Disturbed Discrete Time
Linear-Quadratic Open-Loop Nash games

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Abstract

We examine disturbed linear-quadratic games, where each player chooses his strategy according to a modified Nash equilibrium model under open-loop information structure. We give conditions for the existence and uniqueness of such an equilibrium. We also show how these conditions are related to certain Riccati difference equations and a boundary value problem.

Keywords : Linear quadratic games, disturbance, Riccati difference equation, Nash equilibrium, minimax-strategy, disturbance attenuation.

1 Introduction.

In [1],[6] open-loop Nash equilibria have been considered for linear difference games. This paper deals with open-loop linear difference games where an additional disturbance input is present and the costs functionals are of quadratic type.
The considered equilibrium concept of the underlying game was introduced in [7], [8] for differential games with disturbance input. Although a Nash game approach is used to choose the strategies of the players, no constraints on the disturbance term are made. This means that the players have to find an equilibrium strategy without knowing anything about the disturbance. More precisely, this equilibrium is obtained in the following way:
First each player can calculate a "worst-case" disturbance against any action of the players and then under consideration of this worst-case disturbances (depending on the chosen controls of all players), a Nash equilibrium is sought. For the notion of a Nash equilibrium we refer the reader to [1]. We mainly follow methods developed for the undisturbed continuous time case in [10], [2].

Consider a discrete-time linear system with M decision makers (or players) and a disturbance term of the following type:

$$x(k+1) = A(k)x(k) + \sum_{j=1}^{M} B_j(k)u_j(k) + C(k)w(k), \quad x(0) = x_0, \tag{1.1}$$

with $x(k) \in \mathbb{R}^n, k = 0, \ldots, N,\ A(k) \in \mathbb{R}^{n\times n}, B_j(k) \in \mathbb{R}^{n\times r_j}, C(k) \in \mathbb{R}^{n\times m},\ u_j(k) \in \mathbb{R}^{r_j}, w(k) \in \mathbb{R}^m,\ 1 \le j \le M,\ 0 \le k \le N-1$.
A state x is now defined as a function $x : \mathcal{T} \to \mathbb{R}^n$, a control $u_i, i = 1, \ldots, M$ is a function

$u_i : \mathcal{T} \to \mathbb{R}^{r_i}$ and a disturbance is a function $w : \mathcal{T} \to \mathbb{R}^m$, with time set $\mathcal{T} = \{0, 1, \ldots, N\}$. By $\mathcal{U}$ we denote the set of all n-tuples $(u_1, \ldots, u_n)$ of controls and with $\mathcal{W}$ the set of all disturbances, while $\mathcal{U}_i$ denotes the set of all controls u_i of player i, $\quad i = 1, \ldots, M$.
The solution of equation (1.1) is obtained by (see for instance [9], p. 452):

$$\begin{aligned} x(k) = \Phi(k,0)x_0 + \sum_{i=1}^{M} \left(\sum_{j=0}^{k-1} \Phi(k, j+1) B_i(j) u_i(j) \right) \\ + \sum_{j=0}^{k-1} \Phi(k, j+1) C(j) w(j), \quad k = 0, \ldots, N, \end{aligned} \tag{1.2}$$

where $\sum_0^{-1} = 0$ and $\Phi(k,l), k \geq l$, is the (n,n)-matrix

$$\Phi(k,l) = \begin{cases} A(k-1)A(k-2)\cdots A(l) & \text{for} \quad k \geq l+1 \\ I & \text{for} \quad k = l. \end{cases}$$

Notice that $\Phi(k,l)$ is a solution of the homogeneous difference equation $\Phi(k+1,l) = A(k)\Phi(k,l), k \geq l$, with $\Phi(l,l) = I$, I the n-dimensional unit matrix.

Moreover, we introduce the Hilbert space $\mathcal{X}^p$ of mappings from $\mathcal{T}$ to $\mathbb{R}^p$, equipped with the scalar product

$$\langle f, g \rangle_p := \sum_{k=0}^{N} f^T(k) g(k), \tag{1.3}$$

for $f, g \in \mathcal{X}^p$. Hence, $\mathcal{X}^n$ contains all states, $\mathcal{U}_i = \mathcal{X}^{r_i}$ and $\mathcal{W} = \mathcal{X}^m$.

