore , $dim(Im(X_{N+1}) \geq N+1$. And this is a contradiction . ▮

Corollary 3.8: *Suppose that S is a compact set of dominant positive substochastic operators . Then $LDI(S)$ is(AAS) iff $\rho(T) < 1$ for all operators $T \in S$.*

Proposition 3.9: *If T is dominant positive stochastic operator , then the only eigenvalue of T with magnitude one is 1 , which is equivalent to the convergence of sequence of powers T^n.*

Proof: There exists $\epsilon > 0$ such that $T(X) \succeq \epsilon X$ for all $X \succeq 0$. Thus $T = \epsilon I + (1-\epsilon)((1-\epsilon)^{-1}(T - \epsilon I))$ and $Q =: (1-\epsilon)^{-1}(T - \epsilon I)$ is a positive stochastic operator. Thus the spectrum $Sp(T) = \{\epsilon 1 + (1-\epsilon)z : z \in Sp(Q)\}$. But sinse Q is a positive stochastic operator hence $|z| \leq 1$. This ends the proof. ▮

3.1 Polyquadratic Lyupunov functions

Theorem 3.10: *Consider a finite set S of finite square matrices, $S = \{A_1, \ldots, A_k\}$. ($A_i : C^n \to C^n$).*
If $LDI(S)$ is (AAS) and its generalized spectrum radius $\bar{\rho}(S) = a < 1$ then there exists an integer m and positive-definite $n^m \times n^m$ matrix P such that $\|A_i \otimes ... \otimes A_i\|_P < 1, 1 \leq i \leq k$. Here $\|.\|_P$ is an operator norm induced by the $< Px, x >^{\frac{1}{2}}$, $\otimes$ is a tensor product. We use tensor products of length m. Moreover, one can choose any $m > \frac{\log k}{-2\log a}$.

Proof: First, assume that there exists a norm such the induced norms $\|A_i\| \leq b < 1$ and $b^2 k < 1$. Then the following matrix series converges absolutely:

$$P = \sum_{|\omega|<\infty} A_\omega^* A_\omega$$

(Recall that if ω is empty , i.e. $|\omega| = 0$, then $A_\omega = I$.)

The matrix P equal to this sum above is positive definite and $\|A_i\|_P < 1$.

Now, if we define $B_i = A_i \otimes ... \otimes A_i$ (m times), then $\bar{\rho}(\{B_1, ..., B_k\}) = \bar{\rho}(S)^m$.
(Here $\bar{\rho}(.)$ is a generalized spectrum radius .)
It is well known and quite easy to prove (see, for instance [1]) that for any $\epsilon > 0$ there exists an induced norm sush that

$$\|B_i\|_\epsilon \leq (\bar{\rho}(\{B_1, ..., B_k\}) + \epsilon).$$

Thus there exists an induced norm such that $\|B_i\|^2 k < 1, 1 \leq i \leq k$. This last norm gives the proof. ▮

Corollary 3.11: *The following class of functions provides Lyupunov function for any absolutely stable linear discrete inclusion generated by finite sets S:*
$\{L(x) =< Px^{\otimes m}, x^{\otimes m} >^{\frac{1}{2m}}\}$,
here P is positive-definite matrix and $x^{\otimes m}$ stands for an m-fold tensor power of a vector x.

Remark 3.12: The theorem above is valid also for finite sets of linear bounded operators acting in any Hilbert space. Another , finite dimesional , way to use quadratic Lyapunov functions is to check whether $\|A_\omega\|_P < 1$ for all words $\omega : |\omega| > m \geq \frac{\log n}{-2\log a}$. Here n is a dimension of matrices involved. ∎

4 Absolute asymptotic stability of linear continuous time inclusions

Let us first define quantum continuous time linear inclusions.

