Optimal control and Riccati equation for a degenerate parabolic system

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Abstract

In this paper, we consider a stabilization problem for a fluid flow. For a perturbation of the velocity of an incoming flow on a flat plate, the laminar-to-turbulent transition location varies. We want to stabilize it by a suction velocity trough the plate. The linearization of the nonlinear model around a steady state solution leads to a linear degenerate parabolic equation. We look for a suction velocity in a feedback form, determined by solving a LQR problem with an infinite time horizon. We derive the associated optimality system and the optimal control. The study of the Riccati equation is difficult because the state equation is a degenerate parabolic equation for which the results in the literature cannot be directly applied. The existence of solution is established by studying the asymptotic behaviour of the minimal solution to a Differential Riccati Equation. Numerical tests show that the feedback law stabilizes the laminar-to-turbulent transition location of the flow.

1 Introduction.

This paper deals with the stabilization of the laminar-to-turbulent transition location developped by a fluid flow on a flat plate. The control is a suction velocity through a small slot near the leading edge. As described in [3], the instationary flow in the laminar boundary layer can be described by the Prandtl's equations. Using the so-called Crocco transformation, this system is reduced to a nonlinear degenerate parabolic equation (called the Crocco equation) [8]. By linearizing the Crocco equation around a stationary solution, we obtain the following linear degenerate equation :

$$
\begin{cases}
\dfrac{\partial z}{\partial \tau} + a(\eta)\dfrac{\partial z}{\partial \xi} - b(\xi,\eta)\dfrac{\partial^2 z}{\partial \eta^2} + (c(\xi,\eta) + ka(\eta))z = f, & (\tau,\xi,\eta) \in \ Q,\\
z(0,\xi,\eta) = z_0(\xi,\eta) & (\xi,\eta) \in \ \Omega,\\
\sqrt{a}(\eta)z(\tau,0,\eta) = \sqrt{a}(\eta)z_1(\tau,\eta), & (\tau,\eta) \in (0,T)\times(0,1),\\
(bz)(\tau,\xi,1) = 0, \quad \dfrac{\partial z}{\partial \eta}(\tau,\xi,0) = v_s(\tau)\chi_\gamma(\xi) + g(\tau,\xi) & (\tau,\xi) \in (0,T)\times(0,L),
\end{cases}
\tag{1.1}
$$

with $Q = (0,T)\times\Omega$, $\Omega = (0,L)\times(0,1)$, $(0,L)$ represents the length of the flat plate, $(0,1)$ is the thickness of the boundary layer in Crocco variable. The final time T can be finite or infinite, χ_γ is characteristic function of $\gamma = [x_0, x_1] \subset (0,L)$ and v_s is the suction velocity through the slot γ. The coefficients a, b, c depend on the stationnary solution to the Crocco equation. They have the following behaviour :

$$
\begin{cases}
a(\eta) = U^0_\infty \eta, \quad \eta \in [0,1],\\
b \in C^1(\Omega), \quad C_2(1-\eta)^2\sigma \le b(\xi,\eta) \le C_1(1-\eta)^2\sigma, \ \forall(\xi,\eta)\in\Omega, \quad b(\xi,\eta) > 0 \text{ in } \Omega,\\
c \in C_b(\Omega), \ c(\xi,\eta) \ge 0 \quad \forall(\xi,\eta) \text{ in } \bar{\Omega},
\end{cases}
$$

with $\sigma = \sqrt{-ln(\mu(1-\eta))}$, $0 < \mu < 1$, $C_1, C_2 > 0$ and U^0_∞ corresponds to the stationary velocity of the incoming flow. We suppose that

- $f(\tau,\xi,\eta) = d(\xi,\eta)u_\infty(\tau) + e(\xi,\eta)\frac{du_\infty}{d\tau}(\tau) \in L^2(Q)$ where u_∞ represents a smooth perturbation of U^0_∞,
- $g \in L^2((0,T)\times(0,L))$, $z_0 \in L^2(\Omega)$, $z_1 \in L^2((0,T)\times(0,1))$.

This paper is organized as follows. In section 2, we state an existence, uniqueness and regularity result for the linearized Crocco equation (1.1). In section 3, we formulate a LQR problem with an infinite time horizon. We prove the existence of a unique optimal solution. In section 4, we study the corresponding Riccati equation. In the last section, we numerically show that the feedback control law applied to the nonlinear model, stabilizes the transition location.

Notations.

Let $H^1(0,1;d)$ be the closure of $C^\infty([0,1])$ in the norm :

$$
||z||_{H^1(0,1;d)} = \left(\int_0^1 |z|^2 + |1-\eta|^2\sigma^2 \left|\frac{\partial z}{\partial \eta}\right|^2 \, d\eta \right)^{1/2}. \tag{1.2}
$$

To take the Dirichlet boundary condition into account, we denote by $H^1_{\{1\}}(0,1;d)$ the closure of $C_c^\infty([0,1)))$ in the norm $\|\cdot\|_{H^1(0,1;d)}$. According to Triebel theorem 2.9.2 [9]

$$H^1(0,1;d) = H^1_{\{1\}}(0,1;d).$$

We observe that the Dirichlet boundary condition at $\eta = 1$ is lost.

