A Numerically Reliable Method for Noninteracting Control with Stability for (A, B, C, D) Quadruples

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Abstract

The noninteracting control problem with stability for (A, B, C, D) quadruples has been studied for a long time. But, there are no numerically verifiable solvability conditions and no numerically implementable methods for solving it in the existing literatures. Hence, indeed it is still an unsolved problem from theoretical and numerical points of view. In this paper we develop a numerically reliable method to solve this unsolved problem. The main tool that we use is numerical linear algebra technique and our numerical method can be implemented in a numerically reliable way.

1 Introduction

Consider an (A, B, C, D) quadruple and its associated linear time-invariant system of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

where $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{m\times n}$, $D \in \mathbf{R}^{m\times m}$, D is the direct feedthrough matrix, $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ is the control input and $y \in \mathbf{R}^m$ is the output. Let $u = Fx + Hv$, then the closed loop system becomes

$$\dot{x} = (A + BF)x + BHv, \quad y = (C + DF)x + DHv. \tag{2}$$

The transfer matrix from output y to input v in (2) is $(C + DF)(sI - A - BF)^{-1}BH + DH$. Hence, the noninteracting control problem (i.e., row by row decoupling problem) with stability studied in this paper can be formulated as follows:

Definition 1 *The noninteracting control problem with stability for an (A, B, C, D) quadruple is solvable if there exist matrices $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times m}$ such that*

$$(C + DF)(sI - A - BF)^{-1}BH + DH \text{ is nonsingular and diagonal,} \tag{3}$$

and the matrix $A + BF$ is stable.

The noninteracting control problem with stability for the linear time-invariant system

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{4}$$

which has no direct feedthrough matrix, has been investigated and numerically verifiable coordinate-free solvability conditions have been given in [4, 5, 6, 7] based on geometric approach and structural approach in the last three decades.

The results in [4, 5, 6, 7] are very important theoretically. But, as pointed out in [9], they can not lead to numerically reliable methods for computing a solution for the noninteracting control problem with stability for systems of the form (4). Only very recently, a numerically reliable method has been developed in [9] based on orthogonal transformations for solving this problem. After a hard study, we found that it is very difficult to generalize the works in [4, 5, 6, 7] even to characterize the solvability conditions for the noninteracting control problem with stability for (A, B, C, D) quadruples. Similarly, the result in [9] can not be generalized easily for this purpose, too.

The only existing works that we can find on the noninteracting cotrol problem with stability for (A, B, C, D) quadruples are [1, 2, 3]. In [2, 3] only sufficient conditions are given. While the necessary and sufficient conditions in [1] are based on the existences of some particular but unknown invariant subspaces of system (1) and some particular but also unknown solutions of the noninteracting control problem *without stability* for system (1), hence, the conditions given in [1] are not numerically verifiable. Furthermore, the results in [1, 2, 3] can not give any direction to establish a numerically implementable method for solving the underlying problem in the general setting.

Based on the above observations, we can conclude that the noninteracting control problem with stability for (A, B, C, D) quadruples is still an unsolved problem from both theoretical and numerical points of view. The main purpose of this paper is to develop a numerically reliable method, based on numerical linear algebra technique, to solve this unsolved problem.

2 Main Results

In this section we will establish a numerically reliable algorithm for solving the noninteracting control problem with stability for (A, B, C, D) quadruples. Our method consists of four different stages:

(1) Stage 1 is to reduce the underlying problem for the (A, B, C, D) quadruple into a simultaneous problem of disturbance decoupling and "unusual" noninteracting control with stability for a reduced system with *nonsingular* direct feedthrough matrix and a noninteracting control problem with stability for a reduced system without direct feedthrough matrix.

(2) In Stage 2, we only consider the simultaneous problem with stability arised in Stage 1. This simultaneous problem with stability will be reduced into an "unusual" noninteracting control problem with stability for a linear time-invariant system with *nonsingular* direct feedthrough matrix.

(3) We will present a numerically reliable algorithm in Stage 3 to solve the "unusual" noninteracting control problem with stability produced in Stage 2.

(4) Stage 4 consists of the backtransformations of the results in Stages 1, 2 and 3 to the desired solution for the original noninteracting control problem. An outline of the overall algorithm is given in this stage.