The cost functional of the i^{th} player is then, for $i = 1, \ldots, M$, defined by:

$$\begin{aligned} J_i(u_1, \ldots, u_M, w) = \frac{1}{2} x^T(N) K_{iN} x(N) \\ + \frac{1}{2} \sum_{k=0}^{N-1} \left[x^T(k) Q_i(k) x(k) + \sum_{j=1}^{M} u_j^T(k) R_{ij}(k) u_j(k) + w^T(k) P_i(k) w(k) \right], \end{aligned} \tag{1.4}$$

where for $i, j = 1, 2, \ldots, M$ and each $k = 0, \ldots, N-1$ the matrices $Q_i(k),\ K_{iN} \in \mathbb{R}^{n \times n},\ R_{ij}(k) \in \mathbb{R}^{r_j \times r_j},\ P_i(k) \in \mathbb{R}^{m \times m}$ are symmetric and additionally

$$R_{ii}(k) > 0.$$

It is assumed here that the information structure of all players is of open-loop type, i.e. no state measurements are available during the optimization period and each player computes its optimal policy at the beginning of the game and is committed to follow that policy during the whole period. We define a Nash/worst-case equilibrium in the same way as it was done in [8]. The strategy of each player is defined the following way:

Definition 1.1. *We define the Nash/worst-case equilibrium in two steps:*

1. *Given the controls* $(u_1, u_2, \ldots, u_M) \in \mathcal{U}$. *A disturbance function* $\hat{w}_i(u_1, u_2, \ldots, u_M) \in \mathcal{W}$ *is called* **worst-case disturbance from the point of view of the** i^{th} **player** *according to these controls if*

$$J_i\Big(u_1, u_2, \ldots, u_M, \hat{w}_i(u_1, u_2, \ldots, u_M)\Big) \geq J_i\Big(u_1, u_2, \ldots, u_M, w\Big)$$

   holds for each $w \in \mathcal{W}$.

2. *We say that the controls* $(\tilde{u}_1, \ldots, \tilde{u}_M) \in \mathcal{U}$ *form a* **Nash/worst-case equilibrium** *if for all* $i = 1, \ldots, M$

   (i) there exists a worst-case disturbance from the point of view of the i^{th} *player according to all controls* $(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots, \tilde{u}_M) \in \mathcal{U}$ *and*

   (ii)

$$J_i\Big(\tilde{u}_1, \ldots, \tilde{u}_M, \hat{w}_i(\tilde{u}_1, \ldots, \tilde{u}_M)\Big) \leq$$
$$J_i\Big(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots, \tilde{u}_M, \hat{w}_i(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots, \tilde{u}_M)\Big)$$

   holds for each worst-case disturbance $\hat{w}_i(\tilde{u}_1, \ldots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \ldots, \tilde{u}_M) \in \mathcal{W}$ *and control function* $u_i \in \mathcal{U}_i$.

In order to simplify (1.2) and (1.4) we introduce the following linear operators:

$$\Phi : \mathbb{R}^n \to \mathcal{X}^n, \;\; x_0 \mapsto \Phi(., 0)x_0,$$

$$\mathcal{B}_i : \mathcal{U}_i \to \mathcal{X}^n, \;\; u_i(.) \mapsto \left(k \mapsto \sum_{j=0}^{k-1} \Phi(k, j+1)B_i(j)u_i(j), \;\; k = 0, \ldots, N \right), i = 1, \ldots, M,$$

and

$$\mathcal{C} : \mathcal{W} \to \mathcal{X}^n, \;\; w \mapsto \left(k \mapsto \sum_{j=0}^{k-1} \Phi(k, j+1)C(j)w(j), \;\; k = 0, \ldots, N \right),$$

as well as for $i, j = 1, \ldots M$ the operators

$$\bar{Q}_i : \mathcal{X}^n \to \mathcal{X}^n, \;\; x(.) \mapsto \left(k \mapsto \begin{cases} Q_i(k)x(k) & k = 0, \ldots, N-1 \\ K_{iN}x(N) & k = N \end{cases} \right),$$

$$\bar{R}_{ij} : \mathcal{X}^{r_j} \to \mathcal{X}^{r_j}, \;\; x(.) \mapsto \left(k \mapsto \begin{cases} R_{ij}(k)x(k) & k = 0, \ldots, N-1 \\ 0 & k = N \end{cases} \right),$$

$$\bar{P}_i : \mathcal{X}^m \to \mathcal{X}^m, \;\; x(.) \mapsto \left(k \mapsto \begin{cases} P_i(k)x(k) & k = 0, \ldots, N-1 \\ 0 & k = N \end{cases} \right).$$

