Control and Algebra - An Introduction

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Abstract

The concept of bisimulation has been developed in computer science by R. Milner for the study of equivalences of transition systems. By now it has been generalized to coalgebra and to category theory. In control and system theory the concept of bisimulation is useful in the study of relations of dynamic systems, in particular of observationally equivalent systems. The exploration of categories of subclasses of dynamic systems are a major approach to advance control theory and in such categories bisimulation relations and coalgebras play a major role. This short paper is an introduction to the session *Control and Algebra*.

1 Introduction

The purpose of this lecture is to provide to the audience of the session an introduction about concepts and results of coalgebra and category theory, and about the application of these concepts to control and system theory.

Remarks about coalgebra follow. The concepts of bisimulation and of bisimilarity have been developed by Robin Milner in his books [7, 8]. It has its roots in set theory and algebra. Bisimulation is a relation between two dynamic systems. The main theorem states that a tuple of collections of deterministic processes are identical to an outside observer or behaviorally equivalent if and only if there exists a bisimilarity relation on these collections. Later the concept of bisimulation was generalized to coalgebra and category theory. Bisimulation is the dual in coalgebra of the concept of congruence in algebra. Coalgebra seems thus of great interest to control and system theory for the classes of systems: discrete-event systems, hybrid systems, but also for linear systems, and subclasses of nonlinear systems. Control and system theoretic problems for which the concepts of bisimulation, coalgebra, and category theory may be used include: equivalences of dynamic systems, inclusion relations on system behaviors, definition of specific subclasses of dynamic systems, composition of systems, decentralized control, and hierarchical control. Abstract algebra has been used in system theory since the research of R.E. Kalman and his co-workers. See for references [4, 14].

2 Concepts

Consider then a dynamic system as understood in system theory, see [15]. As a concrete example, consider a discrete-event system in the form of an automaton with a state set, an event set, an output set, and partial observations of events. Research on control of discrete-event systems is motivated by the use of computers for engineering systems such as communication networks, manufacturing systems, traffic systems, etc. Two discrete-event systems that represent the same input-output behavior may be related by a relation on their state sets. An example is an order relation between two such realizations in which the relation between the state sets may be taken to be a surjective map. In the following the reader may think of a discrete-event system in the form of an automaton when the term dynamic system is used.

Below terminology of process algebra is used rather than of control theory. A discrete-event system then corresponds to a collection of processes and any state in the system corresponds to a process. A *bisimulation* between two collections of processes is a relation on these collections such that: if System 1 executes a transition from a process that is in the relation then System 2 does so also from the corresponding process in the relation, and the tuple of processes after the transitions is again a member of the relation; and conversely, if System 2 makes a transition from the process that is in the relation then so does System 1 from the corresponding process and the tuple of processes after the transitions is again a member of the relation. The union of all bisimulations of two collections of processes is again a bisimulation and it is called the *bisimilarity* relation. If there exists a final coalgebra (see definition below) then existence of a bisimulation relation between a tuple of collections of processes is equivalent to an equality relation on their behaviors.

The concept of coalgebra can be put in the setting of category theory and this may help its understanding. However, knowledge of category theory is not at all required for the application of coalgebra. In category theory one considers functors, a kind of functions of sets with an algebraic structure. An *algebra* of a functor is defined as a map from a functor of a set to the set; symbolically, an algebra of a functor T is a pair (U, a) consisting of a set U and a function $a : T(U) \to U$. A *coalgebra* of a functor is defined as a map from a set to the functor of the set; symbolically, a coalgebra is a pair (U, c) consisting of a set U and of a function $c : U \to T(U)$. The set U often corresponds to the state set of a dynamic system. An algebra and a coalgebra are thus dual concepts. According to this duality relation a congruence for an algebra corresponds to a bisimulation for a coalgebra.

It is well known that induction on sets corresponds in category theory to initiality for an algebra of a functor. An algebra is called *initial* if for any algebra of the same kind there is an unique homomorphism (structure-preserving map) from the initial algebra to this algebra. If an algebra is initial then one can define functions by induction and one can prove by induction that two functions are identical. Similarly it has been proven that coinduction on sets corresponds in category theory to finality of a coalgebra. A coalgebra of a functor is said to be *final* for this functor if for any coalgebra of this functor there exists an unique homomorphism of coalgebras from the considered coalgebra to the final coalgebra. If a coalgebra is final then one can define functions by coinduction and one can prove by coinduction that two functions are identical.

The concept of bisimulation has been formulated in a categorical setting by M. Nielsen,

A. Joyal, and G. Winskel, see [9]. The paper by G. Pappas etal in this session is based on the concepts provided in the quoted paper.

Several problems of control and system theory have been treated with concepts of coalgebra. J.J.M.M. Rutten has formulated control of discrete-event systems with complete observations by coalgebra, see [12]. Such an approach requires the formulation of new concepts and theorems. Thus, the concept of a partial bisimulation on discrete-event systems is defined which can be proven to be equivalent to the concept of controllability introduced by P. Ramadge and W.M. Wonham, see [11]. The main result is then that existence of a supervisor for the control problem is equivalent to the existence of a controllability relation, which is a weakened version of a bisimulation relation. The proof is by coinduction. Further research on control of discrete-event systems with partial observations is carried out by J. Komenda, see [5]. G. Pappas has published about bisimilarity of linear systems, see [10]. Research on bisimilarity of hybrid systems is carried out by Pappas and co-workers. A. Lewis investigates affine-connection control systems in which a particular category is developed and its decompositions are to be determined.

Application of the concept of bisimulation to any class of dynamic systems may require restrictions on the class of homomorphisms between dynamic systems of the class. In two of the papers of this session, see [2, 6], this is illustrated. For discrete-event systems no restrictions are required for the use of bisimulation, except for those that relate to control.

3 Outlook

What can be expected from the combination of control and system theory on one hand and coalgebra on the other hand? The current results indicate that benefits of this approach are a uniformization of concepts and results for different classes of dynamic systems, for example discrete-event systems, hybrid systems, linear systems, and subclasses of nonlinear systems. In addition, it seems that the concepts lead to new algorithms in particular for discrete-event systems. The usefulness of coalgebra and category theory for control and system theory consists primarily of increased conceptual understanding and of the generalization of concepts and results from one class of dynamic systems to another.

4 Session papers

The second lecture in this session is by George J. Pappas and coauthors, see [2], and addresses algebraic system theory of control systems on manifolds with bisimulation in categorical setting. The third lecture is by A. Lewis, see [6], and presents results for a category of affine-connection control systems. The fourth lecture in this session is by J. Komenda, see [5], and describes control of discrete-event systems with partial observations in terms of coalgebra.

5 Further reading

Information on references follows. The books by Milner on bisimulation are [7, 8]. Papers on bisimulation in a categorical setting are [9, 1]. A tutorial on algebra and coalgebra is presented by B. Jacobs and J.J.M.M. Rutten in [3]. An application of coalgebra to control of discrete-event systems with complete observations has been developed by Rutten, see [12]. Results on universal coalgebra are presented in [13] and these are relevant for realization theory. The other papers of this MTNS2002 session on *Control and Algebra* are [2, 5, 6]. Control of discrete-event systems with partial observations with coalgebra has been investigated by Jan Komenda and publications are available or in preparation, see [5].

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