We set $\mathcal{O} = \Omega_X \times \Omega_\Xi$ with $\Omega_X = \Omega_\Xi = (0,L) \times (0,1)$. A point in Ω_X (resp. Ω_Ξ), will be denoted by X (resp. Ξ). The space of square integrable functions on $\mathcal{O}$ satisfying $z(X,\Xi) = z(\Xi,X)$ is denoted by $L^2_s(\mathcal{O})$.

2 The linear degenerate equation.

In this section, we study the system (1.1). First, we define weak solutions for the system (1.1) by the transposition method.

Definition 2.1. *Let $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2((0,T)\times(0,L))$, $v_s \in L^2(0,T;L^2(x_0,x_1))$, $z_1 \in L^2(0,T;L^2(0,1))$, and $z_0 \in L^2(\Omega)$. A function $z \in L^2(0,T;L^2(\Omega))$ is a weak solution to problem (1.1) if only if it satisfies the following identity*

$$\begin{aligned}\int_Q z\psi \, d\tau d\xi d\eta \;=\; & \int_Q fp \, d\tau d\xi d\eta - \int_0^T \int_0^L b(\xi,0)(v_s(\tau)\chi_\gamma(\xi) + g(\tau,\xi))p(\tau,\xi,0)\, d\tau d\xi \\ & + \int_0^T \int_0^1 a(\eta) z_1(\tau,\eta) p(\tau,0,\eta)\, d\tau d\eta + \int_\Omega p(0,\xi,\eta) z_0(\xi,\eta)\, d\xi d\eta,\end{aligned} \tag{2.3}$$

for all $\psi \in L^2(0,T;L^2(\Omega))$ where p is solution to equation:

$$\begin{cases} -\dfrac{\partial p}{\partial \tau} - a\dfrac{\partial p}{\partial \xi} - \dfrac{\partial^2(bp)}{\partial \eta^2} + (c+ka)p = \psi & \text{in } Q, \\ \dfrac{\partial(bp)}{\partial \eta}(\tau,\xi,0) = 0, \qquad (bp)(\tau,\xi,1) = 0 & \text{in } (0,T)\times(0,L), \\ \sqrt{a}(\eta) p(\tau,L,\eta) = 0 & \text{in } (0,T)\times(0,1), \\ p(T,\xi,\eta) = 0 & \text{in } \Omega. \end{cases} \tag{2.4}$$

The existence and uniqueness of a solution to system (1.1) is stated in the following theorem.

Theorem 2.1. *Let $f \in L^2(0,T;L^2(\Omega))$, $g \in L^2((0,T)\times(0,L))$, $v_s \in L^2(0,T;L^2(x_0,x_1))$, $z_1 \in L^2(0,T;L^2(0,1))$, and $z_0 \in L^2(\Omega)$, then equation (1.1) admits a unique weak solution $z \in L^2(0,T;L^2(\Omega))$. Moreover*

$$z \in L^2((0,T)\times(0,L);H^1(0,1;d)) \cap L^\infty(0,T;L^2(\Omega)),$$

$$\sqrt{a} z \in L^\infty(0,L;L^2((0,T)\times(0,1))),$$

and the solution obeys:

$$\begin{aligned}||z||_{L^\infty(0,T;L^2(\Omega))} + ||\sqrt{a}z||_{L^\infty(0,L;L^2((0,T)\times(0,1)))} + ||z||_{L^2((0,T)\times(0,L);H^1(0,1;d))} \\ \le C\Big(||f||_{L^2(Q)} + ||v_s||_{L^2((0,T);L^2(\gamma))} \\ +||g||_{L^2((0,T)\times(0,L)} + ||z_1||_{L^2((0,T)\times(0,1))} + ||z_0||_{L^2(\Omega)}\Big),\end{aligned} \tag{2.5}$$

where $C > 0$ is independent of T.

Sketch of the proof.

Step 1. First, we study the evolution equation

$$\begin{cases} a\dfrac{\partial z}{\partial \xi} - b\dfrac{\partial^2 z}{\partial \eta^2} + (c+ka)z = f, \\ (bz)(\xi,1) = 0, \quad b\dfrac{\partial z}{\partial \eta}(\xi,0) = 0, \\ \sqrt{a}z(0,\eta) = 0. \end{cases} \tag{2.6}$$

Due to the degeneracy of a in $\eta = 0$, the classical results for at parabolic equations cannot be used. With a point fixed method, and the results of [2] and [6], we prove in [4] that the system (2.6) admits a unique solution such that

$$\sqrt{a}z \in L^\infty(0,L;L^2(0,1)),\ z \in L^2(0,L;H^1(0,1;d)).$$

Step 2. We define the unbounded operator A in $L^2(\Omega)$ by :

- $Az = -a(\eta)\dfrac{\partial z}{\partial \xi} + b(\xi,\eta)\dfrac{\partial^2 z}{\partial \eta^2} - (c+ka)(\xi,\eta)z,$
- $D(A) = \Big\{z \in L^2(0,L;H^1(0,1;d)),\ \sqrt{a}z \in L^\infty(0,L;L^2(0,1)),\ \sqrt{a}z(0,\cdot) = 0,$
  $N(az, -b\dfrac{\partial z}{\partial \eta}) = 0\ ;\ Az \in L^2(\Omega)\Big\},$

where the operator N denotes the normal trace operator on $(0,L)\times\{0\}\cup\{0\}\times(0,1)$. For $k > 0$ enough large, we can prove that the operator $(A, D(A))$ is the generator of a contraction semigroup on $L^2(\Omega)$. We can obtain the same results for the adjoint system of (2.6).