Definition 4.1: A linear operator $A : M(N) \to M(N)$ called C-positive if $exp(At)$ is a positive operator for all $t \geq 0$.
If $exp(At)$ is a completely positive operator for all $t \geq 0$ then operator A is called Lindblad operator .
If $I + At$ is a dominant positive operator for some $t > 0$ then operator A is called CD-positive .
(It is easy to show that if A is CD-positive then $exp(At)$ is dominant positive for all $t \geq 0$.)
C-positive operator A called C-stochastic (C-substochastic) if $0 = A^*(I)(0 \succeq A^*(I))$.
Quantum continuous time linear inclusion is $LCI(S)$ where S is a set of C-substochastic operators . ∎

It is not difficult to prove using Proposition 3.9 that if A is CD-positive C-substochastic operator then spectrum of A does not contain nonzero pure imaginary eigenvalues and thus $\lim_{t\to\infty} exp(At)$ does exists .

The next Proposition follows in a rather straight way from Choi's characterization of complete positivity in terms of Choi's matrix .

Proposition 4.2: *Linear operator $A : M(N) \to M(N)$ is Lindblad iff*

$$A(X) = aX + CX + XC^* + \sum_{1\leq i\leq (N^2-1)} B_i X B_i^*, \tag{9}$$

where a is a real number and matrices $(C; B_i, 1 \leq i \leq (N^2-1))$ are traceless .
If C is a linear combination of $B_i, 1 \leq i \leq (N^2-1))$ then A is CD-positive .

We need the following simple result which was essentialy proved in [2].

Proposition 4.3: *Let consider $LCI(Z)$, wher Z is a compact set of bounded linear operators in some Banach Space. Then $LCI(Z)$ is (AAS) iff there exists $\tau > 0$ such that Linear Discrete Inclusion $LDI(I + \tau Z)$ is(AAS).*
Here $I + \tau Z =: \{I + \tau A : A \in Z\}$.

Of course , if $LDI(I + \tau Z)$ is(AAS) then also $LDI(I + tZ)$ is(AAS) for all $0 < t \leq \tau$. Proposition(4.3) is not usefull in general : absolute asymptotic stability is hard enough in discrete case too and we have to figure out a proper τ. Still , it is the only , known to the author , finite discretization procedure which turns(AAS) property of linear continuous time inclusions to the discrete case .
It turns out that for two important cases Proposition(4.3) is very constructive . First case is when there exists a polytope a apriori norm , second one is when in Quantum continuous time linear inclusion $LCI(S)$ a set S consists of CD-positive C-substochastic operators .

Theorem 4.4: *Consider Quantum continuous time linear inclusion $LCI(S)$, where S ia compact set consisting of CD-positive C-substochastic operators . Then $LCI(S)$ is(AAS) if and only if all operators in S are nonsingular.*

Proof: "Only if part" is trivial. Let us proof "if" part . First , it is easy to show that CD-positive C-substochastic operator A is nonsingular iff $\rho(I + tA) < 1$ for all sufficiently small positive t. Using compactness of S , it follows that there exists $\tau > 0$ such operators $I + \tau A$ are dominant positive for all $A \in S$. As $0 \succeq A^*(I)$, thus $I + \tau A$ are dominant positive substochastic operators for all $A \in S$ and $\rho(I + \tau A) < 1, A \in S$. Now we just apply Proposition(4.3) and Corollary 3.8 . ∎

The next theorem is proved on the same lines as Theorem(4.4) with some extra efforts in spirits of Proposition 3.3 .

Theorem 4.5: *Consider continuous time linear inclusion $LCI(S)$, where S ia compact set consisting of finite square matrices which are non-strict contractions respect to some polytope norm $\|.\|$. Then $LCI(S)$ is(AAS) if and only if all matrices in S are nonsingular.*

