Positive systems in the behavioral approach: main issues and recent results

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Abstract

The aim of this paper is to provide an overview of the most significant results about positive systems, obtained within the behavioral approach, and to present some new results concerned with the positive realization problem for controllable behaviors.

1 Introduction

In recent years, research interests aiming at developing a general theory of positive linear systems within the behavioral framework [8] resulted in a few contributions which laid on firm foundations the concepts of positive behavior and of positive realizable autonomous behavior. The first original ideas and definitions appeared in a very nice paper [7] by Nieuwenhuis, where the notion of positive (discrete) behavior (whose trajectories are defined on the time axis $\mathbb{Z}_+$) and some preliminary results were presented. These definitions and results stimulated a special interest in autonomous behaviors, thus leading first to a complete characterization of positive autonomous behaviors [10] and, later, to the solution of the positive realization problem for autonomous behaviors [11].

The most recent research efforts in this area are represented by the analysis of the positive realization problem for controllable behaviors. The research on this subject is still a work in progress, even though some significant results have already been obtained in [12].

The aim of this paper is to provide an overview of the previously mentioned results and hence to present the state-of-the-art of the research on positive systems within the behavioral approach. The paper is quite concise and the interested reader is referred to the aforementioned papers for the details.

2 Basic facts from behavior theory

In this paper, by a *dynamic system* we mean a triple $\Sigma = (\mathbb{Z}_+, \mathbb{R}^q, \mathfrak{B})$, where $\mathbb{Z}_+$ represents the *time set*, $\mathbb{R}^q$ is the *signal alphabet*, namely the set where the system trajectories take values, and $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is the *behavior*, namely the set of trajectories which are compatible with the system laws. We will restrict our attention to the class of *linear left shift-invariant* and *complete* behaviors [8]. Such behaviors are kernels of polynomial matrices in the left shift operator σ, which amounts to saying that the trajectories $\mathbf{w} = \{\mathbf{w}(t)\}_{t\in\mathbb{Z}_+}$ of $\mathfrak{B}$ can be identified with the set of solutions in $(\mathbb{R}^q)^{\mathbb{Z}_+}$ of a system of difference equations

$$R_0\mathbf{w}(t) + R_1\mathbf{w}(t+1) + \cdots + R_L\mathbf{w}(t+L) = 0, \qquad t \in \mathbb{Z}_+, \tag{2.1}$$

with $R_i \in \mathbb{R}^{p\times q}$. A behavior $\mathfrak{B}$ described in this way is denoted, for short, as $\mathfrak{B} = \ker(R(\sigma))$, where $R(z) := \sum_{i=0}^{L} R_i z^i$ belongs to $\mathbb{R}[z]^{p\times q}$.

A behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be *autonomous* if there exists $m \in \mathbb{Z}_+$ such that if $\mathbf{w}^1, \mathbf{w}^2 \in \mathfrak{B}$ and $\mathbf{w}^1|_{[0,m]} = \mathbf{w}^2|_{[0,m]}$, then $\mathbf{w}^1 = \mathbf{w}^2$, and is said to be *controllable* if there exists some positive integer L such that for every $t \in \mathbb{Z}_+$ and every pair of trajectories $\mathbf{w}_1, \mathbf{w}_2 \in \mathfrak{B}$, there exists $\mathbf{w} \in \mathfrak{B}$ such that $\mathbf{w}|_{[0,t)} = \mathbf{w}_1|_{[0,t)}$ and $\mathbf{w}|_{[t+L,+\infty)} = \mathbf{w}_2|_{[t,+\infty)}$.

A behavior $\mathfrak{B} = \ker(R(\sigma))$, with $R \in \mathbb{R}[z]^{p\times q}$, is autonomous if and only if R is of full column rank q. Any autonomous behavior can be described as the kernel of a nonsingular square matrix, which is uniquely determined up to a left unimodular factor. On the other hand, for a controllable behavior $\mathfrak{B}$ there exist an $m \in \mathbb{N}$ and $M \in \mathbb{R}[z]^{q\times m}$, such that $\mathbf{w} \in \mathfrak{B}$ if and only if $\mathbf{w} = M(\sigma)\mathbf{u}$, for some $\mathbf{u} \in (\mathbb{R}^m)^{\mathbb{Z}_+}$. When so, we adopt the notation $\mathfrak{B} = \mathrm{im}(M(\sigma))$ and the behavior $\mathfrak{B}$ is said to be endowed with an *image representation*.

