Positive systems in the state space approach: main issues and recent results

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Abstract

A positive system is a system in which the state variables are always positive in value. In this introductory tutorial paper, basic results on positive systems are reviewed and recent developments and open problems are addressed.

1 Introduction

> The theory of positive systems is deep and elegant – and yet pleasantly consistent with intuition. [...] It is for positive systems, therefore, that dynamic systems theory assumes one of its most potent forms.
>
> (David Luenberger, in *Introduction to Dynamic Systems*, Wiley, 1979)

As stated in the celebrated Professor Luenberger's book on dynamic systems, *a positive system is a system in which the state variables are always positive (or at least nonnegative) in value.* It is somewhat surprising to realize how easily available is the information on positivity of state variables and how strong can be the consequences of such information on system's behaviour! Again, this surprise is very well described in the just cited book:

> Indeed, just the knowledge that the system is positive allows one to make some fairly strong statements about its behaviour: these statements being true no matter what value the parameters may happen to take.

So, going deep further into this *feeling* of surprise, one may find evidences in many and diverse areas of science and technology, since the positivity property just defined, is always nothing but the immediate consequence of the nature of the phenomenon we are dealing with. A huge number of examples are just before our eyes: any possible type of *resource* measured by a quantity: time, money, goods, buffers size, queues, data packets flowing in a network, human, animal and plant populations, concentration of any conceivable substance you may think of and also – if you haven't conceived this – mRNAs, proteins, molecules, electric charge (see [3, 8]), light intensity levels (see [5]) ... Moreover, last, but not least, one has to mention any stochastic model (such as the Hidden Markov Model), since, clearly, also probabilities are positive quantities.

However, in oder to be *politically correct*, it is very important to well define limitations and boundaries of this "extended family" of systems. Other than the classical field of electro-mechanical systems where any value is admissible, we have to point out that in many real situations one is interested in considering the deviations of the state variables from a certain equilibrium or operating point (set point) which may *not* be the origin, so that in this case the positivity property of the original state variables can be assumed to hold provided that the deviations are "small enough". But, still, as we will see very soon hereafter, also in such situations, we are not at all at the end of the story...

In this tutorial paper, to explain the main issues and discuss some interesting new openings in the theory of positive systems, we will concentrate on the simple case of linear time invariant single input/single output discrete-time positive systems. Recently, many issues regarding positive system analysis and control has been studied by a large number of authors. We just cite a few: reachability [10, 11, 14, 17, 27, 33, 36, 38], stabilization [12, 26, 23, 39, 40, 34], 2D systems and behavioural approach [24, 27, 28, 35, 42, 44], optimal control [25], identification [13], realization [1, 2, 4, 6, 15, 16, 19, 20, 21, 22, 29, 31, 36, 41, 42, 45, 46].

However, for the sake of brevity, in this paper we will limit ourselves to discussing a few issues regarding positive systems.

The paper is organized as follows. The next section deals with the homogeneous case and the celebrated fundamental result due to Frobenius and Perron will be stated with emphasis on the dynamical aspects of system's behaviour. In section III the nonhomogeneous case is treated and, in particular, questions related to external (input/output) positivity, stability and realization are briefly discussed.

The reader interested in the topic presented in this paper may find useful reference [18].

2 Homogeneous positive systems: definitions and dominant modes analysis

We begin with basic definitions.

Definition 2.1. *A vector x or a matrix A will be said to be positive [strictly positive], and denoted by $x > 0$, $A > 0$, provided that its entries are nonnegative but at least one of them is positive [provided that its entries are positive]. A vector sequence $x(k)$ will be said to be positive [strictly positive] provided that its entries are nonnegative for any $k \geq 0$ but $x(k)$ is positive for some $k > 0$ [provided that its entries are positive for any $k \geq 0$]. The set defined by $x \in \mathbb{R}^n$ such that $x_i \geq 0$, will be called the nonnegative orthant and denoted by $\mathbb{R}^n_+$.*

Note that the above definition implies that we do not consider the trivial cases of zero vectors, matrices or sequences[1].