2.1 Stage 1

The noninteracting control problem with stability for systems of the form (4) and the noninteracting control problem without stability for systems of the form (1) have been studied and the related numerically reliable methods based on orthogonal transformations have been developed in [9] and [10] respectively. Hence, the results in [9] and [10] will be used as a bridge to achieve the purpose of Stage 1 in this subsection.

Lemma 2 *[10] Given $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{m\times m}$. There exist orthogonal matrices $U, V \in \mathbf{R}^{n\times n}$, $W \in \mathbf{R}^{m\times m}$ and a permutation matrix $P \in \mathbf{R}^{m\times m}$ such that*

$$\begin{bmatrix} U & 0 \\ 0 & P \end{bmatrix} \left[\begin{array}{c|c} sI-A & B \\ \hline C & D \end{array}\right] \begin{bmatrix} V & 0 \\ 0 & W \end{bmatrix} = \begin{array}{cc} \begin{array}{cccc} n_1 & n_2 & m_0 & m-m_0 \end{array} & \\ \begin{bmatrix} sE_{11}-A_{11} & -A_{12} & B_{11} & B_{12} \\ -A_{21} & sE_{22}-A_{22} & B_{21} & B_{22} \\ 0 & sE_{32}-A_{32} & 0 & 0 \\ C_{11} & C_{12} & D_{11} & 0 \\ 0 & C_{22} & D_{21} & 0 \end{bmatrix} & \begin{array}{l} \}n_1 \\ \}\tilde{n}_2 \\ \}\tilde{n}_3 \\ \}m_0 \\ \}m-m_0 \end{array} \end{array}, \tag{5}$$

where

$$\mathrm{rank}(D_{11}) = m_0,\ \mathrm{rank}\begin{bmatrix} B_{21} & B_{22} \end{bmatrix} = \tilde{n}_2,\ \max_{s\in\mathbf{C}} \mathrm{rank}\begin{bmatrix} sE_{32}-A_{32} \\ C_{22} \end{bmatrix} = n_2, \tag{6}$$

$$\mathrm{rank}\begin{bmatrix} sE_{11}-A_{11} & B_{11} & B_{12} \\ -A_{21} & B_{21} & B_{22} \end{bmatrix} = n_1 + \tilde{n}_2, \quad \forall s \in \mathbf{C}. \tag{7}$$

The following conditions (9) and (11) follow directly from [10], and the condition (10) is a direct consequence of the stability of $A + BF$.

Lemma 3 *(c.f. [10]) Given $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{m\times m}$. Assume that orthogonal matrices U, V, W and the permutation matrix P have been determined to satisfy the condensed form (5). For any matrices $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times m}$, denote*

$$W^T F V = \begin{array}{cc} \begin{array}{cc} n_1 & n_2 \end{array} & \\ \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix} & \begin{array}{l} \}m_0 \\ \}m-m_0 \end{array} \end{array}, \quad W^T H P^T = \begin{array}{cc} \begin{array}{cc} m_0 & m-m_0 \end{array} & \\ \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} & \begin{array}{l} \}m_0 \\ \}m-m_0 \end{array} \end{array}. \tag{8}$$

If there exist $F \in \mathbf{R}^{m\times n}$ and $H \in \mathbf{R}^{m\times m}$ such that (3) holds and $A + BF$ is stable, then

$$D_{21} = 0,\ \tilde{n}_2 = m - m_0,\ \mathrm{rank}(B_{22}) = \tilde{n}_2, \tag{9}$$

$$\mathrm{rank}(sE_{32} - A_{32}) = \tilde{n}_3, \quad \forall s \in \mathbf{C}/\mathbf{C}^-, \tag{10}$$

$$\tilde{H}_{12} = 0,\ A_{21} + B_{21}\tilde{F}_{11} + B_{22}\tilde{F}_{21} = 0,\ B_{21}\tilde{H}_{11} + B_{22}\tilde{H}_{21} = 0. \tag{11}$$

Lemma 3 contains a very important fact, that is, the conditions (9) and (10) are necessary for the noninteracting control problem with stability of system (1). With these two necessary conditions, we can refine the condensed form (5) as follows.