Step 3. With the semigroup theory, we easily show that the system (2.4) admits a unique weak solution $p \in C([0,T];L^2(\Omega)) \cap L^2(0,T;L^2(0,L;H^1(0,1;d)))$.

Step 4. The uniqueness of a solution to system (1.1) can be immediately deduced from the definition by transposition. By approximation and a passage to the limit, we can prove the existence of a solution for the system (1.1). □

In [5], the system (1.1) has been studied with a, b, c independent of the ξ variable.

3 The LQR problem in infinite time horizon.

Set $Z = L^2(\Omega)$ the state space, $Y = \mathbb{R}$ the observation space and $U = \mathbb{R}$ the control space. We study the following LQR problem

$$(P) \qquad \text{Inf} \quad \left\{ J(z,u) \mid (z,u) \in L^2(0,\infty;Z) \times L^2(0,\infty;U),\ (z,u) \text{ satisfies } (1.1) \right\},$$

with

$$J(z,u) = \frac{1}{2}\int_0^\infty |y(\tau) - y_d|\, d\tau + \frac{1}{2}\int_0^\infty |u(\tau)|\, d\tau, \tag{3.7}$$

where $y(\tau) = \int_\Omega \phi z(\tau)\, d\xi d\eta + c_2 u_\infty$ with $\phi \in L^2(\Omega)$ and $c_2 \in \mathbb{R}$. The function $y_d \in L^2(Q)$ represents a desired transition location.

By classical arguments, we can prove that the above control problem admits a unique solution $(\bar{z}, \bar{v}_s)$. The optimal control for (P) is characterized by

$$\bar{v}_s(\tau) = \frac{1}{R}\int_\gamma b(\xi,0)\bar{p}(\tau,\xi,0)d\xi, \tag{3.8}$$

where $\bar{p}$ is the solution to system (2.4) with

$$\psi(\tau,\xi,\eta) = \phi(\xi,\eta)\left(\int_\Omega \phi(x,y)\bar{z}(\tau,x,y)\, dxdy + c_2 u_\infty(\tau) - y_d(\tau)\right).$$

4 Feedback control law and Riccati equation.

In this section, we denote by A_X^* (resp. A_Ξ^*) the operator A^* written in variable X (resp. Ξ). We prove that the optimal control $\bar{v}_s$ is characterized by the feedback law

$$\bar{v}_s(\tau) = \frac{1}{R}\int_\gamma b(s,0)\left(\int_\Omega \pi(s,0,\Xi)\bar{z}(\tau,\Xi)\, d\Xi + r(\tau,s,0)\right) ds, \tag{4.9}$$

where π is solution to the Riccati equation

$$\begin{cases} A_X^*\pi + A_\Xi^*\pi - \dfrac{1}{R}\displaystyle\int_\gamma b(s,0)\pi(s,0,\Xi)\, ds \int_\gamma b(s,0)\pi(X,s,0)\, ds & \\ \qquad\qquad\qquad\qquad +\phi(X)\phi(\Xi) = 0 & \forall (X,\Xi) \in \mathcal{O}, \\ \dfrac{\partial(b\pi)}{\partial\eta}(x,0,\Xi) = 0, \quad (b\pi)(x,1,\Xi) = 0 & \forall (x,\Xi) \in (0,L)\times\Omega, \\ \dfrac{\partial(b\pi)}{\partial\eta}(X,\xi,0) = 0, \quad (b\pi)(X,\xi,1) = 0 & \forall (X,\xi) \in \Omega\times(0,L), \\ \sqrt{a}\pi(X,L,\eta) = 0 & \forall (X,\eta) \in \Omega\times(0,1), \\ \sqrt{a}\pi(L,\eta,\Xi) = 0 & \forall (\eta,\Xi) \in (0,1)\times\Omega, \\ \pi(X,\Xi) = \pi(\Xi,X) \geq 0 & \forall (X,\Xi) \in \mathcal{O}, \end{cases} \tag{4.10}$$

and r satisfies the system

$$\begin{cases} -\dfrac{\partial r}{\partial \tau}(\tau,\Xi) = A^* r(\tau,\Xi) - \dfrac{1}{R}\displaystyle\int_\gamma b(\zeta,0) r(\tau,\zeta,0)\,d\zeta \int_\gamma b(s,0)\pi(s,0,\Xi)\,ds \\ \qquad\qquad + \displaystyle\int_\Omega \pi(X,\Xi) f(\tau,X)\,dX + \phi(\tau,\Xi)(c_2 u_\infty(\tau) - y_d(\tau)) \\ \qquad\qquad - \displaystyle\int_0^L \pi(x,0,\Xi) b(\xi,0) g(\tau,x)\,dx + \int_0^1 \pi(0,s,\Xi) a(s) z_1(\tau,s)\,ds, \\ \dfrac{\partial (br)}{\partial \eta}(\tau,\xi,0) = 0, \qquad (br)(\tau,\xi,1) = 0, \\ \sqrt{a} r(\tau,L,\eta) = 0, \\ r(\infty,\xi,\eta) = 0. \end{cases} \tag{4.11}$$

