The category of affine connection control systems

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Abstract

The category of affine connection control systems is one whose objects are control affine systems whose drift vector field is the geodesic spray of an affine connection, and whose control vector fields are vertical lifts to the tangent bundle of vector fields on configuration space. This class of system includes a large and important collection of mechanical systems. The morphisms (feedback transformations) in this category are investigated.

1 Introduction

It is apparent that the study of what we will in this paper call "affine connection control systems" has a significant rôle to play in the field of mechanical control systems. In a series of papers, e.g., [10, 3], the author and various coauthors have shown how the affine connection framework is useful in looking at mechanical systems whose Lagrangian is the kinetic energy with respect to a Riemannian metric, possibly in the presence of constraints linear in velocity [9, 4]. In such an investigation, there appears to be no particular advantage to working with affine connections that come from physics, i.e., from the Riemannian metric and the constraints. Therefore, in this paper we deal with general affine connections. The emphasis here is on the groundwork for the investigation of ways in which one can simplify or alter affine connection control systems using feedback. To do so, we use the language of category theory, since it provides a valuable organisational structure in which many ideas surrounding feedback can be discussed in a systematic manner. More results along the lines of what we present in this paper may be found in [8]. The approach we use is strongly motivated by the approach of Elkin [5] for control affine systems.

Due to limitations of space, none of the stated results are given proofs, although these, along with many additional results, are available in an unpublished version of this paper.

2 Background

First let us introduce the basic notation. If M is a smooth manifold we denote by $C^\infty(M)$ the C^∞ functions on M and by $\Gamma^\infty(TM)$ the C^∞ vector fields on M. For a map $\phi\colon M \to N$ of manifolds M and N, we denote by $T\phi\colon TM \to TN$ the derivative of ϕ, and by $T_x\phi\colon T_xM \to T_{\phi(x)}N$ the restriction of $T\phi$. If $c\colon I \to M$ is a curve and if $\phi\colon M \to N$ is a smooth map, $c_\phi\colon I \to N$ is the curve defined by $c_\phi = \phi \circ c$.

2.1 Affine differential geometry

Due to space limitations, we do not review basic affine differential geometry, but assume the reader is familiar with the material in, say, volume 1 of Kobayashi and Nomizu [7]. For mechanical motivation for the use of affine connections, we refer to those papers cited in the introduction. We should, however, provide the notation we shall use. An affine connection on a manifold Q is denoted ∇, so $\nabla_X Y$ is the covariant derivative of Y with respect to X. By Z we denote the geodesic spray, which is a second-order vector field on TQ whose integral curves, projected to Q, are geodesics for ∇.

Given manifolds Q and $\tilde{Q}$ with affine connections ∇ and $\tilde{\nabla}$, respectively, a map $\phi\colon Q \to \tilde{Q}$ is ***totally geodesic*** if $T_q\phi(\nabla_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)}$, where $\tilde{X}$ is a vector field on $\tilde{Q}$ that is ϕ-related to X. Clearly a totally geodesic mapping has the property that it maps geodesics of ∇ to geodesics of $\tilde{\nabla}$. A submanifold $N \subset Q$ is ***totally geodesic*** if for a geodesic $c\colon I \to Q$, $c'(t_0) \in T_{c(t_0)}N$ for some $t_0 \in I$ implies that $c'(t) \in T_{c(t)}N$ for every $t \in I$.

2.2 The category of control affine systems

What we shall call "affine connection control systems" are examples of a commonly studied class of control systems: those which are affine in their controls. A clear discussion of this class of systems from a category theory perspective may be found in [5]. It is an unfortunate clash of common notation that we will use the word "affine" in two rather different contexts here; in one case we mean the general class of control systems affine in the controls, and in the other we means those specific systems whose drift vector field is the geodesic spray of an affine connection.

An ***object*** in the category CAS of control affine systems is a pair $\Sigma = (M, \mathcal{F})$ where M is a finite-dimensional smooth differentiable manifold, and $\mathcal{F}$ is a finite collection of vector fields $\mathcal{F} = \{f_0, f_1, \dots, f_m\}$. Associated with an object $\Sigma = (M, \mathcal{F})$ in CAS is a control affine system

$$\dot{x}(t) = f_0(x(t)) + u^a(t) f_a(x(t)). \tag{2.1}$$

In this equation we employ a summation convention where there is an implied summation over repeated indices.