Let us consider the homogeneous LTI discrete-time dynamic system

$$x(k+1) = Ax(k) \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$ and the following definition of positivity of a system:

Definition 2.2. *A linear system described by the state space representation (2.1) is said to be an (homogeneous) positive linear system iff for any positive initial state vector, the trajectory remains positive for all times.*

The above definition requires the nonnegative orthant to be invariant for the dynamics, *i.e.* the positivity property as just defined call for the mathematical model (2.1) to be consistent with the *a priori* information on the positivity of the relevant variables involved in the process. In fact, in order to make the model consistent, it must happen that *any* initial state in the nonnegative orthant generates a state trajectory bounded to "live" only in the nonnegative orthant. This requirement may appear - depending on the information available and on the modeler attitude - pretty obvious or unduly demanding. Both choices are to be considered somehow reasonable and the actual decision strongly depends on the specific case at hand (other than personal beliefs and tastes... as usual). Different definitions of positivity may be adopted by referring to specific subsets of initial states of practical interest. In the following we will assume that, even if the real values of initial conditions are bounded by some region of the nonnegative orthant, we have no reason to believe that some (real or fictitious) initial state may produce a nonpositive sequence.

Then, let us suppose that we are dealing with a genuine positive system, then the following result is an immediate (and very easily proved) consequence of the above definition:

Theorem 2.1. *A linear system described by the state space representation (2.1) is a positive linear system iff $A > 0$, i.e. the dynamic matrix A is positive.*

The above theorem enable us to determine whether a system is positive or not by simply looking at the entries of the dynamic matrix. It is important to note, at this point, that even if we are interested in modelling the deviations of the (positive) state variables from an equilibrium point (set point), nevertheless the positivity still plays a key role in the determination of the system's dynamic behaviour! In fact, positivity of the dynamic matrix A is independent of the specific equilibrium we are considering. This is the reason why, positivity is a more general property than one might think. Actually, to be more precise (but not pushing too far the question), positivity of matrix A implies also that two solutions

[1] Adopting this definition, one has that a sequence such that $x(0) > 0$ and $x(k) = 0$ for any $k > 0$ is *not* positive. Clearly, this behaviour is trivial from a dynamic system's point of view, so it is reasonable to remove this possibility from the world of positive systems.

$x_1(k)$ and $x_2(k)$ of the associated dynamic system (2.1) share the following property known as *comparative dynamics* (see [30]):

$$x_1(0) \geq x_2(0) \Longrightarrow x_1(k) \geq x_2(k) \text{ for any } k > 0$$

which defines the so-called *cooperative* or *order preserving flow systems.* For linear systems, positivity implies cooperativity and viceversa, but this not generally true for the nonlinear case.

Going back to positive linear systems and before stating the famous result of Frobenius (and Perron) on positive matrices and its consequences on the dominant dynamics of an homogeneous positive system, we need a simple preliminary definition.

Definition 2.3. *An homogeneous positive system of dimension $n \geq 2$ is said to be reducible iff the evolution of a set of n_1 state variables is independent of the evolution of the remaining $n_2 = n - n_1$ state variables. An homogeneous positive system that is not reducible is said to be irreducible.*

For example, systems composed of two positive systems connected in series or in parallel, are reducible. As one can readily see ([18], [9]), any given homogeneous positive systems such as (2.1), after a reordering of the state variables by means of a change of coordinates defined by an appropriate permutation matrix P, can be written in the equivalent form $z(k+1) = PAP^T z(k)$ where PAP^T is block triangular, that is

$$PAP^T = \begin{pmatrix} A_{11}^* & 0 & \dots & 0 \\ A_{21}^* & A_{22}^* & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{k1}^* & A_{k2}^* & \dots & A_{kk}^* \end{pmatrix}$$

and the k square matrices on the main diagonal defines k positive irreducible systems. Since the spectrum of a block triangular matrix is the union of the spectra of each submatrix on its main diagonal, it is clear that we can study, without loss of generality, only the case of irreducible positive systems. We are now ready to state a reformulation of the Perron-Frobenius theorem:

