Skorokhod-Neumann Boundary Conditions in Robust Queueing Service Models

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Abstract

We introduce the idea of boundary extremals (3.9) in control problems for simple fluid models of queueing systems with Skorokohd dynamics, using the robust formulation of [2, 4]. A simple example is presented in which these extremals play a fundamental role in the construction of the optimal control strategy.

1 Introduction

The performance of fluid models (deterministic) of queueing systems has both theoretical and heuristic implications for stochastic queueing systems; see [1, 3] for instance. It is natural to explore nonlinear optimal control problems for them. The dynamics of queueing systems are complicated by inherent nonnegativity constraints on the state queue length vector. These are often modelled by including *Skorokhod dynamics* in the system equations. This provides appropriate modifications to the dynamics when one or more state components become zero, resulting in a state trajectory that remains in the nonnegative orthant. The work of Dupuis and Ishii [5] allows a class of fluid queueing models with Skorokhod dynamics to be formulated as differential equations:

$$\dot{x}(t) = \pi(x(t), q(t) - Gu(t)). \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n_+$ is the *state* or queue length vector, $q(t) \in \mathbb{R}^n$ describes inflow or *exogenous load* on the system, $u(t)$ is the *control*, taking values in the convex hull $\mathcal{U}$ of some finite set $U_0 = \{u_i\}$ of vectors representing basic server settings, G is a matrix determined by the network topology and maximum service rates. $\pi(x, v)$ is the *velocity projection map* of the associated Skorokhod problem. This takes the form $\pi(x, v) = v + \sum_{i:\ x_i=0} \beta_i d_i$, where the d_i are vectors determined by the network topology and the coefficients $\beta_i \geq 0$ are obtained by solving a certain complementarity problem which depends on v and x:

$$w = v + \sum_{i:\ x_i=0} \beta_i d_i,$$

where for all i with $x_i = 0$,

$$w_i \geq 0, \quad \beta_i \geq 0, \quad \text{and } w_i\beta_i = 0.$$

We won't take space here to delineate the various technical hypotheses needed to justify this and the other assertions made below. But but what we say applies at least to the generality of *feed-forward* networks, for which the served output of any queue x_i joins only higher-numbered queues: x_j for $j > i$. (This translates to $(d_i)_j = 0$ for $j < i$, $= 1$ for $j = i$ and ≤ 0 for $j > i$.) This leads to a piecewise linear representation of $\pi(x, \cdot)$:

$$\pi(x, v) = R_F v,$$

where the *reflection matrix* R_F is determined by the set $F = \{i : \beta_i > 0\}$ resulting from the solution of the complementarity problem; see [4].

Recent papers including [2, 4] have considered robust control problems for simple examples of such systems based on the formulation of Soravia [7] using a differential game with (lower) value

$$V(x(0)) = \inf_{\alpha} \sup_{T, q(\cdot)} \int_0^T \frac{1}{2}\|x(t)\|^2 - \frac{1}{2}\|q(t)\|^2 \, dt.$$

(α denotes a generic non-anticipating control policy). These papers use a semi-explicit construction of the value function V by means of a carefully constructed family of extremals, and an appropriate verification theorem. However the construction of the extremals for the examples of those papers did *not* directly involve the Skorokhod dynamics. Only after the construction was complete were the Skorokhod dynamics considered, showing that the strategy and value function was optimal with respect to them as well.

In this paper we begin to consider situations in which the Skorokhod dynamics *are* involved in the construction of the extremals for the value function, and have a more decisive influence on the optimal strategy. In particular we describe the form of extremals in a face of the nonnegative orthant which involve the Skorokhod dynamics in a nontrivial way, and in §4 present one example in which these extremals are the dominant feature.

2 The Hamilton-Jacobi-Isaacs Equation and Viscosity Neumann Boundary Conditions

Ω will denote a relatively open subset of the nonnegative orthant $\mathbb{R}^n_+$. The *faces* of Ω are

$$\partial_i \Omega = \{x \in \Omega : \; x_i = 0\}, \quad i = 1, \dots n,$$

and $I(x) = \{i : \; x_i = 0\}$ is the *null-index set* of x. We will use $\partial_* \Omega = \cup_i \partial_i \Omega$ to denote the part of $\partial \Omega$ in the faces. Note that we consider $\partial_* \Omega \subseteq \Omega$. The standard basis vectors in $\mathbb{R}^n$ will be denoted e_i.