To study the system (4.10), we firstly etablish the well posedness of a local solution in time to the following Differential Riccati Equation associated to (4.10) :

$$\begin{cases} \dfrac{\partial \pi}{\partial \tau} = A_X^*\pi + A_\Xi^*\pi - \dfrac{1}{R}\displaystyle\int_\gamma b(s,0)\pi(s,0)\,ds \int_\gamma b(s,0)\pi(s,0)\,ds \\ \qquad\qquad\qquad\qquad + \phi(X)\phi(\Xi) \quad \text{in } (0,T)\times\mathcal{O}, \\ \dfrac{\partial (b\pi)}{\partial \eta}(\tau,x,0,\Xi) = 0, \quad (b\pi)(\tau,x,1,\Xi) = 0, \\ \dfrac{\partial (b\pi)}{\partial \eta}(\tau,X,\xi,0) = 0, \quad (b\pi)(\tau,X,\xi,1) = 0, \\ \sqrt{a}\pi(\tau,X,L,\eta) = 0, \\ \sqrt{a}\pi(\tau,L,\eta,\Xi) = 0, \\ \pi(0,X,\Xi) = \pi_0(X,\Xi) \geq 0 \\ \pi(\tau,X,\Xi) = \pi(\tau,\Xi,X) \geq 0., \end{cases} \tag{4.12}$$

where $\pi_0 \in L_s^2(\mathcal{O})$. To prove the existence of a solution for the system (4.12), we must study the following Differential Lyapunov Equation :

$$\begin{cases} \dfrac{\partial \pi}{\partial \tau} = A_X^*\pi + A_\Xi^*\pi + \psi(\tau,X)\psi(\tau,\Xi) & \text{in } (0,T)\times\mathcal{O}, \\ \dfrac{\partial (b\pi)}{\partial \eta}(\tau,x,0,\Xi) = 0, \quad (b\pi)(\tau,x,1,\Xi) = 0 & \text{for } (\tau,x,\Xi) \in (0,T)\times(0,L)\times\Omega, \\ \dfrac{\partial (b\pi)}{\partial \eta}(\tau,X,\xi,0) = 0, \quad (b\pi)(\tau,X,\xi,1) = 0 & \text{for } (\tau,X,\xi) \in (0,T)\times\Omega\times(0,L), \\ \sqrt{a}\pi(\tau,X,L,\eta) = 0 & \text{for } (\tau,X,\eta) \in (0,T)\times\Omega\times(0,1), \\ \sqrt{a}\pi(\tau,L,\eta,\Xi) = 0 & \text{for } (\tau,X,\eta) \in (0,T)\times\Omega\times(0,1), \\ \pi(0,X,\Xi) = \pi_0(X,\Xi) \geq 0 & \text{for } (X,\Xi) \in \mathcal{O}, \\ \pi(\tau,X,\Xi) = \pi(\tau,\Xi,X) \geq 0 & \text{for } (\tau,X,\Xi) \in (0,T)\times\mathcal{O}, \end{cases} \tag{4.13}$$

with $\psi \in L^2(0,T;L^2(\Omega))$ and $\pi_0 \in L^2_s(\mathcal{O})$.

Definition 4.1. *Let $\psi \in L^2(0,T;L^2(\Omega))$. A function $\hat{\pi} \in L^2(0,T;L^2_s(\mathcal{O}))$ is a weak solution to the system (4.13) if for all $z \in D(A_\Xi)$ and all $\zeta \in D(A_X)$, $((\hat{\pi}(\cdot),z),z) \geq 0$, the function $((\hat{\pi}(\cdot),z),\zeta)$ belongs to $H^1(0,T)$ and satisfies*

$$\begin{aligned}\frac{d}{d\tau}\int_{\Omega_X\times\Omega_\Xi}\hat{\pi}(\tau,X,\Xi)z(\Xi)\zeta(X)\,dXd\Xi = \int_{\Omega_X}(\hat{\pi}(\tau,X),z)A_X\zeta\,dX + \int_{\Omega_\Xi}(\hat{\pi}(\tau,\Xi),\zeta)A_\Xi z\,d\Xi \\ + \int_{\Omega_\Xi}\psi(\tau,\Xi)z(\Xi)\,d\Xi\int_{\Omega_X}\psi(\tau,X)\zeta(X)\,dX.\end{aligned} \tag{4.14}$$

The term $((\pi(\cdot),z),\zeta)$ stands for $\int_{\Omega_X}\int_{\Omega_\Xi}\pi(\cdot,X,\Xi)z(\Xi)\zeta(X)\,dXd\Xi$. To prove the uniqueness of a solution for (4.13), we introduce an other equivalent definition.