With the scalar product (1.3) and the operators from above, equation (1.2) can now be written as

$$x = \Phi x_0 + \sum_{i=1}^{M} \mathcal{B}_i u_i + \mathcal{C}w, \tag{1.5}$$

and the cost functionals in (1.4), for $i = 1, \ldots, M$, as

$$J_i(u_1, \ldots, u_M, w) = \frac{1}{2} \left(\langle x, \bar{Q}_i x \rangle_n + \sum_{j=1}^{M} \langle u_j, \bar{R}_{ij} u_j \rangle_{r_j} + \langle w, \bar{P} w \rangle_m \right). \tag{1.6}$$

2 Sufficient existence conditions for open-loop equilibrium controls.

In this section, we derive a sufficient condition for the existence of Nash/worst-case equilibrium controls of discrete-time game under open-loop information structure and show that these controls are related to solutions of Riccati difference equations, similarly as in undisturbed games with open-loop Nash equilibrium. The results obtained are similar to those in [7], [8]. We begin the investigation of a Nash/worst-case equilibrium with the following result.

Theorem 2.1. *For $i = 1, \ldots, M$, let us define the operators $F_i : \mathcal{U}_i \mapsto \mathcal{U}_i$, $G_i : \mathcal{W} \mapsto \mathcal{W}$, $H_i : \mathcal{W} \mapsto \mathcal{U}_i$ by $F_i := \mathcal{B}_i^* \bar{Q}_i \mathcal{B}_i + \bar{R}_{ii}$, $G_i := \mathcal{C}^* \bar{Q}_i \mathcal{C} + \bar{P}_i$, $H_i := \mathcal{B}_i^* \bar{Q}_i \mathcal{C}$.*

1. *There exists a unique worst-case disturbance $\hat{w}_i \in \mathcal{W}$ from the point of view of the i^{th} player if and only if $G_i < 0$. This disturbance then is given by*

$$\hat{w}_i(u_1, \ldots, u_M) = \hat{w}_i = -G_i^{-1}(H_i^* u_i + \mathcal{C}^* \bar{Q}_i(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{M} \mathcal{B}_j u_j)), \tag{2.7}$$

for all $(u_1, \ldots, u_M) \in \mathcal{U}$.

2. *Moreover, for $i = 1, \ldots, M$, let $G_i < 0$ and let furthermore $F_i > 0$. Then $(\tilde{u}_1, \ldots, \tilde{u}_M) \in \mathcal{U}$ form an open-loop Nash/worst-case equilibrium if and only if for each $i = 1, \ldots, M$*

$$\tilde{u}_i = \left(F_i - H_i G_i^{-1} H_i^*\right)^{-1} \left(H_i G_i^{-1} \mathcal{C}^* - \mathcal{B}_i^*\right) \bar{Q}_i(\Phi x_0 + \sum_{\substack{j=1 \\ j\neq i}}^{M} \mathcal{B}_j \tilde{u}_j) \tag{2.8}$$

holds.

Proof. The proof is completely analogue to the proof presented in [8] since the representations (1.5), (1.6) are formally identical with the representation formulae in the differential game situation. Only the Hilbert spaces considered here are different. Therefore we omit the proof. □

Also the next theorem, which allows to describe Nash/worst-case controls in a "feedback" form for a "virtual" worst-case state trajectory, is completely analogue to Theorem 2 in [8].

Theorem 2.2. *Suppose that the matrices $P_i(k)$ and $R_{ii}(k), k = 0, \ldots, N-1$, $i = 1, \ldots, N$, are negative definite and positive definite, respectively. Suppose furthermore that also the operators G_i, and F_i, $i = 1, \ldots, M$, are negative definite and positive definite, respectively. Then, $\tilde{u}_1, \ldots, \tilde{u}_M$ form an open-loop Nash/worst-case equilibrium if and only if the following equations are fulfilled:*

$$\tilde{u}_i(k) = -R_{ii}^{-1}(k) \left(\mathcal{B}_i^* \bar{Q}_i \hat{x}_i\right)(k), \quad k = 0, \ldots, N-1 \tag{2.9}$$

$$\hat{w}_i(k) = -P_i^{-1}(k) \left(\mathcal{C}^* \bar{Q}_i \hat{x}_i\right)(k), \quad k = 0, \ldots, N-1 \tag{2.10}$$

where

$$\hat{x}_i = \Phi x_0 + \sum_{j=1}^{M} \mathcal{B}_j \tilde{u}_j + \mathcal{C}\hat{w}_i, \tag{2.11}$$

$i = 1, \ldots, M$. $\hat{x}_i$ can be seen as a "worst-case" state trajectory from the point of view of the i^{th} player.