Theorem 4 *Given $A \in \mathbf{R}^{n\times n}, B \in \mathbf{R}^{n\times m}, C \in \mathbf{R}^{m\times n}$ and $D \in \mathbf{R}^{m\times m}$. Assume that orthogonal matrices U, V, W and the permutation matrix P have been determined to satisfy the condensed form (5). If the conditions (9) and (10) hold, then there exist nonsingular matrices $X, Y \in \mathbf{R}^{n\times n}$ and $Z \in \mathbf{R}^{m\times m}$ such that*

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc|cc} sE_{11}-A_{11} & -A_{12} & B_{11} & B_{12} \\ -A_{21} & sE_{22}-A_{22} & B_{21} & B_{22} \\ 0 & sE_{32}-A_{32} & 0 & 0 \\ \hline C_{11} & C_{12} & D_{11} & 0 \\ 0 & C_{22} & 0 & 0 \end{array}\right] \begin{bmatrix} Y & 0 \\ 0 & Z \end{bmatrix}$$

$$= \begin{array}{c l} \begin{array}{ccccc} n_1 & \tilde{n}_2 & \tilde{n}_3 & m_0 & \tilde{n}_2 \end{array} & \\ \left[\begin{array}{ccccc} s\mathcal{E}_{11}-\mathcal{A}_{11} & -\mathcal{A}_{12} & -\mathcal{A}_{13} & \mathcal{B}_{11}\mathcal{D}_{11} & 0 \\ -A_{21} & s\mathcal{E}_{22}-\mathcal{A}_{22} & s\mathcal{E}_{23}-\mathcal{A}_{23} & 0 & B_{22} \\ 0 & -\mathcal{A}_{32} & s\mathcal{E}_{33}-\mathcal{A}_{33} & 0 & 0 \\ C_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} & \mathcal{D}_{11} & 0 \\ 0 & \mathcal{C}_{22} & \mathcal{C}_{23} & 0 & 0 \end{array}\right] & \begin{array}{l} \}n_1 \\ \}\tilde{n}_2 \\ \}\tilde{n}_3 \\ \}m_0 \\ \}\tilde{n}_2 \end{array} \end{array}, \tag{12}$$

where

$$\mathrm{rank}(\mathcal{E}_{11}) = n_1,\ \mathrm{rank}(\mathcal{E}_{22}) = \tilde{n}_2,\ \mathrm{rank}(\mathcal{E}_{33}) = \tilde{n}_3,\ \mathrm{rank}(\mathcal{D}_{11}) = m_0, \tag{13}$$

$$\max_{s\in\mathbf{C}} \mathrm{rank} \left[\begin{array}{cc} -\mathcal{A}_{32} & s\mathcal{E}_{33}-\mathcal{A}_{33} \\ \mathcal{C}_{22} & \mathcal{C}_{23} \end{array}\right] = \tilde{n}_2 + \tilde{n}_3, \tag{14}$$

$$\mathrm{rank}(s\mathcal{E}_{33}-\mathcal{A}_{33}) = \tilde{n}_3, \quad \forall s \in \mathbf{C}/\mathbf{C}^-, \tag{15}$$

and

$$\mathrm{rank}\left[\begin{array}{cc} s\mathcal{E}_{11}-\mathcal{A}_{11} & \mathcal{B}_{11} \end{array}\right] = n_1, \quad \forall s \in \mathbf{C}. \tag{16}$$

Furthermore, matrices X, Y, Z and the condensed form (12) are computed only based on orthogonal transformations which can be implemented via an inverse-free and numerically reliable way.

Lemmas 2 and 3 and Theorem 4 offer a springboard to leap forward to the objective of this subsection.

Theorem 5 *The noninteracting control problem with stability for (A, B, C, D) quadruple is solvable if and only if the condition (9) holds and furthermore the following (a) and (b) are satisfied:*

(a) The state feedback decoupling problem with stability for strictly proper system

$$\left[\begin{array}{cc} \mathcal{E}_{22} & \mathcal{E}_{23} \\ 0 & \mathcal{E}_{33} \end{array}\right]\dot{x} = \left[\begin{array}{cc} \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{32} & \mathcal{A}_{33} \end{array}\right]x + \left[\begin{array}{c} B_{22} \\ 0 \end{array}\right]u, \quad y = \left[\begin{array}{cc} \mathcal{C}_{22} & \mathcal{C}_{23} \end{array}\right]x, \tag{17}$$

is solvable, i.e., there exist $F_{22} \in \mathbf{R}^{\tilde{n}_2\times\tilde{n}_2}$, $F_{23} \in \mathbf{R}^{\tilde{n}_2\times\tilde{n}_3}$ and $H_{22} \in \mathbf{R}^{\tilde{n}_2\times\tilde{n}_2}$ such that