One naturally expects the value function $V(x)$ to be described by some sort of Hamilton-Jacobi-Isaacs equation. For x in the interior of $\mathbb{R}^n_+$ the appropriate Hamiltonian is simply

$$H(x, p) = \sup_q \inf_{u \in \mathcal{U}} \left\{ p \cdot (q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right\} = \frac{1}{2}\|p\|^2 - \sup_{u \in \mathcal{U}} p \cdot Gu + \frac{1}{2}\|x\|^2.$$

However if some coordinates of x are 0, the Skorokhod dynamics in (1.1) come into play, so that the Hamiltonian should be

$$H^\pi(x,p) = \sup_q \inf_{u\in\mathcal{U}} \left\{ p\cdot\pi(x, q - Gu) - \frac{1}{2}\|q\|^2 + \frac{1}{2}\|x\|^2 \right\}.$$

P. L. Lions [6] considered a class of differential games quite similar to ours, including Skorokhod dynamics, but for a domain Ω with smooth boundary (so that there is only one d_i to consider). He showed in particular that the Hamilton-Jacobi-Isaacs equation can be formulated in terms of H instead of H^π, with the Skorokhod dynamics entering only in the form of viscosity-sense Neumann boundary conditions. For us, the presence of corners and edges in $\partial\Omega$ makes the Skorokhod problem more involved. We will say $V(x)$ (assumed continuous) is a *viscosity solution of* $-H(x, DV(x)) = 0$ *with Neumann boundary conditions* $-d_i\cdot DV(x) = 0$, $i \in I(x)$ when

$$\min_{i\in I(x)}(-H(x,\xi), -d_i\cdot\xi) \le 0 \quad \text{for all } \xi\in D^+_\Omega V(x) \tag{2.2}$$

$$\max_{i\in I(x)}(-H(x,\xi), -d_i\cdot\xi) \ge 0 \quad \text{for all } \xi\in D^-_\Omega V(x). \tag{2.3}$$

The connection between this and the dynamic programming formulation directly in terms of H^π is not simple, but some easy implications are collected in the following theorem.

Theorem 2.1.

a) *Suppose $V\in C^1(\Omega)$. Then V is a classical solution of $H^\pi(x, DV(x)) = 0$ in Ω if and only if it is a viscosity solution of $-H^\pi(x, DV(x)) = 0$ in Ω.*

b) *If V is a continuous viscosity solution of $-H^\pi(x, DV(x)) = 0$ in Ω, then V is a viscosity solution of $-H(x, DV(x)) = 0$ with Neumann boundary conditions $-d_i\cdot DV(x) = 0$ for all $i\in I(x)$ in Ω.*

c) *If $V\in C^1(\Omega)$ satisfies $H(x, DV(x)) = 0$ for all $x\in\Omega$ and $d_i\cdot DV(x) = 0$ for all $x\in\partial_i\Omega$, then $H^\pi(x, DV(x)) = 0$ for all $x\in\Omega$.*

Proof. Part a) is elementary if x in the interior of Ω. For $x\in\partial_*\Omega$ the key is to recognize that, for any $v\in\mathbb{R}^n$ and $\xi\in D^+_\Omega V(x)$, $\xi\cdot\pi(x,v)\ge DV(x)\cdot\pi(x,v)$. This implies $-H^\pi(x,\xi)\le -H^\pi(x, DV(x))$. So if V is a classical solution of $H^\pi(x, DV(x)) = 0$ then it is a viscosity subsolution of $-H^\pi(x, DV(x)) = 0$. Analogous reasoning implies the supersolution property. The converse follows as usual from the fact that $V\in C^1(\Omega)$ implies that $DV(x)$ belongs to both $D^\pm V(x)$.

In b), for x in the interior of Ω the two notions of solution coincide, so we consider an $x\in\partial_*\Omega$ and $\xi\in D^+_\Omega V(x)$. By hypothesis $-H^\pi(x,\xi)\le 0$. We want to show (2.2). We may suppose that $d_i\cdot\xi\le 0$ for all $i\in I(x)$, else (2.2) is trivial. But then $\pi(x,v) = v + \sum_{I(x)}\beta_i d_i$ with $\beta_i\ge 0$ implies that $\xi\cdot\pi(x,v)\le\xi\cdot v$. Thus $-H(x,\xi)\le -H^\pi(x,\xi)\le 0$, which confirms (2.2). The argument for (2.3) is analogous.