Proof: (Sketch) Our proof consists of the following steps .
Step 1. Show , using Hahn-Banach theorem , that there exists $\tau > 0$ such that $\|I + \tau A\| \leq 1 : A \in S$.
Step 2. Show that under conditions of Theorem (4.5) all matrices in the convex hull $CO(S)$ are Hurwitz .
Step 3. Results from [1] imply that $LDI(I + tS)(0 < t \leq \tau)$ is not (AAS) iff there exists $A_{i,t} \in S(1 \leq i \leq M)$ such that $\rho((I + tA_{1,t})...(I + tA_{M,t})) = 1$ and M is universal finite integer depending only on the polytope norm $\|.\|$.
Step 4. $\rho((I + tA_{1,t})...(I + tA_{M,t})) = \rho(I + t(A_{1,t} + ... + A_{M,t}) + O(t^2)) < 1$ based on Step 2 and compactness of S .
Step 5 . Use Proposition 4.3 to finish the proof . ∎

Remark 4.6: One of the main points of Theorem (4.5) is that when there exists polytope a apriori norm then there is a (trivial) polynomial time algorithm to check whether given $LCI(S)$ is(AAS) for finite S . In a discrete time case the analogous problem even for l_1 a apriori norm is at least NP-HARD [2] and most likely is P-SPASE complete. ∎

In both Theorem(4.4) and Theorem(4.5) , we took advantage of the following property :
if for some (good) norm $\|exp(At)\| \leq 1 : t \geq 0, A \in S$ then there exists $\tau > 0$ such that $\|I + \tau A\| \leq 1 : A \in S$.
But even for Hilbert norms and S consisting of two elements this property does not hold in general .
The next theorem gives only sufficient condition for(AAS) property for general Quantum continuous time linear inclusion .

Theorem 4.7: *Quantum continuous time linear finite inclusion $LCI(A_1, .., A_k)$ is(AAS) if the following conditions hold :*
A. For all $i < j$ the intersections $Ker(A_i^*(I) \cap Ker(A_i^*(I) = \{0\}$.
B. Eigenvalues of all operators A_i have negative real parts.

Proof: Let us consider the following family of operator "differential equations":

$$\dot{X} = \left(\sum_{i=1}^{k} U_i(t) A_i\right) X, \quad X(0) \equiv I.$$

Here $(U_1(t), ..., U_k(t))$ is a measurable vector-function, all its components are nonnegative and their sum $\sum_{i=1}^{K} U_i(t) \equiv 1$. We will denote this class of vector-functions as **SM**. The proper way to define the corresponding solution $X(t)$ is as an unique continuous solution of the corresponding integral equation:

$$X(t) = I + \int_0^t \left(\sum_{i=1}^{k} U_i(\tau) A_i\right) X(\tau) d\tau.$$

Let us denote as X_U the solution corresponding to (weighted switching function) $(U_1(t), ..., U_k(t))$. As it was remarked in [2], if

$$U_i^{(n)}(\omega) \to U_i \text{ in } L_2[0,T] \text{ (converges weakly) then } \max_{t \in [0,T]} \|X_{U^{(n)}}(t) - X(t)_U\| \to 0.$$

It is almost obvious that the set **SM** is a weak compact in $L_2[0,T]$ for any finite $T > 0$. Thus the set $D(1) = \{X_U(1) : U \in \mathbf{SM}\}$ is a compact set of invertible positive substochastic operators. In fact, $D(1)$ is an exact discretization of our problem: continuous absolute asymptotic stability is equivalent to absolute asymptotic stability of linear discrete inclusion $LDI(D)$.
Anyway, we are to prove that $\|B\| < 1$ for all $B \in D(1)$, where $\|.\|$ is an operator norm in induced by the traces norm in $M(N)$.
If $x \in M(N)$ is a complex $N \times N$ matrix and

$$X(t) = I + \int_0^t \left(\sum_{i=1}^{k} U_i(\tau) A_i\right) X(\tau) d\tau,$$

then $x(t) =: X(t)x$ satisfies the following integral equation:

$$x(t) = x(0) + \int_0^t \left(\sum_{i=1}^{k} U_i(\tau) A_i \right) x(\tau) d\tau.$$