Theorem 2.2. *If an irreducible homogeneous discrete-time system (2.1) has h dominant eigenvalues of modulus r, then these numbers are distinct roots of $\lambda^h - r^h = 0$. In particular, one of them is positive real with algebraic multiplicity equal to one and called the Frobenius eigenvalue λ_F. The associated eigenvector is strictly positive and called the Frobenius eigenvector x_F. Moreover, there are no positive eigenvectors other than the Frobenius eigenvector.*

The Perron-Frobenius theorem provides information of the long term behaviour of an homogeneous positive irreducible system, so that we may well agree with Professor Luenberger:

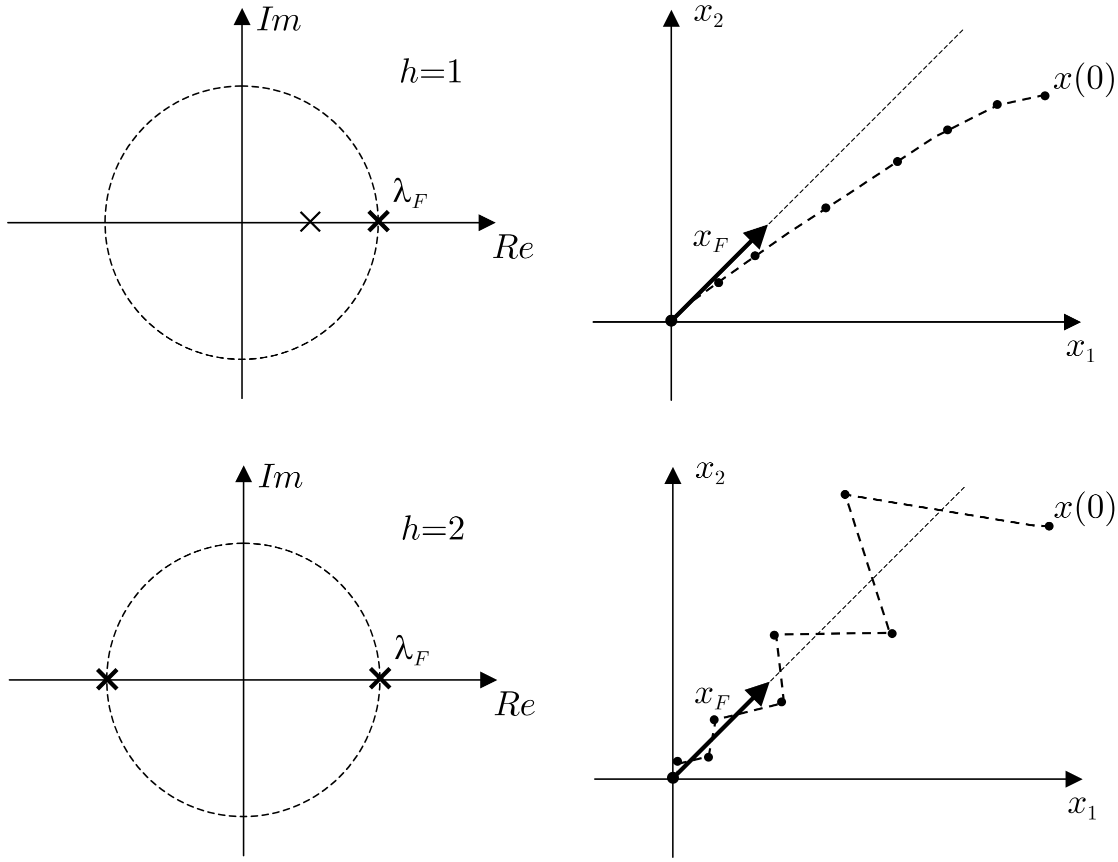

Fig.1: The behaviour of a second order discrete-time irreducible positive linear system for the cases $h = 1$ and $h = 2$

> Just the knowledge that the system is positive allows one to make some fairly strong statements about its behaviour; these statements being true no matter what values the parameters may happen to take.