$$\text{the pencil } (\left[\begin{array}{cc} \mathcal{E}_{22} & \mathcal{E}_{23} \\ 0 & \mathcal{E}_{33} \end{array}\right], \left[\begin{array}{cc} \mathcal{A}_{22}+B_{22}F_{22} & \mathcal{A}_{23}+B_{22}F_{23} \\ \mathcal{A}_{32} & \mathcal{A}_{33} \end{array}\right]) \text{ is stable,} \tag{18}$$

$$\left\{\begin{array}{l} \left[\begin{array}{cc} \mathcal{C}_{22} & \mathcal{C}_{23} \end{array}\right]\left[\begin{array}{cc} s\mathcal{E}_{22}-\mathcal{A}_{22}-B_{22}F_{22} & s\mathcal{E}_{23}-\mathcal{A}_{23}-B_{22}F_{23} \\ -\mathcal{A}_{32} & s\mathcal{E}_{33}-\mathcal{A}_{33} \end{array}\right]^{-1}\left[\begin{array}{c} B_{22} \\ 0 \end{array}\right]H_{22} \\ \text{is nonsingular and diagonal.} \end{array}\right. \tag{19}$$

(b) The simultaneous problem of disturbance decoupling and "unusual" noninteracting control for the linear time-invariant system

$$\left[\begin{array}{cc} \mathcal{E}_{11} & 0 \\ 0 & \mathcal{E}_{33} \end{array}\right]\dot{x} = \left[\begin{array}{cc} \mathcal{A}_{11}-\mathcal{B}_{11}C_{11} & \mathcal{A}_{13}-\mathcal{B}_{11}\mathcal{C}_{13} \\ 0 & \mathcal{A}_{33} \end{array}\right]x + \left[\begin{array}{c} \mathcal{B}_{11} \\ 0 \end{array}\right]u + \left[\begin{array}{c} \mathcal{A}_{12}-\mathcal{B}_{11}\mathcal{C}_{12} \\ \mathcal{A}_{32} \end{array}\right]d, \quad y = u \tag{20}$$

with feedback $u = \mathcal{F}x + v$ is solvable, i.e., there exists $\mathcal{F} \in \mathbf{R}^{m_0\times(n_1+\tilde{n}_3)}$ such that

$$\text{the pencil } (\left[\begin{array}{cc} \mathcal{E}_{11} & 0 \\ 0 & \mathcal{E}_{33} \end{array}\right], \left[\begin{array}{cc} \mathcal{A}_{11}-\mathcal{B}_{11}C_{11} & \mathcal{A}_{13}-\mathcal{B}_{11}\mathcal{C}_{13} \\ 0 & \mathcal{A}_{33} \end{array}\right] + \left[\begin{array}{c} \mathcal{B}_{11} \\ 0 \end{array}\right]\mathcal{F}) \text{ is stable,} \tag{21}$$

$$\mathcal{F}(\left[\begin{array}{cc} s\mathcal{E}_{11}-\mathcal{A}_{11}+\mathcal{B}_{11}C_{11} & -\mathcal{A}_{13}+\mathcal{B}_{11}\mathcal{C}_{13} \\ 0 & s\mathcal{E}_{33}-\mathcal{A}_{33} \end{array}\right] - \left[\begin{array}{c} \mathcal{B}_{11} \\ 0 \end{array}\right]\mathcal{F})^{-1}\left[\begin{array}{c} \mathcal{B}_{11} \\ 0 \end{array}\right] \text{ is diagonal,} \tag{22}$$

$$\mathcal{F}(\left[\begin{array}{cc} s\mathcal{E}_{11}-\mathcal{A}_{11}+\mathcal{B}_{11}C_{11} & -\mathcal{A}_{13}+\mathcal{B}_{11}\mathcal{C}_{13} \\ 0 & s\mathcal{E}_{33}-\mathcal{A}_{33} \end{array}\right] - \left[\begin{array}{c} \mathcal{B}_{11} \\ 0 \end{array}\right]\mathcal{F})^{-1}\left[\begin{array}{c} \mathcal{A}_{12}-\mathcal{B}_{11}\mathcal{C}_{12} \\ \mathcal{A}_{32} \end{array}\right] = 0, \tag{23}$$

2.2 Stage 2

In this subsection we will reject the disturbance and thus transform the simultaneous problem of (21), (22) and (23) into a single problem like that of (21) and (22).