3 Nonnegative autonomous behaviors

Definition 3.1 [7] *A behavior (in particular, an autonomous behavior)* $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ *is said to be* positive *if there exist* $m \in \mathbb{N}$ *and positive trajectories* $\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^m$ *such that* $\mathfrak{B} \equiv \mathfrak{B}(\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^m)$*, namely* $\mathfrak{B}$ *is* the most powerful unfalsified model [4] *explaining* $\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^m$*.*

A fundamental result we will resort to, in order to provide a characterization of positivity for autonomous behaviors, is the fact that every autonomous behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ can be "realized" by means of a state-space model [5], by this meaning that there exist $n \in \mathbb{N}$ and real matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{q\times n}$ such that

$$\mathfrak{B} = \left\{ \mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+} :\ \exists\, \mathbf{x}(0) \text{ s.t. } \mathbf{x}(t+1) = A\mathbf{x}(t),\ \mathbf{w}(t) = C\mathbf{x}(t),\ t \in \mathbb{Z}_+ \right\}.$$

The pair (A, C) is an *n-dimensional realization of* $\mathfrak{B}$. Those realizations of $\mathfrak{B}$ for which n is minimal are called *minimal.* Minimal realizations of an autonomous behavior $\mathfrak{B}$ are those realizations of $\mathfrak{B}$ which are observable [8, 10].

We need, now, to introduce a few important sets, that will turn out to be useful in providing a complete characterization of both positivity and, later, of positive realizability. To this end, we need to consider a general (not necessarily autonomous) n-dimensional state space model with m inputs and p outputs:

$$\mathbf{x}(t+1) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{3.2}$$
$$\mathbf{w}(t) = C\mathbf{x}(t), \qquad t \in \mathbb{Z}_+. \tag{3.3}$$

Such a system is denoted, for sake of brevity, by means of the triple (A, B, C).

Definition 3.2 *Given an n-dimensional state space model (A, B, C), with m inputs and p outputs, the* reachable cone *of the system (or, equivalently, of the pair (A, B)) is the cone*

$$\mathcal{R}(A, B) := \overline{\mathrm{Cone}}\{B, AB, A^2B, \dots\},$$

where the "upper bar" over the word "Cone" denotes the closure operation, while the observable cone *of the system (of the pair (A,C)) is the cone*

$$\mathcal{S}(C,A) := \{\mathbf{x} \in \mathbb{R}^n : CA^t\mathbf{x} \geq 0, \ \forall\, t \geq 0\} = \Big(\mathrm{Cone}\{C^T, A^TC^T, (A^T)^2C^T, \dots\}\Big)^*,$$

where the symbol $*$ denotes the "dual" of the indicated cone.

Definition 3.3 *Given a behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$, we call the* positive part of $\mathfrak{B}$, *and denote it by $\mathfrak{B}_+$, the set of all positive trajectories in $\mathfrak{B}$, namely $\mathfrak{B}_+ := \mathfrak{B} \cap (\mathbb{R}^q_+)^{\mathbb{Z}_+}$.*

Theorem 3.1 [10] *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be an autonomous behavior, let $\mathfrak{B}_+$ be its positive part and let (A,C) be an n-dimensional and minimal realization of $\mathfrak{B}$. The following facts are equivalent:*

1) $\mathfrak{B}$ is a positive behavior;

2) there exists a positive integer m and some matrix $B \in \mathbb{R}^{n\times m}$ such that

2a) (A,B,C) is a minimal realization of its transfer matrix $W(z) := C(zI_n - A)^{-1}B$, and

2b) the Markov coefficients of $W(z)$, i.e. the coefficients W_t of the power series expansion $\sum_{t\geq 0} W_t\, z^{-t}$ of $W(z)$, are all positive matrices;

3) there exists a positive integer m and some matrix $B \in \mathbb{R}^{n\times m}$ such that the reachable cone $\mathcal{R}(A,B)$ is proper and included in the observable cone $\mathcal{S}(C,A)$;

4) there exists a proper A-invariant cone $\mathcal{K} \subset \mathbb{R}^n$ included in $\mathcal{S}(C,A)$;

5) the observable cone $\mathcal{S}(C,A)$ is a proper cone;

6) $\mathfrak{B}$ is the smallest (linear, left shift-invariant and complete) behavior having $\mathfrak{B}_+$ as its positive part;

7) $\mathfrak{B}_+$ generates an n-dimensional real vector space in $(\mathbb{R}^q)^{\mathbb{Z}_+}$ (equivalently, $\mathfrak{B}_+$ is a proper left shift-invariant cone in $(\mathbb{R}^q)^{\mathbb{Z}_+}$).