And $y(t) =: tr(x(t))$ satisfies the following integral equation:

$$y(t) = y(0) + \int_0^t tr(x(\tau)C(\tau))d\tau,$$

$$C(\tau) =: \sum_{i=1}^{k} U_i(\tau) A_i^*(I).$$

Notice that $0 \succeq C(\tau)$.
After all those preparations (compactness of $D(1)$ most of all) we need to prove that for all $U \in \mathbf{SM}$ and $\|x(0)\|_{tr} = 1$ one has that $\|x(1)\|_{tr} < 1$. Using positivity , it is enough to consider only positive semidefinite $x(0) \succeq 0$.
We consider two cases. In the first case we assume that the measure of the following subset Z_U of $[0, 1]$ is positive :

$$Z_U =: \{t \in [0,1] : \text{ there exists } i \text{ such that } 0 < U_i(t) < 1\}.$$

It follows from the positivity that $tr(C(\tau)x(\tau)) \leq 0$, and from the Condition (A) that $0 \succ C(t)$ if $t \in Z_U$.
Therefore $tr(C(t)x) < 0$ for all $t \in Z_U$ and nonzero positive semidefinite matrices x.
Thus $tr(x(1)) \leq tr(x(0)) + \int_{Z_U} < tr(C(\tau)x(\tau))d\tau$.
As Lebeques integral of a strictly positive measurable function is srictly positive, we conclude that $tr(x(1)) < tr(x(0)) = 1$.
In the second case we assume that the measure of Z_U is zero. Let us define the following convex (connected) compact subsets of $M(N)$:

$$T(1) = \{x \in M(N) : x \succeq 0, tr(X) = 1\}; L_i = \{x \in T(1) : Im(x) \subset Ker(A_i^*(I)\}, 1 \leq i \leq k.$$

It is clear that $L_i \cap L_j$ if $1 \leq i < j \leq k$ because of the Condition (A).
Suppose that there exists $x(0) \succeq 0$ such that $tr(x(1)) = 1$. Since operators $A_i(1 \leq i \leq k)$ are C-substochastic hence the following identity holds :

$$tr(x(\tau)) \equiv 1(x(\tau) \succeq 0).$$

We have a continuous curve $x(\tau)$ sitting in a connected compact $T(1)$.
As $L_i \cap L_j$ if $1 \leq i < j \leq k$ thus either for some $t \in [0, 1]$ $x(t)$ belongs to none of $L_i(1 \leq i \leq k)$ (then $tr(C(\tau)x(\tau)) < 0$ in some interval around t and $tr(x(1)) < 1$) ,
or the entire curve belongs to exactly one L_i. The only possibility for the last "branch" is that $C(\tau) = A_i^*(I)$ for some $1 \leq i \leq k$ and almost all τ (up to measure zero set). This possibility contradicts to the Condition (B). ∎

This result is really quite powerful, it provides many important counter examples.
Example(4.8) was first suggested by David Angeli for a different aim and was the main motivation for Theorem 4.7 .

Example 4.8: Consider the folowing compact set of real 2×2 matrices $S = \{U(\alpha)AU(-\alpha) : 0 \leq \alpha \leq 2\pi\}$. Similarly to Example 3.6 , $LCI(S)$ is(AAS) iff $0 \succ A + A^T$.
Take any Hurwitz matrix A with $0 \succeq A+A^T$ but not $0 \succ A+A^T$. . It follows from Theorem(4.7) that $LCI(D)$ is(AAS) for all finite subsets D of S .
(Here we deal with the following C-substochastic operators : $\bar{A}(X) = AX + XA^T$.)
Thus we consrtrusted a compact set S such that $LCI(D)$ is(AAS) for all finite subsets D of S , but $LCI(S)$ is not(AAS) . ∎

Example 4.9: Another counter example , based on Theorem(4.7) , is $LCI(A_1, A_2)$ which is(AAS) , but $LDI(C(A_1), C(A_2))$ is not(AAS) . Here the Cayley transform $C(A) = (I+A)(I-A)^{-1}$. I.e. , Cayley transform can transform absolutely asymptoticaly stable continuous time systems into not absolutely asymptoticaly stable discrete systems . Of course , this also give an example of $LCI(A_1, A_2)$ which is(AAS) but does not have quadratic Lyupunov function :

$$A_1 = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}.$$

It is easy to see that $LCI(A_1, A_2)$ satisfies conditions of Theorem(4.7) , but $\rho(C(A_1)C(A_2)) = 1$ therefore $LDI(C(A_1), C(A_2))$ is not(AAS) . ∎