Definition 4.2. *Let $\psi \in L^2(0,T;L^2(\Omega))$. A function $\hat{\pi} \in L^2(0,T;L^2_s(\mathcal{O}))$ is a weak solution to the Differential Lyapunov Equation (4.13) if for all symmetrical function w such that*

$$w \in L^2(\Omega_X;D(A_\Xi)) \cap L^2(\Omega_\Xi;D(A_X)) \cap L^2_s(\mathcal{O}),$$

the function $(\hat{\pi}(\cdot),w)$ belongs to $H^1(0,T)$ and satisfies

$$\begin{aligned}\frac{d}{d\tau}\int_{\Omega_X\times\Omega_\Xi}\hat{\pi}(\tau,X,\Xi)w(X,\Xi)\,dXd\Xi = \int_{\Omega_X\times\Omega_\Xi}\hat{\pi}(A_Xw + A_\Xi w)\,dXd\Xi \\ + \int_{\Omega_X\times\Omega_\Xi}\psi(\tau,X)\psi(\tau,\Xi)w(X,\Xi)\,dXd\Xi.\end{aligned} \tag{4.15}$$

In the following theorem, we prove the uniqueness and the existence of a solution for the linear system (4.13).

Theorem 4.1. *Let $\psi \in L^2(0,T;L^2(\Omega))$. The system (4.13) admits a unique weak solution $\hat{\pi}$. It obeys :*

$$\hat{\pi} \in L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d)))) \cap C_s([0,T];L^2_s(\mathcal{O}))$$

Moreover, $\hat{\pi}$ satisfies the estimation

$$||\hat{\pi}||_{L^2(0,T;L^2(\Omega;L^2(0,L;H^1(0,1;d))))} \leq C\left(||\pi_0||_{L^2(\mathcal{O})} + ||\psi\otimes\psi||_{L^1(0,T;L^2(\mathcal{O}))}\right), \tag{4.16}$$

where the constant C is independent of T.

Proof. The uniqueness of the solution to (4.13) immediatly follows from the definition 4.2.

Existence. We can verify that the solution to (4.13) is defined by :

$$\begin{aligned}(\hat{\pi}(t)z,\zeta) \quad = \quad & \int_{\mathcal{O}} e^{t(A_X^*+A_\Xi^*)}\pi_0 z\zeta\,dXd\Xi \\ & + \int_0^t\left(\int_{\Omega_X}\left(e^{(t-\tau)A_X^*}\psi\right)\zeta\,dX\right)\left(\int_{\Omega_\Xi}\left(e^{(t-\tau)A_\Xi^*}\psi\right)z\,d\Xi\right)d\tau,\end{aligned} \tag{4.17}$$

for all $z, \zeta \in D(A)$.

With the Cauchy-Schwarz inegality and (4.17), we prove that $\hat{\pi} \in L^\infty(0,T; L^2_s(\mathcal{O}))$. $\square$

To show the existence of a solution for (4.12), we use a point fixed method in the space

$$E_M = \Big\{ \pi \in C_s([0,\bar{t}]; L^2_s(\mathcal{O})) \cap L^2(0,\bar{t}; L^2(\Omega_X; L^2(0,L; H^1(0,1;d)))), \\ ||\pi||_{L^\infty(0,\bar{t};L^2_s(\mathcal{O}))} + ||\pi||_{L^2(0,\bar{t};L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \leq 3M^2 \Big\},$$

with $\bar{t} > 0$ a small enough, $||\phi||_{L^2(\Omega)} \leq M$ and $||\pi_0||_{L^2(\Omega)} \leq M^2$. We have the following theorem

Theorem 4.2. *Let $\phi \in L^2(\Omega)$, $\pi_0 \in L^2_s(\mathcal{O})$ such that $||\phi||_{L^2(\Omega)} \leq M$ and $||\pi_0||_{L^2(\mathcal{O})} \leq M^2$ for some $M > 0$. We denote by $\bar{t}$ a small constant. The system (4.12) with admits a symmetrical solution $\pi(\cdot, X, \Xi) = \pi(\cdot, \Xi, X) \geq 0$ that satisfies :*

$$\pi \in L^2(0,\bar{t}; L^2(\Omega_X; L^2(0,L; H^1(0,1;d))) \cap C_s([0,\bar{t}]; L^2_s(\mathcal{O}))$$

Moreover, we have :

$$||\pi||_{L^2(0,\bar{t};L^2(\Omega;L^2(0,L;H^1(0,1;d))))} + ||\pi||_{L^\infty(0,\bar{t};L^2(\mathcal{O}))} \leq C\left(||\phi \otimes \phi||_{L^2(\mathcal{O})} + ||\pi_0||_{L^2(\mathcal{O})}\right), \quad (4.18)$$

where C is a constant not depending on T and $\bar{t}$.

To prove the existence of solution for the system (4.12) on each interval $[T, \bar{t} + T]$, we use the same point fixed method with π_0 now corresponding to the solution of the system (4.12) at time T. Therefore, the system (4.12) admits a unique solution on $[0, \infty)$.

By studying the asymptotic behaviour of the solution to the DRE, we prove the existence of a solution for the Algebraic Riccati Equation.

Theorem 4.3. *The Riccati equation (4.10) admits a unique symmetrical solution $\pi \geq 0$ such that*

$$\pi \in L^2(\Omega_X; L^2(0,L; H^1(0,1;d))) \cap L^2_s(\mathcal{O}).$$

Moreover, π satisfies :

$$||\pi||_{L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \leq C||\phi \otimes \phi||_{L^2(\mathcal{O})}. \quad (4.19)$$

Proof. It based on the proposition 2.1 p. 309 of [1].