As is always the case with a category, we need to specify its morphisms. We suppose that we have two objects $\Sigma = (M, \mathcal{F})$ and $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathcal{F}})$ where $\tilde{\mathcal{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\}$. We let $L(\mathbb{R}^m, \mathbb{R}^{\tilde{m}})$ denote the set of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^{\tilde{m}}$. A **CAS** ***morphism*** sending Σ to $\tilde{\Sigma}$ is a triple $(\psi, \lambda_0, \Lambda)$ where $\psi\colon M \to N$ is a smooth mapping, and $\lambda_0\colon M \to \mathbb{R}^{\tilde{m}}$ and $\Lambda\colon M \to L(\mathbb{R}^m, \mathbb{R}^{\tilde{m}})$ are smooth mappings satisfying $T_x\psi(f_a(x)) = \Lambda_a^\alpha(x)\tilde{f}_\alpha(\psi(x))$, $a = 1, \dots, m$, and $T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \lambda_0^\alpha(x)\tilde{f}_\alpha(\psi(x))$. An essential feature of this class of morphisms is that there is a morphism $(\psi, \lambda_0, \Lambda)$ from $\Sigma = (M, \mathcal{F})$ to $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathcal{F}})$ if and only if ψ maps trajectories for Σ to $\tilde{\Sigma}$. This is said precisely for affine connection control systems in [8].

3 The category of affine connection control systems

Now we can properly discuss the actual subject of the paper. What we consider in this section is a special class of control affine systems. We begin with a discussion of the objects in this category, and note that it is precisely the systems described here that form the basis for the work of the author and coauthors on "simple mechanical control systems (with constraints)."

3.1 Objects in ACCS

We shall denote by ACCS the ***category of affine connection control systems***. An ***object*** in this category is a triple $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$ where Q is a finite-dimensional manifold, ∇ is an affine connection on Q, and $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ is a collection of vector fields on Q. If $U \subset Q$ is an open submanifold, we may define the ***restricted object*** $\Sigma_{\text{aff}}|U = (U, \nabla|U, \mathcal{Y}|U)$.

To an affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$ we associate a control system given by

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t)). \tag{3.2}$$

A ***controlled trajectory*** for $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$ is a pair (c, u) with $c\colon I \to Q$ having the property that its derivative $t \mapsto c'(t)$ is an absolutely continuous curve on TQ, and $u\colon I \to \mathbb{R}^m$ is an admissible control (say measurable) such that together c and u satisfy (3.2).

The above manner of representing a control system associated with a triple $(Q, \nabla, \mathcal{Y})$ emphasises that the system essentially evolves on the configuration manifold Q. However, since the equations (3.2) are second-order, one may also think of them as a first-order system on TQ, and so as a control affine system. Let us see what this control affine system looks like. First of all, let us state that the object in CAS will have the form $(TQ, \mathcal{F})$. That is, its state space is the tangent bundle TQ. The vector field f_0 is defined to be the geodesic spray Z corresponding to the affine connection ∇. We also need to regard the vector fields $Y_1, \dots, Y_m$ as vector fields on TQ in the appropriate manner. To do this, given a vector field X on Q, define the ***vertical lift*** of X to be the vector field $\operatorname{verlift}(X)$ on TQ defined by $\operatorname{verlift}(X)(v_q) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(v_q + tX(q))$, where $v_q \in T_qM$. In coordinates, if $X = X^i \frac{\partial}{\partial q^i}$, we have $\operatorname{verlift}(X) = X^i \frac{\partial}{\partial v^i}$. We then define $f_a = \operatorname{verlift}(Y_a)$, $a = 1, \dots, m$. Thus, to an affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \{Y_1, \dots, Y_m\})$ we associate the control affine system $\Sigma = (TQ, \{Z, \operatorname{verlift}(Y_1), \dots, \operatorname{verlift}(Y_m)\})$. The associated first-order control affine system on TQ is then