Remark 4.10: If $LCI(S)$ is (AAS) then all matrices in the convex hull $CO(S)$ are Hurwitz . But this condition (COH) is not sufficient even for two 2×2 matrices . Theorem 4.5 proves that the condition (COH) is sufficient provided there exists a apriori polytope norm and S is compact . Example 4.8 shows that the condition (COH) is not sufficient for compact S and Hilbert a apriori norms even in 2-dimensional case .
From the other hand , a slight modification of Theorem (4.7) gives that the condition (COH) is sufficient for finite S and Hilbert a apriori norms in 2-dimensional case .
All this said indicates a number of possible conjectures . ∎

5 Controllabilities of Switching Systems

Let us consider controlled switched systems : $x(n+1) = A_{t(n)}x(n) + B_{t(n)}u(n)$, where all the matrices involved act on some linear finite-dimensional space X.
Here $t : N \rightarrow 1, 2, .., k$ is a switching rule. Let us associate with pairs (A, B) the following map which maps linear subspaces of X to linear subspaces of X:

$$D_{(A,B)}(Y) =: A(Y) + Im(B) \text{ (discrete case) },$$

For a given finite set of pairs $\{(A_1, B_1), ..., (A_k, B_k)\}$ we will use the following notations :

$$D_i =: D_{(A_i, B_i)}.$$

Similarly , we define D_ω for a word ω in the alphabet $\{1, ..., k\}$.
Using these notations the switching controllability in discrete case means that there exists a

word ω such that

$$D_\omega(0) = X \text{ (full space) .}$$

The absolute controllability means that for any infinite word $\omega = \omega_1\omega_2..\omega_m...$ there exists N such that

$$D_{\omega_1\omega_2..\omega_N}(0) = X.$$

Let us first make obvious observations:
1. Linear feedbacks don't change the problem , i.e. $D_{(A,B)} = D_{(A+BG,B)}$.
2. Absolute controllability implies standard controllability of each pair (A_i, B_i) , but not vice versa.
3. Switching controllability implies that there exists $1 \leq j \leq k$ such that $D_j(X) = X$.
4. A feedback G exists such that $A + BG$ is nonsingular iff $D_{(A,B)}(X) = X$, and this is true for any commutative field of scalars.
5. It is necessary for the switching controllability in both continuous and discrete time case that

$$L(A_1, ..., A_k; Im(B_1 + ... + Im(B_k)) = X.$$

Here $L(A_1, ..., A_k; Z)$ is a minimal linear subspace invariant respect to all $A_i, 1 \leq i \leq k$ and containing Z.

Theorem 5.1: *Assume that there exist feedbacks G_i such that $A_i + B_iG_i$ is nonsingular for all $i \in 1, 2, .., k$. Then the discrete time controlled switching system is Switching controllable iff*

$$L(A_1, ..., A_k; Im(B_1 + ... + Im(B_k)) = X.$$

Proof: Using the first observation above we can assume WLOG that A_i is nonsingular for all $1 \leq i \leq k$.
Suppose that discrete time controlled switching system is not Switching controllable , or $D_\omega(0) \neq X$ for all
finite words ω. As the state space X is finite-dimensional thus there exists a word ω^* such that

$$dim(D_\omega(0)) \leq dim(D_{\omega^*}(0)) < dim(X) < \infty \text{ for all finite words } \omega.$$