Lemma 6 *Let* $\begin{bmatrix} \mathcal{E}_{11} & 0 \\ 0 & \mathcal{E}_{33} \end{bmatrix}$, $\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{13} \\ 0 & \mathcal{A}_{33} \end{bmatrix}$, $\begin{bmatrix} \mathcal{A}_{12} \\ \mathcal{A}_{32} \end{bmatrix}$, $\begin{bmatrix} \mathcal{B}_{11} \\ 0 \end{bmatrix}$, $\begin{bmatrix} C_{11} & \mathcal{C}_{12} & \mathcal{C}_{13} \end{bmatrix}$ *come from the condensed form (12). There exist orthogonal matrices* $\mathcal{U}, \mathcal{V} \in \mathbf{R}^{(n_1+\tilde{n}_3)\times(n_1+\tilde{n}_3)}$ *such that*

$$(\mathcal{U}\begin{bmatrix} s\mathcal{E}_{11} - \mathcal{A}_{11} + \mathcal{B}_{11}C_{11} & -\mathcal{A}_{13} + \mathcal{B}_{11}\mathcal{C}_{13} \\ 0 & s\mathcal{E}_{33} - \mathcal{A}_{33} \end{bmatrix}\mathcal{V}, \mathcal{U}\left[\begin{array}{c|c} \mathcal{A}_{12} - \mathcal{B}_{11}\mathcal{C}_{12} & \mathcal{B}_{11} \\ \mathcal{A}_{32} & 0 \end{array}\right])$$

is in its controllability staircase form, i.e., the following properties hold:

$$\mathcal{U}\left[\begin{array}{cc|c|c} s\mathcal{E}_{11} - \mathcal{A}_{11} + \mathcal{B}_{11}C_{11} & -\mathcal{A}_{13} + \mathcal{B}_{11}\mathcal{C}_{13} & \mathcal{A}_{12} - \mathcal{B}_{11}\mathcal{C}_{12} & \mathcal{B}_{11} \\ 0 & s\mathcal{E}_{33} - \mathcal{A}_{33} & \mathcal{A}_{32} & 0 \end{array}\right]\begin{bmatrix} \mathcal{V} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{array}{c} \\ \left[\begin{array}{ccccc} n_1+\tilde{n}_3-\tau-\nu & \tau & \nu & \tilde{n}_2 & m_0 \\ s\tilde{\Theta}_{11} - \tilde{\Phi}_{11} & s\tilde{\Theta}_{12} - \tilde{\Phi}_{12} & s\tilde{\Theta}_{13} - \tilde{\Phi}_{13} & \Delta & \tilde{\Psi} \\ 0 & s\Theta - \Phi & s\tilde{\Theta}_{23} - \tilde{\Phi}_{23} & 0 & \Psi \\ 0 & 0 & s\tilde{\Theta}_{33} - \tilde{\Phi}_{33} & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} \\ \}n_1+\tilde{n}_3-\tau-\nu \\ \}\tau \\ \}\nu \end{array}, \tag{24}$$

$$\operatorname{rank}\begin{bmatrix} s\tilde{\Theta}_{11} - \tilde{\Phi}_{11} & \Delta \end{bmatrix} = n_1 + \tilde{n}_3 - \tau - \nu, \quad \forall s \in \mathbf{C}, \tag{25}$$

$$\operatorname{rank}\begin{bmatrix} s\Theta - \Phi & \Psi \end{bmatrix} = \tau, \ \forall s \in \mathbf{C}. \tag{26}$$

Theorem 7 *Assume that the form (24) has been determined. The simultaneous problem of (21), (22) and (23) is solvable if and only if*

$$\text{the pencils } (\tilde{\Theta}_{11}, \tilde{\Phi}_{11}) \text{ and } (\tilde{\Theta}_{33}, \tilde{\Phi}_{33}) \text{ are stable,} \tag{27}$$

and the "unusual" noninteracting control problem with stability for system

$$\Theta\dot{x} = \Phi x + \Psi u, \ y = u \tag{28}$$

with state feedback

$$u = \mathcal{K}x$$

is solvable, i.e., there exists a $\mathcal{K} \in \mathbf{R}^{m_0\times\tau}$ *such that*

$$\text{the pencil } (\Theta, \Phi + \Psi\mathcal{K}) \text{ is stable,} \quad \mathcal{K}(s\Theta - \Phi - \Psi\mathcal{K})^{-1}\Psi \text{ is diagonal.} \tag{29}$$