For c), simply notice that $d_i\cdot DV(x) = 0$ implies that $DV(x)\cdot\pi(x,v) = DV(x)\cdot v$ for all $v\in\mathbb{R}^n$. This implies $H^\pi(x, DV(x)) = H(x, DV(x))$ which is =0 by hypothesis. ∎

3 Extremals and Boundary Extremals

Suppose now that V is a viscosity solution of $-H(x, DV(x) = 0$ with Neumann conditions, as in (2.2) and (2.3) above. For this exposition we assume $V \in C^1(\Omega)$, although there is no reason to expect that much regularity in general. The constructions of V in previous work [2, 4] employ a family of extremals $x(t)$, $p(t) = DV(x(t))$ for which the $x(t)$ cover an appropriate region Ω, and provide the structure needed to prove a verification theorem for $V(x)$. In brief, each extremal $x(t)$ should be a system trajectory

$$\dot{x}(t) = \pi(x(t), q^*(t) - Gu_*(t))$$

for a load, control pair $q^*(t), u_*(t)$ which is a saddle-point in the construction of $H^\pi(x(t), p(t))$:

$$p \cdot \pi(x, q^* - Gu_*) \le p \cdot \pi(x, q^* - Gu) \qquad \text{all } u \in \mathcal{U} \tag{3.4}$$

$$p \cdot \pi(x, q^* - Gu*) - \frac{1}{2}\|q^*\|^2 \ge p \cdot \pi(x, q - Gu_*) - \frac{1}{2}\|q\|^2 \qquad \text{all } q. \tag{3.5}$$

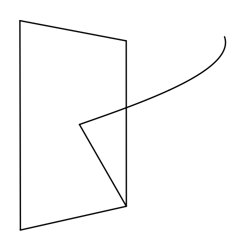
Moreover, along $x(t), p(t)$ we require the Hamiltonian equation, $H^\pi(x, p) = 0$, and a positivity condition, $\|p\| < \|x\|$ for $x \ne 0$. Finally, all extremals must reach the origin in finite time: $x(T) = p(T) = 0$, at which time they terminate. By stipulating that $T = 0$ we consider the extremals defined on some interval $T_0 < t \le 0$. The construction procedure is to solve the equations which describe extremals backwards in time, starting from $x(0) = p(0) = 0$, to identify a family that has all the desired properties.

For $x(t)$ in the interior $\Omega \setminus \partial_*\Omega$, (3.5) implies $q^*(t) = p(t) = DV(x(t))$ and one may deduce the equations of an interior extremal: $\dot{x} = p - Gu_*, \quad \dot{p} = -x$, subject to the optimality condition of (3.4). We are particularly interested, however, in extremals which move from the interior to contact a face at a point $x_0 = x(t_0) = \partial_i\Omega$ with negative normal velocity $(e_i \cdot (p(t_0) - Gu_*) < 0)$ as illustrated, followed by a section $(t_0 < t)$ on which the Skorokhod dynamics become active:

$$\pi(x(t), q^*(t) - Gu_*) = R_{\{i\}}(q^*(t) - Gu_*)$$

with

$$e_i \cdot (q^*(t) - Gu_*) < 0.$$



Extremals with active Skorokhod dynamics are what we call *boundary extremals*. To explore how this might happen, we make the simplifying assumption that only a single state coordinate is 0 at the contact point, $I(x_0) = \{i\}$, and that $u_* \in \mathcal{U}$ is the *unique* maximizer of $p(t_0) \cdot Gu$. We claim that under these assumptions it is necessary that $d_i \cdot DV(x_0) = 0$.

Let $p_0 = DV(x_0)$. The assumption on u_* implies that it is one of the extreme points of $\mathcal{U}$, $u_* \in U_0$, so that for all $\|\xi - p_0\|$ sufficiently small,

$$H(x_0, \xi) = \frac{1}{2}\|\xi\|^2 - \xi \cdot Gu_* + \frac{1}{2}\|x\|^2. \tag{3.6}$$

Moreover by letting $x \to x_0$ through the interior in $H(x, DV(x)) = 0$ we know that $H(x_0, p_0) = 0$. At the contact point we are assuming $e_i \cdot (p_0 - Gu_*) < 0$. Now $D^{\pm}_{\Omega}V(x_0) = \{p_0 \pm ce_i : \ c \geq 0\}$. Suppose $d_i \cdot p_0 < 0$. Then for all $0 \leq c < -d_i \cdot p_0$ we will have $\xi = p_0 + ce_i \in D^+_{\Omega}V(x_0)$ and $d_i \cdot \xi < 0$. Thus (2.2) requires that $H(x, p_0 + ce_i) \geq 0$ for all $0 \leq c < -d_i \cdot p_0$. Moreover, for $c > 0$ sufficiently small

$$H(x, p_0 + ce_i) \geq 0. \tag{3.7}$$

Using (3.6) and differentiating with respect to c, it follows that

$$0 \leq e_i \cdot \frac{\partial}{\partial p} H(x_0, p_0) = e_i \cdot (p_0 - Gu_*),$$

contrary to our hypothesis. Supposing $d_i \cdot p_0 > 0$ leads to a contradiction with the supersolution property (2.3) in a similar way. Thus if $0 > e_i \cdot (p_0 - Gu_*)$ we are forced to conclude that $d_i \cdot p_0 = d_i \cdot DV(x_0) = 0$.