Let us denote $Y =: D_{\omega^*}(0)$. Then for all $1 \leq i \leq k$ we have that $dim(A_i(Y) + Im(B_i)) \leq dim(Y)$.
As A_i is nonsingular , we conclude that $Im(B_i) \subset A_i(Y), 1 \leq i \leq k$.
Via standard induction argument we get that $Im(B_i) \subset A_i(A_\omega Y), 1 \leq i \leq k$
for all finite words ω , including empty word. Thus , it follows that

$$Im(B_i) \subset \bigcap_\omega A_i((A_\omega(Y)), 1 \leq i \leq k; Z =: \bigcap_\omega A_\omega(Y).$$

As A_i is a nonsingular matrix (and a injective map from X on to X) , therefore

$$A_i(Z) = A_i(\bigcap_\omega A_\omega(Y)) = \bigcap_\omega A_i((A_\omega(Y)) \supset Z.$$

So, we have that $Z \subset A_i(Z)$. It follows from finite dimensionality that $Z = A_i(Z))$

From the conditions of the theorem , it follows that there exists at least one nonzero B_i. Thus the subspace Z is nonzero subspace, invariant respect to all A_i and containing $Im(B_i), 1 \leq i \leq k$.
From the other hand, $dim(Z) \leq dim(Y) < dim(X)$. We got a contradiction, therefore $D_{\omega^*}(0) = X$. ∎

Corollary 5.2: *The condition $L(A_1, ..., A_k; Im(B_1 + ... + Im(B_k))) = X$ is necessary and sufficient for switching controllability in continuous time case without any assumptions.*

It is clear that Theorem 5.1 provides polynomial in $(dim(X), k)$ algorithm .

Example 5.3: Let us show that the nonsingularity is essential. We need the following example of controlled switched system
$\{(A_1, B_1), ..., (A_k, B_k)\}$ which is not switching controllable:
there exists j such that $A_j(X) + Im(B_j) = X$ and
$L(A_1, ..., A_k; Im(B_1 + ... + Im(B_k)) = X$.
Consider the following three pairs of 3 by 3 matrices :

$$A_1 = I, A_2 = 0, A_3 = 0; B_1 = Diag(1, 0, 0), B_2 = Diag(0, 1, 0),$$

$$B_3 = Diag(0, 0, 1).$$

It is easy to see that in this case $dim(D_\omega(0)) \leq 2$ for all finite words ω (the system is not switching controllable) , $A_1(X) + Im(B_1) = X$, and $L(A_1, ..., A_k; Im(B_1 + ... + Im(B_k)) = X$.
From the other hand Conditions(3,5) are sufficient for the switching controllability in the case when $A_i \equiv A, rank(B_i) \equiv 1$ and the field is sufficiently large . This not very difficult result is based on elementary matroidal considerations . ∎

It is not clear (to the author) if it is decidable to check whether given controlled switching system is switching controllable , say for the field of rational numbers.
For finite fields it is certainly decidable. In this case we have a (very) particular case of the following graph decision problem (GDP):
given a finite set of matrices C_i with nonnegative entries (directed graphs) to check whether there exists some product of them having all positive entries. This (GDP) problem is known to be "hard" in general , i.e. it is P-SPACE Complete [4] .
The matrices C_i corresponding to the controlled switching system
$\{(A_1, B_1), ..., (A_k, B_k)\}$ are boolean $card(X) \times card(X)$ matrices with :

$$C_i(x, y) = 1 \text{ iff there exists } u \text{ such that } y = A_i x + B_i u.$$

A polynomial-time (but not "combinatorial") algorithm to solve (GDP) has been described in [2] in one special case :

all the directed graphs associated in a standard way with the products of matrices C_i have symmetric transitive closure. For linear controlled switching systems over finite fields it is exactly the nonsingularity assumption.
The case of absolute controllability fits nicely to the general framework of monotone automatons and the next result is very similar to Corollary 3.2. Moreover if $\|A_i\|_{l_2} \leq 1, 1 \leq i \leq k$ then $LDI(A_1, ..., A_k)$ is(AAS) iff the controlled switched system with pairs $\{(A_i^*, I - A_i A_i^*), 1 \leq i \leq k\}$ is absolutely controllable .