In fact, we know that for a linear system like (2.1), the state vector tends to align itself with the dominant eigenvector, and consequently the behaviour of *any* irreducible positive system is just like one of those depicted in Figure 1 for the case of a second order system.

3 Nonhomogeneous positive systems: External positivity, stability, and realization

In this section we will focus on the more general case in which, other than state variables, also inputs and outputs have to be modelled. Let us then consider for the sake of simplicity the homogeneous SISO LTI discrete-time dynamic system:

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k) \\ y(k) &= c^T x(k) \end{aligned} \tag{3.2}$$

where $A \in \mathbb{R}^{n \times n}$ and $b, c \in \mathbb{R}^n$ and the following definition of positivity of a system:

Definition 3.1. *A linear system described by the input/state/output representation (3.2) is said to be a (nonhomogeneous) positive linear system iff for any positive initial state vector and positive input, the output and the state trajectory are positive.*

The following theorem enables one to recognize whether a given system in state space form is positive or not.

Theorem 3.1. *A linear system described by the input/state/output representation (3.2) is a positive linear system iff $A, b, c > 0$.*

3.1 External positivity and stability

When considering, as in this case, also input and outputs, it is natural to define properties regarding them. By doing so, we arrive at the following "natural" definition of input/output positivity (that can be called *external positivity*):

Definition 3.2. *A linear system is said to be externally positive iff for any positive input, the output is positive.*

The above definition leads to the following theorem, whose proof is straightforward:

Theorem 3.2. *A linear system is an externally positive system iff its impulse response is positive.*

Also in the case of input/output representation, the condition of positivity can be simply checked by inspection on the impulse response. Clearly, a systematic (and minimal in some sense) way to check positivity, given the transfer function would be preferable. However, external positivity is an *a priori* information which is easily available and stems directly from the specific problem at hand. It is plain that, for example, when modelling the kinetics of a drug in the human body, typically the input is the drug input rate and the output is some concentration measure in a specific organ of the body. If this is the case, the system under study is certainly an externally positive system (and *internally* as well).

An interesting problem is that of estimating model parameters (identification) while imposing external positivity. A first step of this hard task is described in [13, 7].

What about specific properties of the *poles*? Well, a basic result is a direct consequence of Theorem 2.2:

Theorem 3.3. *The poles of the transfer function of a positive system are a subset of those which are admissible eigenvalues of some positive matrix. One of the dominant poles is positive real (and called the Frobenius pole).*

The above theorem admits the possibility that some of the eigenvalues (possibly dominant) may be *hidden*, *i.e.* they do not appear as poles of the transfer function. It is important to point out that one cannot consider only coprime transfer functions, since it might be well the case that positivity "forces" the presence of hidden modes, as we will see soon in the subsequent section. For this reason, the concept of "minimality" of a state space representation (realization) of a positive system is inherently different from that of ordinary linear systems.

For externally positive systems, the following result hold:

Theorem 3.4. *One of the dominant poles of an externally positive system is positive real (and called the Frobenius pole).*

The proof is found immediately by considering that the long term behaviour of the impulse response must remain positive for all times.

We end this section with a remarkable result on stability of externally positive systems (whose proof can be found in [18]):

Theorem 3.5. *An externally positive system is asymptotically stable iff the denominator* $d(z)$ *of the transfer function of the given system is such that the coefficients of* $d(z-1)$ *are all positive.*

It is known that the above condition is only *necessary* for ordinary linear systems, but for *positive systems* is also sufficient! This fact greatly simplifies the calculations and the only information needed is externally positivity...

So, it's worth recalling hereafter Professor Luenberger statement: *...just the knowledge that the system is positive allows one to make some fairly strong statements about its behaviour...*

3.2 Positive realization

Given a strictly proper rational transfer function $G(z)$, the triple $\{A, b, c^T\}$ is said to be a *positive realization* if

$$G(z) = \sum_{k \geq 1} c^T A^{k-1} b z^{-k}$$

with A, b, c^T positive. The positive realization problem consists of providing answers to the questions:

- (*The existence problem*) Is there a positive realization $\{A, b, c^T\}$ of some finite dimension N and how it may be found?