Next consider what $d_i \cdot DV(x) = 0$ on a face $\partial_i\Omega$ would mean for the equations which describe boundary extremals. For $F = \{i\}$ the reflection matrix is simply $R_{\{i\}} = I - \frac{1}{e_i \cdot d_i} d_i e_i^T$ (d_i and e_i viewed as columns). It follows from this that

$$DV(x) = R^T_{\{i\}} DV(x), \quad x \in \partial_i\Omega. \tag{3.8}$$

In particular along a boundary extremal in this face, $p(t) = R^T_{\{i\}} p(t)$. As in Theorem 2.1 c), this implies that $p(t) \cdot \pi(x(t), v) = p(t) \cdot v$ for all v, from which it follows that $H^{\pi}(x, p) = H(x, p)$ and $q^*(t) = DV(x(t)) = p(t)$. The x-equation for a boundary extremal is therefore $\dot{x} = R_{\{i\}}(p - Gu_*)$.

To derive the p-equation, differentiate $H(x, R^T_{\{i\}} DV(x)) = 0$ with respect to x_j, $j \neq i$ (which are tangential to the face $\partial_i\Omega$). After a little algebra, this leads to the conclusion that $\dot{p}_j$ agress with the j-component of $-R^T_{\{i\}} x$, for all $j \neq i$. On the other hand we know $\dot{p} = R^T_{\{i\}} \dot{p}$, and the right side does not depend on $\dot{p}_i$, because $R^T_{\{i\}} e_i = 0$. Thus we find the following equations characterizing boundary extremals on $\partial_i\Omega$:

$$\dot{x} = R_{\{i\}}(p - Gu_*), \quad \dot{p} = -R^T_{\{i\}} x, \tag{3.9}$$

with the additional side condition $e_i \cdot (p - Gu_*) \leq 0$ to insure validity of the Skorokhod projection (i.e. that $F = \{i\}$ is correct in the complementarity problem).

We should emphasize that we made some strong assumptions in the above discussion, namely that $V \in C^1(\Omega)$ and that there was a unique maximizer of $DV(x) \cdot Gu$ over $u \in \mathcal{U}$. However what we have derived at least gives us an idea of what to look for in simple examples, such as that of the next section. Also, while our discussion is far from exhaustive, what we have found under these hypotheses is that an extremal which (in forward time) contacts a face $\partial_i\Omega$ from the interior and then follows a boundary extremal is associated with V satisfying the Neumann boundary condition $d_i \cdot DV(x) = 0$ in the classical sense. In contrast, the examples of [4] and [2, §4] exhibit extremals which move in the opposite direction, from $\partial_i\Omega$ into the interior, and there Neumann condition was only satisfied in the viscosity sense, not classically.

4 An Example in 2 Dimensions

Consider the following elementary example:

$$G = \begin{bmatrix} 2 & 1/2 \\ 1 & 2 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad d_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

with $U_0 = \{e_1, e_2\}$. This example is artificial – in a real network the reflection directions d_i should agree with the basic service options Ge_i, i.e. the columns of G. That is not the case here. However the example does involve boundary extremals in an essential way. We consider it for that reason.