Step . We show that the solution to (4.12) admits a limit when τ tends to ∞. Let π the solution to (4.12) with $\pi_0 = 0$ and $\tilde{\pi}$ the solution to (4.12) with $\tilde{\pi}(0) = \pi(\varepsilon)$, $\varepsilon > 0$. With proposition 2.2 p. 147 [1], we have

$$((\pi(t+\varepsilon), z), z) = ((\tilde{\pi}(t), z), z) \geq ((\pi(t), z), z).$$

So, the map $t \to ((\pi(t)z), z))$ is increasing. We denote by $\Pi \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ the operator :

$$\Pi(t)z(X) = \int_\Omega \pi(t, X, \Xi)z(\Xi)\, d\Xi.$$

Since $||\pi||_{L^\infty(0,\infty;L^2(\mathcal{O}))} < \infty$, we have

$$\underset{t\geq 0}{Sup}\, |(\Pi(t)z, z)| < \infty,$$

for all $t \geq 0$ and all $z \in L^2(\Omega)$. Consequently, for all $z \in L^2(\Omega)$, the limit $\underset{t\to\infty}{lim}\, (\Pi(t)z, z)$ exists and is finite. Let Π_{min} defined by :

$$\underset{t\to\infty}{lim}\, (\Pi(t)z, z) = (\Pi_{min}z, z). \tag{4.20}$$

We have

$$(\Pi(t)z, \zeta) = \frac{1}{4}(\Pi(t)(z+\zeta), (z+\zeta)) - \frac{1}{4}(\Pi(t)(z-\zeta), (z-\zeta)). \tag{4.21}$$

With (4.20) and (4.21), we obtain

$$\underset{t\to\infty}{lim}\, (\Pi(t)z, \zeta) = (\Pi_{min}z, \zeta),$$

for all $z, \zeta \in L^2(\Omega)$. By applying two times the Banach Steinhaus theorem, we find that

$$\underset{t\geq 0}{Sup}\, ||(\Pi(t)\cdot, \cdot)||_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} < \infty.$$

Thus, $\Pi_{min} \in \mathcal{L}(L^2(\Omega))$ and $\Pi_{min} = \Pi^*_{min}$. By the uniqueness of the limit of $(\Pi(t)z, z)$ when t tends to ∞, Π_{min} can be represented by a function $\pi_{min} \in L^2_s(\mathcal{O})$ such that

$$\underset{t\to\infty}{lim} \int_\mathcal{O} \pi(t, X, \Xi)z(\Xi)\zeta(X)\, dXd\Xi = \int_\mathcal{O} \pi_{min}(X, \Xi)z(\Xi)\zeta(X)\, dXd\Xi,$$

for all $z, \zeta \in L^2(\Omega)$.

Step 2. We show that π_{min} is solution to the ARE. Let $\tilde{\pi}$ be the solution to (4.12) with $\pi_0 = \pi_{min}$ and $\tilde{\pi}_n$ the solution to (4.12) with $\pi_0 = \bar{\pi}(n)$ where $\bar{\pi}$ is the solution to (4.12) with $\pi_0 = 0$. By using the dynamic programming principle, we have

$$\tilde{\pi}_n(t) = \tilde{\pi}(t+n), \quad t > 0.$$

Due to the first step, we have

$$((\tilde{\pi}_n(0), z), z) \underset{n\to\infty}{\longrightarrow} ((\tilde{\pi}(0), z), z) = ((\pi_{min}, z), z),$$

for all $z \in L^2(\Omega)$. Therefore

$$\begin{aligned} ((\tilde{\pi}(t), z), z) &= \underset{n\to\infty}{lim}((\tilde{\pi}_n(t), z), z) \\ &= \underset{n\to\infty}{lim}\, ((\bar{\pi}(t+n), z), z) \\ &= ((\pi_{min}, z), z), \end{aligned}$$

for all $t > 0$ and all $z \in L^2(\Omega)$. Finally, $\tilde{\pi}$ is constant and equal to π_{min}. As $\tilde{\pi}$ is the solution to (4.12) with $\pi_0 = 0$, we have

$$\begin{aligned} 0 &= \frac{d}{d\tau}((\tilde{\pi}, z), z) \\ &= (A_X z, (\pi_{min}, z)) + ((\pi_{min}, z), A_\Xi z) - \frac{1}{R}\left(\int_\gamma b(s,0)(\pi_{min}(s,0), z)\, ds\right)^2 + (\phi z, \phi z). \end{aligned}$$

Consequently, π_{min} is the solution to the ARE (4.10). □.

Now, we can prove the existence of solution r for the system (4.11).