Proof. The proof is omitted here (see [8]). □

We want to remark that the conditions on the positivity of the $R_{ii}(\cdot)$ and the negativity of $P_i(\cdot)$ can be relaxed by exploiting the structure of the operators F_i and G_i. This will be discussed in detail in a forthcoming paper, which will contain also aspects from infinite horizon case.
Note that in general each player gets associated a different worst-case trajectory $\hat{x}_i$. Therefore, it is not possible to treat this problem as a standard Nash game.
Our aim now is to describe the relation for equilibrium controls (2.9) more explicitly, i.e. by solutions of certain difference equations. For this we first need to construct the adjoint operators $\mathcal{B}_i^*$ and $\mathcal{C}^*$.

Lemma 2.1. *For $p, n \in \mathbb{N}$, let $L : \mathcal{T} \to \mathbb{R}^{n \times p}$. Supposing that $\mathcal{L}$ denotes the linear operator*

$$\mathcal{L} : \mathcal{X}^p \to \mathcal{X}^n, \ \ u \mapsto \left(k \mapsto \sum_{j=0}^{k-1} \Phi(k, j+1) L(j) u(j), \ \ k = 0, \ldots, N \right),$$

the adjoint operator is obtained by

$$\mathcal{L}^* : \mathcal{X}^n \to \mathcal{X}^p, \quad y \mapsto \left(k \mapsto L^T(k) \sum_{j=k+1}^{N} \Phi^T(j, k+1) y(j), \quad k = 0, \ldots, N \right). \tag{2.12}$$

Proof. The proof is straightforward, verifying the identity $< u, \mathcal{L}^* y > = < y, \mathcal{L} u >$. □

Furthermore, for $i = 1, \ldots, M$, we will need the following set of terminal value problems:

$$E_i(k) \ = \ Q_i(k) + A^T(k) E_i(k+1) \underbrace{\left[I + (S_i(k) + T_i(k)) E_i(k+1) \right]^{-1}}_{=: \Omega_i^{-1}(k)} A(k) \tag{2.13}$$

for $k = N-1, \ldots, 0$ and $E_i(N) = K_{iN}$, where $E_i(k) \in \mathbb{R}^{n \times n}$, $k = 0, \ldots, N$, is a symmetric matrix.

Theorem 2.3. *Suppose that the assumptions on the matrices R_{ii} and P_i and on the operators F_i, G_i in Theorem 2.2 are fulfilled. Further assume that there exist the solutions of equations* (2.13) *for $i = 1, \ldots, M$.*

i) The boundary value problem

$$
\begin{aligned}
e_i(k) &= A^T(k)\Omega_i^{-T}(k)e_i(k+1) \\
&\quad - A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)[E_j(k+1)\hat{x}_j(k+1)+e_j(k+1)],
\end{aligned}
\tag{2.14}
$$

with terminal condition $e_i(N)=0$ and where $e_i(k)\in \mathbb{R}^n$, $k\in\mathcal{T}$, and

$$
\begin{aligned}
\hat{x}_i(k+1) &= \Omega_i^{-1}(k)A(k)\hat{x}_i(k) - \Omega_i^{-1}(k)(S_i(k)+T_i(k))e_i(k+1) \\
&\quad - \Omega_i^{-1}(k)\sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)[E_j(k+1)\hat{x}_j(k+1)+e_j(k+1)],
\end{aligned}
\tag{2.15}
$$

with initial condition $\hat{x}_i(0)=x_0$, is equivalent to equations (2.9),(2.10),(2.11). *Here, for each $k\in\mathcal{T}$ the matrices $S_i(k) := B_i(k)R_{ii}^{-1}(k)B_i^T(k)$, $T_i(k) := C(k)P_i^{-1}(k)C^T(k)$, for $i = 1,\ldots,M$.*

ii) The control functions

$$
\tilde{u}_i(k) = -R_{ii}^{-1}(k)B_i^T(k)[E_i(k+1)\hat{x}_i(k+1)+e_i(k+1)],
\tag{2.16}
$$

$k=0,\ldots,N-1$, form an open-loop Nash/worst-case equilibrium if and only if $\hat{x}_i$ and e_i are solutions of (2.14),(2.15). *Moreover, the corresponding worst-case disturbance of the i^{th} player ($i=1,\ldots,M$) is then given by*

$$
\hat{w}_i(k) = -P_i^{-1}(k)C^T(k)[E_i(k+1)\hat{x}_i(k+1)+e_i(k+1)],
\tag{2.17}
$$