- (*The minimality problem*) What is a minimal value for N?
- (*The generation problem*) How can we generate all possible positive realizations?

In references [1, 15], the existence problem has been completely solved and a means of constructing such realizations is there also given. In this section we shall not consider the interesting question of characterizing the relationship between equivalent realizations, and we shall concentrate on what we have termed the minimality problem.

3.2.1 Minimality in the positive setting

It is known that, for transfer functions of degree 1 or 2, positivity of the impulse response is a necessary and sufficient condition for the existence of a positive realization. Moreover, in those two cases, the minimal dimension of a positive realization coincides with the degree of the transfer function. On the other hand, the situation for the case of transfer functions of degree $n > 2$ is totally different. To show that the minimality problem for positive linear systems is inherently different from that of ordinary linear systems we shall make use of the following two examples.

Example 1 Consider the following positive realization

$$A = \overbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & & & 0 & 0 \\ 0 & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}}^{2^{q+1}} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \tag{3.3}$$

$$c^T = \underbrace{\begin{pmatrix} 0 & \dots & 0 \end{pmatrix.}}_{2^q} \underbrace{\begin{matrix} 1 & \dots & 1 \end{matrix} \Big)}_{2^q}$$

where the parameter q is an integer greater than or equal to 1. The dimension of the realization is $N(q) = 2^{q+1}$ while the corresponding transfer function

$$G(z, q) = \frac{1}{(z-1)\left(z^{2^q}+1\right)}, \qquad q \geq 1$$

is of McMillan degree $n(q) = 2^q + 1$. By exploiting the rotational symmetry of the dominant poles of $G(z, q)$ as required by the Frobenius theorem, it can be easily proved that for any integer $q \geq 1$, the realization (3.3) is minimal as a positive linear system. The poles pattern of $G(z, q)$ for $q = 1$ and $q = 2$ is depicted in Figure 2.

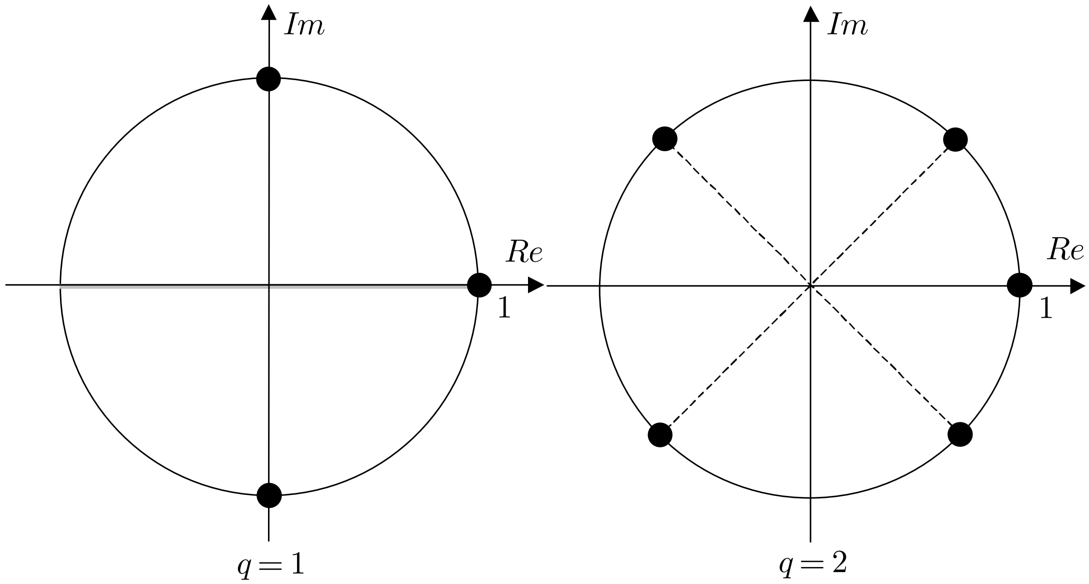

Fig.2: Poles pattern of $G(z, q)$ for $q = 1$ and $q = 2$

Note that the difference between the dimension $N(q)$ of the minimal positive realization of the system and the corresponding transfer function McMillan degree $n(q)$

$$N(q) - n(q) = 2^q - 1$$

goes exponentially to ∞ as q increases.