Theorem 4.4. *Let $g \in L^2((0,\infty)\times(0,L))$, $z_0 \in L^2(\Omega)$, $f \in L^2((0,\infty)\times\Omega)$, $u_\infty \in L^2(0,\infty)$, $z_1 \in L^2((0,\infty)\times(0,1))$. The system (4.11) admits a unique weak solution r such that*

$$r \in L^2(0,\infty; L^2(0,L; H^1(0,1;d))) \cap L^\infty(0,\infty; L^2(\Omega)),$$

$$\sqrt{a} r \in L^\infty(0,L; L^2((0,\infty)\times(0,1))).$$

Moerover r obeys :

$$\begin{aligned} &||r||_{L^\infty(0,\infty;L^2(\Omega))} + ||\sqrt{a}r||_{L^\infty(0,L;L^2((0,\infty)\times(0,1)))} + ||r||_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))} \\ &\qquad \le C\Big(||f||_{L^2((0,\infty)\times\Omega)} + ||z_1||_{L^2(0,\infty)\times(0,1))} + ||g||_{L^2(0,\infty,L^2(0,L))}\Big). \end{aligned} \tag{4.22}$$

The proof can be found in [4].

5 Numerical results.

In this section, we solve numerically the LQR problem. Following the numerical scheme given in [3], the system (1.1) is semi-discretized in space. We obtain a finite dimensional system of the form :

$$\begin{cases} \dfrac{dz^n}{d\tau} = Az^n + Bv_s + f^n(\tau), \\ z^n(0) = z_0^n. \end{cases} \tag{5.23}$$

where $f^n(\tau) = E_1 u_\infty(\tau) + E_2 \frac{du_\infty}{d\tau}(\tau) + E_3\, g^n(\tau) + E_4\, z_1^n(\tau)$ and $z^n(\tau)$ represents a vector of $\mathbb{R}^n$. The operator A now belongs to $\mathcal{L}(\mathbb{R}^n)$. The feedback control law is

$$v_s(\tau) = -Kz^n(\tau) - R^{-1}B^T r^n(\tau), \tag{5.24}$$

where

- $K = R^{-1}B^T\Pi$ and Π is the unique symmetrical non negative solution to the ARE

$$A^T\Pi + \Pi A - \Pi B R^{-1} B^T \Pi + C^T C = 0. \tag{5.25}$$

- The vector r^n is the solution of

$$\begin{cases} -\dfrac{dr^n}{d\tau} = (A - BK)^T r^n + C^T(C_2 u_\infty - y_d) + \Pi f^n, \\ r^n(\infty) = 0. \end{cases} \tag{5.26}$$

- The state z^n is the solution of

$$\begin{cases} \dfrac{dz^n}{d\tau} = (A - BK)z^n - BR^{-1}B^T r^n + f^n, \\ z^n(0) = z_0^n. \end{cases} \tag{5.27}$$

Notice that numerically, all eigenvalues of A have a negative real part. Therefore, A is exponentially stable and the ARE (5.25) admits a unique solution Π.

Algorithm of control. Before the beginning of the time loop, we build the matrixes $A, B, C, E_1, E_2, E_3, E_4$. We solve the Riccati equation (4.10) to determine Π and build K. Since the perturbation u_∞ is known on $[0, \infty]$, we calculate the solution to (5.26). During the time loop, we solve the system (5.27) and we apply the control law given by (5.24) to the nonlinear model.

The nonlinear model (Prandtl's system) is solved on a fine grid with 20000 points [4]. The compensator (5.27) is calculated on a coarse grid of $n = 63$ points. The constant R defined in (3.7) is taken equal to 0.001 and $y_d = 0$.

For the first example, we consider a sinusoidal perturbation $u_\infty = 0.5 * sin(4\pi t) * cos(24\pi t)$. The figure (1) represents the variation of the controlled and uncontrolled transition location determined with the nonlinear model and the values of the suction velocity (the control). The maximum amplitude of the transition location is reduced of 65%.

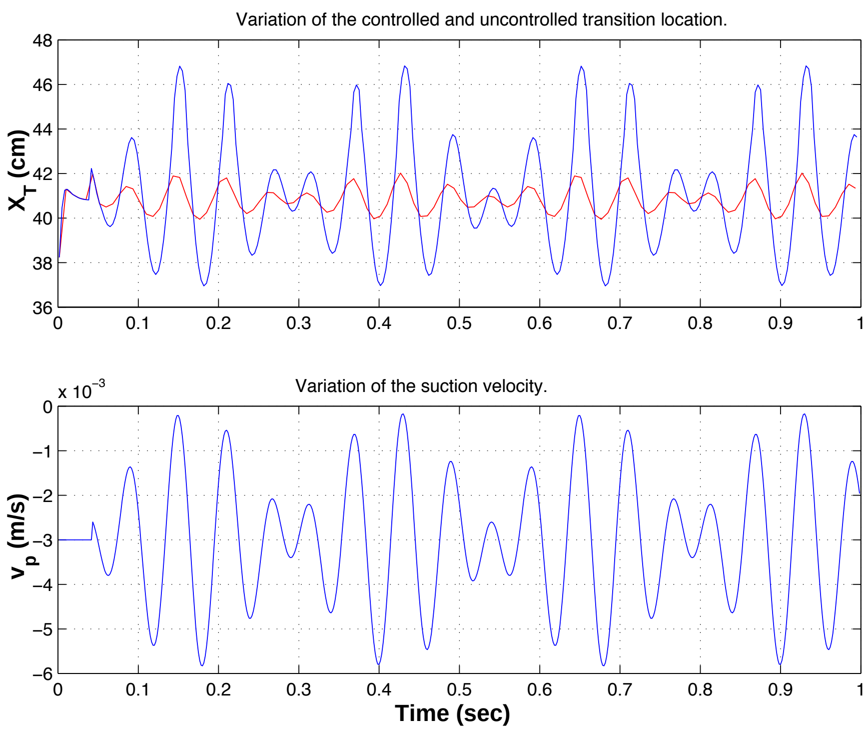

Figure 1: At the top, controlled (blue) and uncontrolled (red) variations of the transition location in response to a perturbation $u_\infty = 0.5 * sin(4\pi t) * cos(24\pi t)$. At the bottom, variation of the control in response to the same perturbation.