$k=0,\ldots,N-1$.

iii) The Nash/worst-case equilibrium represented by (2.16) *is unique if and only if the boundary value problem* (2.14),(2.15) *has a unique solution.*

Proof. Given $\hat{x}_i$ and e_i solutions of (2.14)–(2.15) then the functions as defined in (2.16)–(2.17) fulfill the formulae (2.9),(2.10) and (2.11). That means nothing else, in view of Lemma 2.1, that

$$
\Lambda_i(k) := \sum_{j=k}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) = E_i(k)\hat{x}_i(k)+e_i(k)
$$

for any $k\in\{0,\ldots,N\}$. To prove this we show that the auxiliary sequence $\Psi_i(k) := \Lambda_i(k) -$

$E_i(k)\hat{x}_i(k) - e_i(k)$ is zero. For that we deduce a difference equation for Ψ_i in the following way:

$$
\begin{aligned}
A^T(k)\Psi_i(k+1) = A^T(k)\left(\sum_{j=k+1}^{N} \Phi^T(j,k+1)Q_i(j)\hat{x}_i(j) - E_i(k+1)\hat{x}_i(k+1) - e_i(k+1)\right) \\
&= \sum_{j=k+1}^{N} A^T(k)\Phi^T(j,k+1)Q_i(j)\hat{x}_i(j) - A^T(k)e_i(k+1) \\
&\quad - A^T(k)E_i(k+1)\Big[\Omega_i^{-1}(k)A(k)\hat{x}_i(k) - \Omega_i^{-1}(k)(S_i(k)+T_i(k))e_i(k+1) \\
&\quad - \Omega_i^{-1}(k)\sum_{\substack{p=1\\p\neq i}}^{M} S_p(k)(E_p(k+1)\hat{x}_p(k+1) + e_p(k+1))\Big] \\
&= \sum_{j=k+1}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) - A^T(k)E_i(k+1)\Omega_i^{-1}(k)A(k)\hat{x}_i(k) \\
&\quad - A^T(k)\left[I - E_i(k+1)\Omega_i^{-1}(k)(S_i(k)+T_i(k))\right]e_i(k+1) \\
&\quad + A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{p=1\\p\neq i}}^{M} S_p(k)[E_p(k+1)\hat{x}_p(k+1) + e_p(k+1)] \\
&= \sum_{j=k+1}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) - A^T(k)E_i(k+1)\Omega_i^{-1}(k)A(k)\hat{x}_i(k) \\
&\quad - A^T(k)\Omega_i^{-T}(k)e_i(k+1) \\
&\quad + A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{p=1\\p\neq i}}^{M} S_p(k)(E_p(k+1)\hat{x}_p(k+1) + e_p(k+1)) \\
&= \sum_{j=k+1}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) + [Q_i(k) - E_i(k)]\hat{x}_i(k) - e_i(k)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=k+1}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) + [Q_i(k) - E_i(k)]\hat{x}_i(k) - e_i(k) \\
&= \sum_{j=k}^{N} \Phi^T(j,k)Q_i(j)\hat{x}_i(j) - E_i(k)\hat{x}_i(k) - e_i(k) \\
&= \Psi_i(k).
\end{aligned}
$$

As terminal condition we have $\Psi_i(N) = K_{iN}\hat{x}_i(N) - K_{iN}\hat{x}_i(N) = 0$ and therefore the unique solution of that difference equation is $\Psi_i \equiv 0$.
If there is on the other hand a solution $\tilde{u}_i$ and $\hat{w}_i$ and $\hat{x}_i$ to equation (2.9),(2.10) and (2.11) then the sequence $(e_i(k))_{k\in\mathcal{T}}$ is well defined by

$$
e_i(k) = A^T(k)\Omega_i^{-T}(k)e_i(k+1) - A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{j=1\\j\neq i}}^{M} B_j(k)\tilde{u}_j(k)
$$

with terminal condition $e_i(N) = 0$. Using the representation of the adjoints in Lemma 2.1 it is evident that

$$\tilde{u}_i(k) = -R_{ii}^{-1}(k)B_i^T(k)\Lambda_i(k+1), \quad \hat{w}_i(k) = -P_i^{-1}(k)C^T(k)\Lambda_i(k+1).$$