Moreover, if (27) and (29) hold true with $\mathcal{K} \in \mathbf{R}^{m_0\times\tau}$*, then (21), (22) and (23) hold with*

$$\mathcal{F} = \begin{bmatrix} 0 & \mathcal{K} & \hat{\mathcal{K}} \end{bmatrix}\mathcal{V}^T,$$

where $\hat{\mathcal{K}} \in \mathbf{R}^{m_0\times\nu}$ *is arbitrary.*

2.3 Stage 3

To our knowledge, the problem (29) has not been studied yet. In this subsection we will develop a numerically reliable algorithm for solving the problem (29).

The following result is trivial.

Corollary 8 *Let (Θ, Φ, Ψ) be from the condensed form (24). If $m_0 = 1$, then the problem (29) is always solvable and all its solutions are given by all matrices $\mathcal{K}$ satisfying that the pencil $(\Theta, \Phi + \Psi\mathcal{K})$ is stable.*

In the following we consider the case that $m_0 > 1$.

Theorem 9 *Assume that (Θ, Φ, Ψ) is from the condensed form (24) with $m_0 > 1$. There exist nonsingular matrices $\mathcal{X}, \mathcal{Y} \in \mathbf{R}^{\tau\times\tau}$ and a permutation matrix $\mathcal{P} \in \mathbf{R}^{m_0\times m_0}$ such that*

$$\mathcal{X}\left[\begin{array}{c|c} s\Theta - \Phi & \Psi \end{array}\right]\begin{bmatrix} \mathcal{Y} & 0 \\ 0 & \mathcal{P} \end{bmatrix} = \tag{30}$$

$$\begin{array}{cccccccccc}
\tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 & m_1 & \tau_5 & m_0 - m_1 - \tau_5 & & \\
\left[\begin{array}{c} s\Theta_{11} - \Phi_{11} \\ s\Theta_{21} - \Phi_{21} \\ s\Theta_{31} - \Phi_{31} \\ 0 \\ 0 \end{array}\right. &
\begin{array}{c} 0 \\ s\Theta_{22} - \Phi_{22} \\ s\Theta_{32} - \Phi_{32} \\ 0 \\ 0 \end{array} &
\begin{array}{c} 0 \\ 0 \\ s\Theta_{33} - \Phi_{33} \\ 0 \\ 0 \end{array} &
\begin{array}{c} 0 \\ 0 \\ 0 \\ s\Theta_{44} - \Phi_{44} \\ s\Theta_{54} - \Phi_{54} \end{array} &
\begin{array}{c} 0 \\ 0 \\ -\Phi_{35} \\ s\Theta_{45} - \Phi_{45} \\ s\Theta_{55} - \Phi_{55} \end{array} &
\begin{array}{c} \Psi_{11} \\ \Psi_{21} \\ \Psi_{31} \\ 0 \\ 0 \end{array} &
\begin{array}{c} 0 \\ 0 \\ 0 \\ \Psi_{42} \\ 0 \end{array} &
\left.\begin{array}{c} 0 \\ \Psi_{23} \\ \Psi_{33} \\ \Psi_{43} \\ 0 \end{array}\right] &
\begin{array}{l} \}\tau_1 \\ \}\tau_2 \\ \}\tau_3 \\ \}\tau_5 \\ \}\tau_4 \end{array} & ,
\end{array}$$

where

$$0 < m_1 < m_0, \quad 0 \le \tau_5 \le 1, \tag{31}$$

$$\text{if } \tau_5 = 0, \text{ then } \tau_3 = \tau_4 = 0, \tag{32}$$

$$\operatorname{rank}\begin{bmatrix} s\Theta_{11} - \Phi_{11} & 0 & 0 & \Psi_{11} \\ s\Theta_{21} - \Phi_{21} & s\Theta_{22} - \Phi_{22} & 0 & \Psi_{21} \\ s\Theta_{31} - \Phi_{31} & s\Theta_{32} - \Phi_{32} & s\Theta_{33} - \Phi_{33} & \Psi_{31} \end{bmatrix} = \tau_1 + \tau_2 + \tau_3, \quad \forall s \in \mathbf{C}, \tag{33}$$

$$\operatorname{rank}\begin{bmatrix} s\Theta_{22} - \Phi_{22} & \Psi_{23} \end{bmatrix} = \tau_2, \quad \forall s \in \mathbf{C}, \tag{34}$$

and furthermore, if $\tau_5 = 1$, we also have

$$\Psi_{42} \neq 0, \tag{35}$$

$$\operatorname{rank}\begin{bmatrix} s\Theta_{33} - \Phi_{33} & \Phi_{35} \end{bmatrix} = \tau_3, \quad \operatorname{rank}(s\Theta_{54} - \Phi_{54}) = \tau_4, \quad \forall s \in \mathbf{C}. \tag{36}$$