Remarks For a more exhaustive discussion on the previous set of characterizations, we refer the interested reader to [10, 11]. We here aim to introduce only a few specific comments.

i) It is well-known that an autonomous behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is a finite-dimensional vector subspace of $(\mathbb{R}^q)^{\mathbb{Z}_+}$, whose dimension coincides with the dimension of a minimal realization of $\mathfrak{B}$. In fact, it is easily seen that, under the minimality (namely observability) assumption on the pair (A,C), there exists a bijective correspondence between the set of behavior trajectories and the (vector) space $\mathbb{R}^n$ of initial conditions $\mathbf{x}(0)$. In particular, there exists a bijective correspondence between the positive part of a behavior $\mathfrak{B}$, $\mathfrak{B}_+$, and the observable cone $\mathcal{S}(C,A)$, which is just the set of initial conditions corresponding (by means of (A,C)) to the trajectories of $\mathfrak{B}_+$. Thus the above theorem states that the positivity of $\mathfrak{B}$ corresponds to the fact that such a cone $\mathcal{S}(C,A)$ is a solid one, or, in a sense, is "dense" in $\mathbb{R}^n$, just as the positivity of $\mathfrak{B}$ means that the set $\mathfrak{B}_+$ is rich enough to carry on all the information about $\mathfrak{B}$.

ii) By Theorem 3.1, an autonomous behavior $\mathfrak{B}$, having (A,C) as an n-dimensional and minimal realization, is positive if and only if there exists a proper A-invariant cone $\mathcal{K}$ included in the observable cone $\mathcal{S}(C,A)$. As is well-known, an $n\times n$ matrix A leaves a proper cone invariant if and only if it satisfies the following two conditions (known as *Perron-Schaefer conditions*) [2]:

a) the spectral radius of A, $\rho(A)$, is an eigenvalue of A;

b) any other eigenvalue λ of A whose modulus $|\lambda|$ is equal to $\rho(A)$ satisfies the inequality $\deg\lambda \leq \deg\rho(A)$ (and hence the size of the largest Jordan block corresponding to λ in the Jordan form of A is not bigger than the largest Jordan block corresponding to $\rho(A)$).

This amounts to saying that $\rho(A)$ is the greatest positive real zero of $\psi_A(z)$, the minimal polynomial of A, and is dominant as a zero of ψ_A (if A is cyclic, then $\psi_A(z) = \det(zI_n - A)$ and hence the same properties hold true in terms of characteristic polynomial of A).

4 Nonnegative realizability of autonomous behaviors

Definition 4.1 [7, 11] *An autonomous behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* positive realizable *if there exists a realization of $\mathfrak{B}$, say (A_+, C_+), with A_+ and C_+ positive matrices.*

Nonnegative realizability admits a wide set of equivalent characterizations, that strictly remind of those obtained for positivity and make use of the same distinguished sets: the reachable cone, the observable cone and the positive part of the behavior.

Theorem 4.1 [11] *Let $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ be an autonomous behavior, let $\mathfrak{B}_+$ be its positive part and let (A, C) be an n-dimensional and minimal realization of $\mathfrak{B}$. The following facts are equivalent:*

1) $\mathfrak{B}$ is a positive realizable behavior;

2) there exists a positive integer m and some matrix $B \in \mathbb{R}^{n\times m}$ such that

2a) (A, B, C) is a minimal realization of its transfer matrix $W(z) := C(zI_n - A)^{-1}B$,

2b) the Markov coefficients of $W(z)$, i.e. the coefficients W_t of the power series expansion $\sum_{t\geq 0} W_t\, z^{-t}$ of $W(z)$, are all positive matrices, and

2c) the reachable cone of the pair (A, B) is proper polyhedral;

3) there exists a positive integer m and some matrix $B \in \mathbb{R}^{n\times m}$ such that the reachable cone of the pair, $\mathcal{R}(A, B)$ is proper polyhedral and included in $\mathcal{S}(C, A)$;

4) there exists an A-invariant proper polyhedral cone $\mathcal{K} \subset \mathbb{R}^n$ included in $\mathcal{S}(C, A)$;

5) the set $\mathfrak{B}_+$ includes a proper polyhedral left shift-invariant cone.

Remarks i) It is immediately seen, from the above theorem, that every positive realizable behavior is positive. Simple examples can be given showing that the converse is not true [11].

ii) Conditions 3) and 4) strictly remind of analogous characterizations obtained for the positive realizability of a strictly proper rational transfer function [1, 3, 6]. In fact, if $w(z)$ is a strictly proper rational transfer function and (A, B, C) is an n-dimensional and minimal realization of $w(z)$, then

$w(z)$ admits a positive realization if and only if there exists an A-invariant proper polyhedral cone $\mathcal{K}$ satisfying $\mathcal{R}(A,B) \subseteq \mathcal{K} \subseteq \mathcal{S}(C,A)$.