We show that $\hat{x}_i$ is a solution of (2.15). Again we use the well-defined auxiliary sequence $(\Psi_i(k))_{k\in\mathcal{T}}$ from above and compute

$$\begin{aligned}
A^T(k)\Omega_i^{-T}(k)\Psi_i(k+1) =& A^T(k)\Omega_i^{-T}(k)\Lambda_i(k+1) - A^T(k)\Omega_i^{-T}(k)E_i(k+1)\hat{x}_i(k+1) \\
&- A^T(k)\Omega_i^{-T}(k)e_i(k+1) \\
=& A^T(k)\Omega_i^{-T}(k)\Lambda_i(k+1) - A^T(k)\Omega_i^{-T}(k)E_i(k+1) \times \\
&\times \Big[A\hat{x}_i(k) - [S_i(k)+T_i(k)]\Lambda_i(k+1) - \sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)\Lambda_j(k+1)\Big] \\
&- [e_i(k) + A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)\Lambda_j(k+1)] \\
=& A^T(k)\Omega_i^{-T}(k)(I + E_i(k+1)[S_i(k)+T_i(k))]\Lambda_i(k+1) \\
&- A^T(k)\Omega_i^{-T}(k)E_i(k+1)[A(k)\hat{x}_i(k) - \sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)\Lambda_j(k+1)] \\
&- e_i(k) - A^T(k)E_i(k+1)\Omega_i^{-1}(k)\sum_{\substack{j=1\\ j\neq i}}^{M} S_j(k)\Lambda_j(k+1) \\
=& A^T(k)\Lambda_i(k+1) - e_i(k) - A^T(k)\Omega_i^{-T}(k)E_i(k+1)A(k)\hat{x}_i(k) \\
\\
=& A^T(k)\Lambda_i(k+1) - e_i(k) - A^T(k)\Omega_i^{-T}(k)E_i(k+1)A(k)\hat{x}_i(k) \\
=& A^T(k)\Lambda_i(k+1) + [Q_i(k) - E_i(k)]\hat{x}_i(k) - e_i(k) \\
=& \Lambda_i(k) - E_i(k)\hat{x}_i(k) - e_i(k) \\
=& \Psi_i(k).
\end{aligned}$$

According to that relation and the terminal condition $\Psi_i(N) = 0$ it is clear that $\Psi_i(k) = 0$ for any $k \in \mathcal{T}$ and hence $\Lambda_i(k+1) = E_i(k+1)\hat{x}_i(k+1) + e_i(k+1)$. The latter relation shows that $\hat{x}_i$ satisfies also (2.15). This proves part *i)* of the Theorem.

The remaining statements are now obvious from the first part and Theorem 2.2. □

Remark 2.1. In the single player case ($M = 1$) equation (2.14) becomes homogenous and hence $e_1 \equiv 0$ is the solution. Suppose further that all conditions in Theorem 2.3 are fulfilled, which means in particular that the Riccati equation $E_1(k) = Q_1(k) + A^T(k)E_1(k+1)\big[I + (S_1(k) + T_1(k))E_1(k+1)\big]^{-1}A(k)$, admits a solution. Then, there exists a unique solution to each of the following variational problems:

(i) (worst-case disturbance) For any $u \in \mathcal{U}$ find a $\hat{w}(u) \in \mathcal{W}$ such that $J(u, \hat{w}(u)) \geq J(u, w)$ for all $w \in \mathcal{W}$.

(ii) (worst-case control) Find a control $\tilde{u} \in \mathcal{U}$ such that $J(\tilde{u}, \hat{w}(\tilde{u})) \leq J(u, \hat{w}(u))$ for all $u \in \mathcal{U}$.

The equilibrium control $\tilde{u}$ can be written in the form $\tilde{u}(k) = -R_{11}^{-1}(k)B_1^T(k)E_1(k+1)\hat{x}(k+1)$, with $\hat{x}$ denoting the worst-case trajectory $\hat{x}(k+1) = [I+(S_1(k)+T_1(k))E_1(k+1)]^{-1}A(k)\hat{x}(k), \quad \hat{x}(0) = x_0$. In other words, the above remark yields the unique solvability of the variational problem

$$\inf_{u \in U} \sup_{w \in \mathcal{W}} J(u, w)$$

under the constraint that $x(k)$ fulfills the difference equation

$$x(k+1) = A(k)x(k) + B_1(k)u_1(k) + C(k)w(k).$$

Thus we have obtained a solution of a min-max-problem which is well known in $\mathcal{H}^\infty$-optimization problems as they appear in disturbance attenuation control problems (see [1], [4], [5]).
For conditions on global solvability of the system of (coupled) matrix Riccati difference equations (2.13) see for instance [3].

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