Theorem 5.4: *It is decidable to check absolute controllability in discrete case. Moreover there exists (very rapidly growing) function $F(dim(X), k)$ such that a controlled switching system is absolutely controllable iff $D_\omega(0) = X$ for all finite words ω of length $F(dim(X), k)$.*

6 Rank one matrices

Let us consider a bounded set of $n \times n$ rank one matrices $Z = \{x_\alpha y_\alpha^*, \alpha \in S\}$, here S is some , possibly infinite ,"index" set. We associate with this set the following matrix or function of two variables:

$$M_Z(\alpha, \beta) =:< x_\alpha, y_\beta > .$$

We also define for any square , possibly infinite , matrix B the following quantity:

$$\gamma(B) =: sup_{i_1,i_2,...,i_k} |B(i_1, i_2)B(i_1, i_2)...B(i_k, i_1)|^{\frac{1}{k}}.$$

As spectral radius $\rho(xy^*) =< x, y >$, we the following identity :

$$\bar{\rho}(Z) = \gamma(M_Z).$$

If Z is a finite set then $\gamma(M_Z)$ can be computed in poly-time. Also in this finite case it is obvious that a norm $\|.\|$ exists such that induced operator norms $\|x_\alpha y_\alpha^*\| \leq 1$ iff $\gamma(M_Z) \leq 1$. Next result showes that the last statement holds for infinite sets also.

Proposition 6.1: *Let us consider a bounded set of $n \times n$ rank one matrices $Z = \{x_\alpha y_\alpha^*, \alpha \in S\}$. Then norm $\|.\|$ exists such that induced operator norms $\|x_\alpha y_\alpha^*\| \leq 1$ iff $\gamma(M_Z) \leq 1$.*

Corollary 6.2: *Consider (an infinite) finite rank matrix (function of two variables) $F = f(\alpha, \beta); \alpha, \beta \in S$. If $\gamma(F) \leq 1$ and $(|f(\alpha, \beta)|) < \infty$ then there exist a function $d(\alpha)$ such that*

$$0 < a \leq d(\alpha) \leq b < \infty \text{ and } |d(\alpha)^{-1} f(\alpha, \beta) d(\beta)| \leq 1 (\alpha, \beta \in S).$$

Proof: First we factorize $F = LR$ where L is a "$S \times n$" and R is "$n \times S$ matrices with uniformly bounded elements, n is a rank of F. We associate with this factorization a natural set of rank one $n \times n$ matrices : $Z = \{x_\alpha y_\alpha^*$, where x_α is "α"raw of L and y_α is "α" column of R. It follows from Proposition 6.1 that there exists a norm $\|.\|$ such that induced operator norms $\|x_\alpha y_\alpha^*\| \leq 1 (\alpha \in S)$.

But $\|x_\alpha y_\alpha^*\| = \|x_\alpha\|\|y_\alpha\|_*$, where $\|.\|_*$ is a dual norm.
Recall that L, R have uniformly bounded elements , thus for some positive constant D we get that

$$\|x_\alpha\| \leq D, \|y_\alpha\|_* \leq D.$$

Define $d(\alpha) = \|x_\alpha\|$ if $\|x_\alpha\| \geq D^{-1}$ and $d(\alpha) = D^{-1}$ otherwise.
It follows that $d(\alpha)^{-1}\|x_\alpha\| \leq 1$ and $d(\alpha)\|y_\alpha\|_* \leq 1$ for all $\alpha \in S$.
Therefore we get that

$$|d(\alpha)^{-1} f(\alpha,\beta) d(\beta)| = |d(\alpha)^{-1} < x_\alpha, y_\beta > d(\beta)| \leq d(\alpha)^{-1}\|x_\alpha\| d(\beta)\|y_\beta\|_* \leq 1.$$

∎The last Corollary generalizes ("finite") Theorem 7.2 from [7]. Of course, it would be interesting to say something similar about continuous functions $f(x, y)$ defined on the unit square. Our proof works for polynomials.

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