Moreover, matrices $\mathcal{X}$, $\mathcal{Y}$ and the condensed form (30) are computed only based on orthogonal transformations which can be implemented via an inverse-free and numerically reliable way.

We are now ready to derive a very useful reduction property of the problem (29) based on the condensed form (30).

Theorem 10 *Let that (Θ, Φ, Ψ) be from the condensed form (24). Assume that nonsingular matrices $\mathcal{X}$, $\mathcal{Y}$ and the permutation matrix $\mathcal{P}$ have been determined satisfying the condensed form (30). Then the problem (29) is solvable if and only if*

$$\text{pencils } (\Theta_{22}, \Phi_{22}) \text{ and } (\Theta_{33}, \Phi_{33}) \text{ are stable,} \tag{37}$$

$$\text{the pencil } \left(\begin{bmatrix} \Theta_{44} & \Theta_{45} \\ \Theta_{54} & \Theta_{55} \end{bmatrix}, \begin{bmatrix} \Phi_{44} & \Phi_{45} \\ \Phi_{54} & \Phi_{44} \end{bmatrix} \right) \text{ is stable if } \Psi_{43} \neq 0, \tag{38}$$

and there exists a matrix $\mathcal{K}_{11} \in \mathbf{R}^{m_1 \times \tau_1}$ *such that*

$$\text{pencil } (\Theta_{11}, \Phi_{11} + \Psi_{11}\mathcal{K}_{11}) \text{ is stable, } \mathcal{K}_{11}(s\Theta_{11} - \Phi_{11} - \Psi_{11}\mathcal{K}_{11})^{-1}\Psi_{11} \text{ is diagonal.} \tag{39}$$

Moreover, if (37), (38) and (39) hold, then one solution $\mathcal{K}$ *of the problem (29) is given by*

$$\mathcal{K}\mathcal{Y} = \mathcal{P} \begin{array}{c@{}l} \begin{array}{ccccc} \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \end{array} & \\ \begin{bmatrix} \mathcal{K}_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}_{24} & \mathcal{K}_{25} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \}m_1 \\ \}\tau_5 \\ \}m_0 - m_1 - \tau_5 \end{array} \end{array}, \tag{40}$$

where

$$\begin{bmatrix} \mathcal{K}_{24} & \mathcal{K}_{25} \end{bmatrix} = 0 \text{ if } \Psi_{43} \neq 0, \tag{41}$$

$$\text{the pencil } \left(\begin{bmatrix} \Theta_{44} & \Theta_{45} \\ \Theta_{54} & \Theta_{55} \end{bmatrix}, \begin{bmatrix} \Phi_{44} & \Phi_{45} \\ \Phi_{54} & \Phi_{55} \end{bmatrix} + \begin{bmatrix} \Psi_{42} \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{K}_{24} & \mathcal{K}_{25} \end{bmatrix} \right) \text{ is stable if } \Psi_{43} = 0. \tag{42}$$

Theorem 10 leads to the following numerically reliable algorithm, which is only based on orthogonal transformations and solutions of some linear systems of equations, for solving the problem (29).