In the next test, we consider a perturbation with a greater amplitude but with a smaller frequency

$$u_\infty = 1.5 * sin(4\pi t) * cos(8\pi t).$$

The figure (2) shows that the maximal amplitude of the variation of the transition location is reduced by 66%.

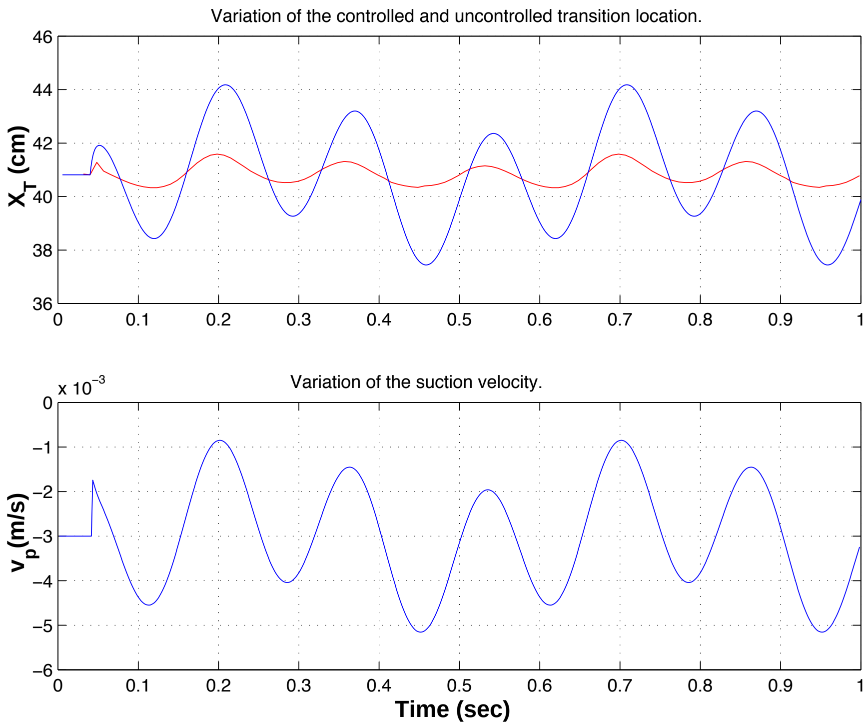

Figure 2: At the top, controlled (blue) and uncontrolled (red) variations of the transition location in response to a perturbation $u_\infty = 1.5 * sin(4\pi t) * cos(8\pi t)$. At the bottom, variation of the control in response to the same perturbation.

6 Conclusions and further works.

In this paper, we have considered the numerical and theoretical stabilization of the transition location obtained with the Prandtl's system. The proof of the existence of a solution for the ARE differs from [7] and permits to deal with problems where the semigroup generated by the operator A is exponentially stable.

We have supposed that the perturbation u_∞ is known. Therefore, the action of the perturbation on the transition location is taken into account in the feedback law with an extra term r solution of a backward equation.

Similar numerical results for the LQG problem have been obtained in [4]. In this case, the longitudinal and vertical velocities in the boundary layer can be estimated.

In a future paper, a H^∞-approach will be used to determine a feedback law robust with respect to perturbations of the velocity of the incoming flow U_0^∞.

References

[1] A. Bensoussan, G. Da. Prato, M. C. Delfour, S. K. Mitter", *Representation and Control of Infinite Dimensional Systems*, Systems & Control: Fondations & Applications, Boston, Birkhäuser, Volume 2, 1993.

[2] R. W. Carroll, R. E. Showalter, *Singular and degenerate Cauchy problems*, Mathematics in science and engineering, Academic Press, New-York, San Francisco, London, 127, 1976.

[3] J.M. Buchot, *Stabilization of the laminar turbulent transition location*, Proceedings MTNS 2000, El Ja Ed., 2000.

[4] J.M. Buchot, *Stabilisation et contrle optimal des quations de Prandtl*, Ph-D thesis, submit, 2002.

[5] J.M. Buchot, J.P. Raymond, *A linearized model for boundary layer equations*, International Series of Numerical Mathematics, Birkhäuser, 139, pp 31-42, 2001.

[6] A. Favini, A. Yagi, *Degenerate differential equations in Banach spaces*, Monograph and textbooks in Pure in Applied Mathematics 215, 1999, Marcel Deker, New-York.

[7] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: II Abstract hyperbolic systems*, Cambridge University Press, vol 1 & 2, 2000.

[8] O. A. Oleinik, *Mathematical Problems of Boundary Layer Theory*, Lectures Notes, Spring Quarter, 1969.

[9] H. Triebel, *Interpolation theory, Functions spaves, Differential operators*, North Holland, 1978.