iii) If $\mathfrak{B}$ is positive realizable, then, by point 4) of the previous theorem, the matrix A appearing in a minimal realization of $\mathfrak{B}$ leaves a proper polyhedral cone $\mathcal{K}$ invariant. As a consequence [9], A satisfies the following three conditions: 1) the spectral radius of A, $\rho(A)$, is an eigenvalue of A, and, when $\rho(A) \neq 0$, 2) $\lambda \in \sigma(A)$ with $|\lambda| = \rho(A)$ implies $\deg \lambda \leq \deg \rho(A) =: m$, namely the size of the largest Jordan block corresponding to λ in the Jordan form of A is not bigger than the largest Jordan block corresponding to $\rho(A)$; 3) $\lambda \in \sigma(A)$ with $|\lambda| = \rho(A)$ implies $\lambda/\rho(A)$ is a root of unity.

5 Controllable behaviors and their positive state-space realizations

Definition 5.1 *The state-space model $\Sigma_{DV} = (F,G,H,J)$ described by*

$$\begin{aligned} \mathbf{x}(t+1) &= F\mathbf{x}(t) + G\mathbf{v}(t) && (5.4)\\ \mathbf{w}(t) &= H\mathbf{x}(t) + J\mathbf{v}(t), \qquad t \geq 0, && (5.5) \end{aligned}$$

with $\mathbf{x}(t)$ the state vector, $\mathbf{v}(t)$ the driving input and $\mathbf{w}(t)$ the output vector, $n = \dim \mathbf{x}, m = \dim \mathbf{v}$ and $q = \dim \mathbf{w}$, is said to be a driving variable state-space representation *(for short, a (DV) representation) of the behavior $\mathfrak{B}$ if*

$$\mathfrak{B} \equiv \{\mathbf{w} \in (\mathbb{R}^q)^{\mathbb{Z}_+} : \exists\ \mathbf{x} \in (\mathbb{R}^n)^{\mathbb{Z}_+}, \mathbf{v} \in (\mathbb{R}^m)^{\mathbb{Z}_+},\ \textit{such that } (\mathbf{w},\mathbf{x},\mathbf{v}) \textit{satisfies } (5.4) \div (5.5)\}.$$

Definition 5.2 *A controllable behavior $\mathfrak{B} \subseteq (\mathbb{R}^q)^{\mathbb{Z}_+}$ is said to be* positive realizable *if there exists a (DV) realization of $\mathfrak{B}$, $\Sigma_{DV} = (F_+, G_+, H_+, J_+)$, with F_+, G_+, H_+ and J_+ positive matrices.*

The following proposition provides a complete characterization of positive realizability for a controllable behavior. The proof, which is here omitted for the sake of brevity, is based on some results about the positive realization problem in its more traditional setting, namely that of input/output models or, equivalently, rational transfer matrices, that can be found in [1, 3, 6].

Theorem 5.1 [12] *Let $\mathfrak{B} = \mathrm{im}(M(\sigma))$ be a controllable behavior, and suppose that $M(z) \in \mathbb{R}[z]^{q\times(q-p)}$ is a right prime column reduced polynomial matrix, with column degrees $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{q-p}$. Set*

$$W(z) := M(z) \begin{bmatrix} z^{-\nu_1} & & & \\ & z^{-\nu_2} & & \\ & & \ddots & \\ & & & z^{-\nu_{q-p}} \end{bmatrix}$$

and let Σ_{DV} be a minimal realization of (the proper rational matrix) $W(z)$. The following facts are equivalent:

i) $\mathfrak{B}$ is positive realizable;

ii) there exists a nonsingular square rational transfer matrix $\bar{Q}(z)$ such that $\bar{W}(z) := M(z)\bar{Q}(z)$ is a proper rational transfer matrix with positive Markov coefficients;

iii) there exist $q - p$ positive trajectories $\mathbf{w}^i$, $i = 1, 2, \dots, q - p$, generated by Σ_{DV} in forced evolution, such that the corresponding power series $\hat{\mathbf{w}}^i(z^{-1})$ are rational and

$$\mathrm{rank}[\, \hat{\mathbf{w}}^1(z^{-1}) \quad \hat{\mathbf{w}}^2(z^{-1}) \quad \dots \quad \hat{\mathbf{w}}^{q-p}(z^{-1})\,] = q - p;$$

iv) there exists some nonsingular square polynomial matrix $N(z)$ such that the leading coefficients of each nonzero (i, j)th entry of $M(z)N(z)$ is positive, i.e.

$$[M(z)N(z)]_{i,j} \neq 0 \qquad \Rightarrow \qquad \mathrm{lc}[M(z)N(z)]_{i,j} > 0. \tag{5.6}$$

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