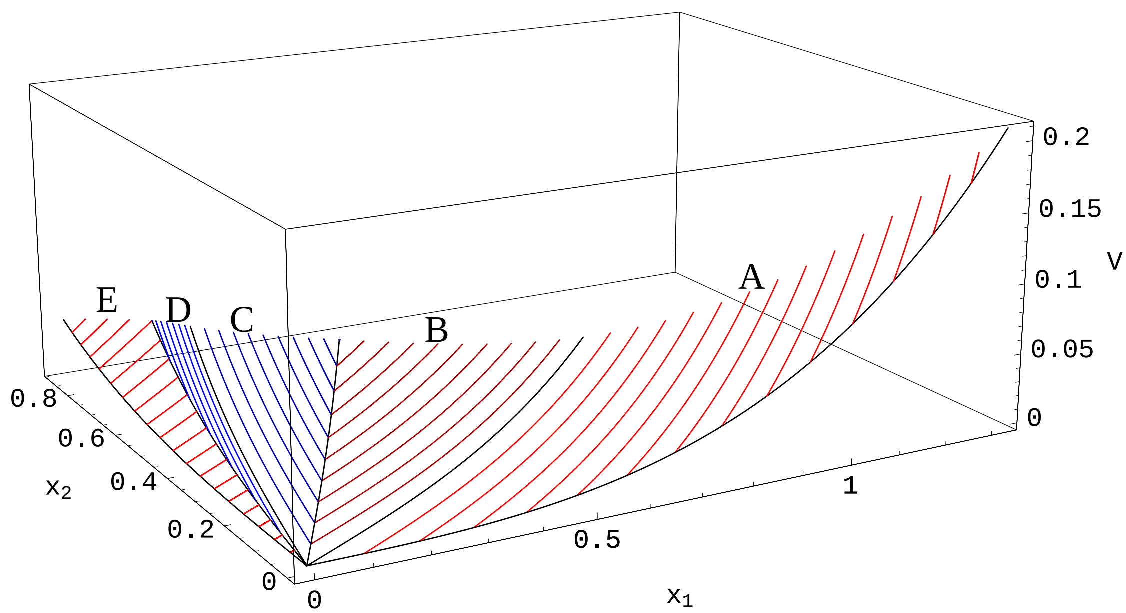


Figure 1: $V(x)$ and extremals for the example

The behavior on the boundary $\partial_1\Omega$ (the x_2-axis) is particularly interesting. There are two possible solutions of the boundary extremal equations (3.9) with $i = 1$ and terminal conditions $x(0) = p(0) = 0$: one each for $u_* = e_1, e_2$. One might anticipate that since $-Ge_2$ has a stronger vertical component than $-Ge_1$, that $u_* = e_2$ ought to be the optimal control in this region. However when one calculates the boundary extremal for $u_* = e_2$ it turns out that the optimality requirement (3.4) fails. On the other hand, the boundary extremal using $u_* = e_1$,

$$x^{(1)}(t) = (0, \frac{-3}{2\sqrt{2}} \sin(\sqrt{2}t)), \quad p^{(1)}(t) = \frac{3}{2}(1 - \cos(\sqrt{2}t)(1, 1), \quad 2\sqrt{2} < t \leq 0$$

does satisfy all the necessary conditions. On the boundary $\partial_2\Omega$ we again find that only the boundary extremal using $u_* = Ge_1$ satisfies all the necessary conditions:

$$x^{(2)}(t) = (-2\sin(t), 0), \quad p^{(2)}(t) = (2(1 - \cos(t), 1), \quad -\pi/2 < t \leq 0.$$

With these two boundary extremals $x^{(1)}$ and $x^{(2)}$ as a beginning, one must construct the rest of a family of saddle-point extremals satisfying all the necessary conditions in some region Ω of $\mathbb{R}^2_+$. A plot of the resulting value function $V(x)$ along a selection of the extremals of the family is in Figure 1. The extremals in sectors A, B, and E are those along which $u_* = e_1$ is optimal, while $u_* = e_2$ is optimal in sectors D and C . In sectors A and E the extremals run to one of the two boundaries and then follow the boundary extremal $x^{(1)}$ or $x^{(2)}$ to the origin. Thus in both these regions the resulting value $V(x)$ is dependent on the boundary extremals $x^{(i)}$. Sector D is particularly interesting. As one follows the $u_* = e_1$ extremals E backwards from the boundary into the interior, the optimality condition (3.4) begins to fail at some point, and one must switch to $u_* = e_2$ to continue in D. The curve along at which this control switching occurs is the boundary between sectors E and D. The extremals in sectors B and C follow the pattern of [2], joining along a relaxed extremal with $u_* = (\frac{5}{13}, \frac{8}{13})$ to reach the origin.

The region shown in the figure is $\Omega = \{2x_1 + 3x_2 < 2.64575\}$. In this region all the ingredients of a simple verification theorem can be established from the extremal constructions described above: the associated $V(x)$ is a C^1 (classical) solution of $H^\pi(x, DV(x)) = 0$ in Ω, and is the value function of the game described in §1. The optimal control policy is to use $u_* = e_1$ when the state x is in the sectors A, B, or E of the figure; and to use $u_* = e_2$ when x is in C or D.

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