As it will appear in the sequel, this rotational symmetry of the spectrum of a positive matrix, due to the specific dominant poles pattern, is not the only reason for non minimality (i.e. not jointly reachable and observable systems) in the positive realization problem. Roughly speaking, we show next that the dimension of a positive realization may be "large" although the dominant eigenvalue is unique, so that no symmetry of the spectrum is required by the Frobenius theorem. In fact, since a positive matrix cannot have arbitrary eigenvalues, then the non dominant poles also have limitations. For this consider the sets Θ_n^ρ denoting the set of points in the complex plane that are eigenvalues of positive $n \times n$ matrices with Frobenius eigenvalue ρ. A full characterization of these sets has been given by Karpelevic (see [32]). For example, the set Θ_2^ρ consists of points on the segment $[-\rho,\ \rho]$ and the set Θ_3^ρ consists of points in the interior and on the boundary of the triangle with vertices ρ, $\rho e^{2\pi i/3}$, $\rho e^{4\pi i/3}$ and on the segment $[-\rho,\ \rho]$. The sets Θ_3^ρ and Θ_4^ρ are depicted in Figure 3.

Example 2 Consider the following positive realization

$$A = \begin{pmatrix} 0 & 0.95 & 0 & 0.05 \\ 0.05 & 0 & 0.95 & 0 \\ 0 & 0.05 & 0 & 0.95 \\ 0.95 & 0 & 0.05 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{3.4}$$

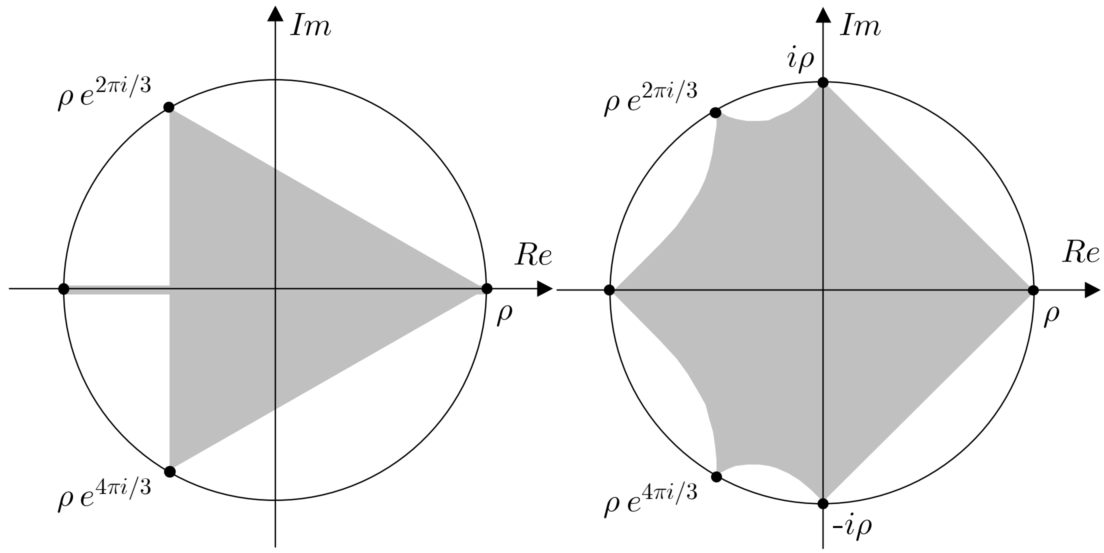

Fig.3: The sets Θ_3^ρ and Θ_4^ρ

of dimension 4. The corresponding transfer function

$$G(z) = \frac{*}{(z-1)(z^2+0.81)}$$

is of McMillan degree 3. Since the poles of $G(z)$ are 1, $\pm 0.9\,i$ and they lie inside Θ_4^1 and not in Θ_3^1, then the dynamic matrix of any minimal positive realization must be of dimension greater than 3. Therefore the fourth order positive realization (3.4) is minimal as a positive system.