Algorithm 1
Input*:* $\Theta, \Phi \in \mathbf{R}^{\tau \times \tau}$, $\Psi \in \mathbf{R}^{\tau \times m_0}$ *satisfying (26).*
Output*:* $\mathcal{K} \in \mathbf{R}^{m_0 \times \tau}$ *(if possible) solving the problem (29).*
Step 0*. Set* $K := \emptyset$, $M := I_{m_0}$, $\mathcal{N} := I_\tau$, $l := 0$.
Step 1*. If* $m_0 = 1$*, compute* $\mathcal{K}$ *such that the pencil* $(\Theta, \Phi + \Psi\mathcal{K})$ *is stable, and then set* $K := \begin{bmatrix} \mathcal{K} & 0 \\ 0 & K \end{bmatrix}$. *Go to Step 3. Otherwise, if* $m_0 > 1$*, go to Step 2.*
Step 2*. Compute the condensed form (30). If (37) or (38) fails, print "The Problem (29) Is Unsolvable" and stop. Otherwise, compute* $\begin{bmatrix} \mathcal{K}_{24} & \mathcal{K}_{25} \end{bmatrix}$ *based on (41) and (42). Set*

$$K := \begin{array}{c@{}l} \begin{array}{ccccc} \tau_2 & \tau_3 & \tau_4 & \tau_5 & \end{array} & \\ \begin{bmatrix} 0 & 0 & \mathcal{K}_{24} & \mathcal{K}_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K \end{bmatrix} & \begin{array}{l} \}\tau_5 \\ \}m_0 - m_1 - \tau_5 \\ l \end{array} \end{array}, \quad M := M \begin{bmatrix} \mathcal{P} & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{N} = \mathcal{N} \begin{bmatrix} \mathcal{Y} & 0 \\ 0 & I \end{bmatrix},$$

and $\Theta := \Theta_{11}$, $\Phi := \Phi_{11}$, $\Psi := \Psi_{11}$, $l = l + m_0 - m_1$, $\tau = \tau_1$, $m_0 := m_1$. *Go to Step 1.*
Step 3*. Compute* $\mathcal{K}$ *by solving the linear system* $\mathcal{K}\mathcal{N} = MK$*. Output* $\mathcal{K}$.

2.4 Stage 4

The results in Subsections 2.1, 2.2 and 2.3 can be combined to provide an overall numerically reliable algorithm for solving the noninteracting control problem with stability for (A, B, C, D) quadruples as follows.

Algorithm 2
Input*:* $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{m \times n}$ *and* $D \in \mathbf{R}^{m \times m}$.
Output*: Matrices* $F \in \mathbf{R}^{m \times n}$ *and* $H \in \mathbf{R}^{m \times m}$ *(if possible) such that the matrix* $A + BF$ *is stable and (3) holds.*

Step 1. *Compute the condensed form (5) using the algorithm given in [10]. If the conditions (9) and (10) are not true, print "The Studied Problem Is Unsolvable" and stop. Otherwise, compute the condensed form (12).*
Step 2. *Solve the noninteracting control problem with stability for system (17) using the algorithm in [9]. If it is unsolvable, then print "The Studied Problem Is Unsolvable" and stop. Otherwise, compute one of its solutions* $(\left[\begin{array}{cc} F_{22} & F_{23} \end{array}\right], H_{22})$ *such that (18) and (19) hold.*
Step 3. *Compute the staircase form (24). If the condition (27) does not hold, print "The Studied Problem Is Unsolvable" and stop. Otherwise, compute a solution* $\mathcal{K}$ *of the problem (29), if it exists.*
Step 4. *Set* $\mathcal{F} = \begin{array}{c} \begin{array}{ccc} n_1+\tilde{n}_3-\tau-\nu & \tau & \nu \end{array} \\ \left[\begin{array}{ccc} \qquad 0 \qquad & \mathcal{K} & 0 \end{array}\right] \end{array} \mathcal{V}^T$, *and compute* F *and* H *by*

$$B_{22}F_{21} = -A_{21},\ \mathcal{D}_{11}F_{12} = -\mathcal{C}_{12},\ \mathcal{D}_{11}\left[\begin{array}{cc} F_{11} & F_{13} \end{array}\right] = \mathcal{F} - \left[\begin{array}{cc} C_{11} & \mathcal{C}_{13} \end{array}\right],$$

$$FVY = WZ\left[\begin{array}{ccc} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \end{array}\right], \qquad H = WZ\left[\begin{array}{cc} H_{11} & 0 \\ 0 & H_{22} \end{array}\right]P$$

with $\mathcal{D}_{11}H_{11}$ being nonsingular and diagonal.

Output F *and* H.

3 Conclusions and Remarks

We have developed a numerical algorithm–Algorithm 2 to verify the solvability conditions and compute a solution for the noninteracting control problem with stability for (A, B, C, D) quadruples. Algorithm 2 involves only orthogonal transformations and the solutions of several linear systems of equations and thus it can be implemented in a numerically reliable way.

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