This last mechanism, related to a specific poles pattern, is – again – not the only reason for non minimality in the positive realization problem even when the dominant eigenvalue is unique. In fact, the dimension of a positive realization may be "large" although the dominant eigenvalue is unique and no complex eigenvalues are present. This should be not surprising since positivity of the system implies restrictions not only on the dynamic matrix but on the input and output vectors also. In fact, it has been shown in reference [4] that the transfer function

$$G(z,N) = \frac{1}{z-1} - 25 \cdot \frac{0.4^{4-N}}{z-0.4} + 75 \cdot \frac{0.2^{4-N}}{z-0.2}$$

admits a minimal positive realization of state space dimension not smaller than N, where the parameter N is an integer greater than or equal to 4. This is quite surprising since, in spite of the fact that we are dealing with the seemingly simple case of a third order transfer function with distinct positive real poles, the minimal positive realization may possibly have a "large" state space dimension.

In what follows we present a partial insight into the positive minimality problem dealing with the case of third order transfer functions with distinct positive real poles. In this case, in reference [2], the following result is proved:

Theorem 3.6. *Let*

$$G(z) = \frac{r_1}{z-\lambda_1} + \frac{r_2}{z-\lambda_2} + \frac{r_3}{z-\lambda_3}$$

be a third order transfer function (i.e. $r_1,\ r_2,\ r_3 \neq 0$*) with distinct positive real poles* $\lambda_1 > \lambda_2 > \lambda_3 > 0$*. Then,* $G(z)$ *has a third order positive realization if and only if the following conditions hold:*

1. $r_1 > 0$
2. $r_1 + r_2 + r_3 \geq 0$
3. $(\lambda_1 - \overline{\eta})\, r_1 + (\lambda_2 - \overline{\eta})\, r_2 + (\lambda_3 - \overline{\eta})\, r_3 \geq 0$
4. $(\lambda_1 - \eta)^2\, r_1 + (\lambda_2 - \eta)^2\, r_2 + (\lambda_3 - \eta)^2\, r_3 \geq 0$ *for all* η *such that* $\overline{\eta} \leq \eta \leq \lambda_3$

$$\textit{where } \overline{\eta} = \max\left\{0, \frac{\lambda_1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{3}\right\}$$

It is worth noting that the proof of the previous result, as presented in [2], is mainly geometric and heavily relies on the third–order assumption. For this reason, it appears very difficult to us to extend that kind of proof to the higher order case. Nevertheless, this geometric approach may be fruitfully applied to the case in which either the assumption on the poles location is removed or the order of the minimal positive realization is not limited to equal the McMillan degree.

4 Open problems

As it is clear from the issues so far discussed, there are a considerable number of open problems related to positive systems. We just name a few of them. First of all, it is not clear what kind of mathematical "instruments" should be used to effectively tackle the minimality problem. In fact, the geometric approach (*i.e.* that of working with invariant cones) has proved to be the right choice for determining the existence of a positive realization. By contrast, such approach, has lead to the determination of necessary and sufficient conditions for the third order case only. A different formulation, such as the factorization approach proposed by Picci and van Schuppen in reference [37], can be a viable and promising possibility.

Another important issue related to minimality of positive systems is the study of "hidden modes", *i.e.* of the eigenvalues which possibly one has to add in order to obtain a minimal positive realization. A full characterization of this property may lead to a deeper and valuable insight into the problem. Lastly, we mention the MIMO case, which is not a straightforward extension of the SISO case, as for the existence problem.

Some other open problems related to minimality of positive systems are listed below:

- how are all positive minimal realizations connected?
- how can one simply figure, directly from the system's parameters (say, residues and eigenvalues), the minimum number of samples of the impulse response to be checked in order to infer positivity of the whole impulse response and how is this number related to minimality?
- how can one approximate a positive realization by a lower dimension one?
- find "tight" lower and upper bounds to minimal order of a positive realization[2].

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