$$\dot{v}(t) = Z(v(t)) + u^a(t)\operatorname{verlift}(Y_a(v(t))).$$

3.2 Morphisms in ACCS

Now let us look at morphisms in the category ACCS. Thus we need to specify a way to send an affine connection control system to another affine connection control system. We consider morphisms that are special forms of morphisms in CAS. This is sensible since, as we noted in the previous section, we may think of ACCS as a subcategory of the category CAS. We let $\Sigma_2(TQ)$ denote the bundle of symmetric $(0,2)$ tensors on Q, and we denote by $\mathbb{R}^m_Q$ the trivial vector bundle $Q \times \mathbb{R}^m$ over Q. If S is a section of $\mathbb{R}^m_Q \otimes \Sigma_2(TQ)$, then for $a = 1, \dots, m$, we define a section S^a of $\Sigma_2(TQ)$ by $S^a(X,Y) = S(\boldsymbol{e}^a \otimes (X,Y))$, where $\boldsymbol{e}^a$ is the ath standard basis vector for $(\mathbb{R}^m)^*$.

We consider affine connection control systems denoted $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathcal{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathcal{Y}})$ with $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ and $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{m}}\}$. Recall our notation that if c is a curve on Q and $\phi \colon Q \to \tilde{Q}$ is a smooth map, we let c_ϕ be the curve on $\tilde{Q}$ given by $c_\phi = \phi \circ c$. An **ACCS** ***morphism*** sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$ is a triple (ϕ, S, Λ) with the following properties:

1. $\phi \colon Q \to \tilde{Q}$ is a smooth mapping;
2. S is a smooth section of $\mathbb{R}^{\tilde{m}}_Q \otimes \Sigma_2(TQ)$ and $\Lambda \colon Q \to L(\mathbb{R}^m, \mathbb{R}^{\tilde{m}})$ is a smooth map that together satisfy the following conditions:
   (a) $T_q\phi(Y_a(q)) = \Lambda^\alpha_a(q)\tilde{Y}_\alpha(\phi(q))$;
   (b) $T_q\phi(\nabla_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)} + S^\alpha(X(q), X(q))\tilde{Y}_\alpha(\phi(q))$ where $\tilde{X}$ is a vector field on $\tilde{Q}$ that is ϕ-related to the vector field X on Q.

If $\Lambda(q)$ is an isomorphism for each $q \in Q$ then the ACCS morphism (ϕ, S, Λ) is called ***control nondegenerate***. If

$$T_q\phi(\operatorname{span}_{\mathbb{R}}\{Y_1(q), \dots, Y_m(q)\}) = \operatorname{span}_{\mathbb{R}}\{\tilde{Y}_1(\phi(q)), \dots, \tilde{Y}_{\tilde{m}}(\phi(q))\}$$

for all $q \in Q$, the ACCS morphism (ϕ, S, Λ) is called ***complete***.

Let us look at what are isomorphisms in this category. Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathcal{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathcal{Y}})$ be two affine connection control systems, and suppose that (ϕ, S, Λ) sends Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. We say that (ϕ, S, Λ) is an ***isomorphism*** of Σ_{aff} and $\tilde{\Sigma}_{\mathrm{aff}}$ if $\phi \colon Q \to \tilde{Q}$ is a diffeomorphism and if ϕ^{-1} has the property that for every controlled trajectory $(\tilde{c}, \tilde{u})$ of $\tilde{\Sigma}_{\mathrm{aff}}$, there exists an admissible input u for

Σ_{aff} so that $(\tilde{c}_{\phi^{-1}}, u)$ is a controlled trajectory of Σ_{aff}. Given two affine connection control systems $\Sigma_{\mathrm{aff}} = (U, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{U}, \tilde{\nabla}, \tilde{\mathscr{Y}})$, we say that they are ***locally equivalent by*** $(\boldsymbol{\phi}, \boldsymbol{S}, \boldsymbol{\Lambda})$ if for each $q \in Q$ there exists a neighbourhood U of q and a neighbourhood $\tilde{U}$ of $\phi(q)$ so that $(\phi|U, S|U, \Lambda|U)$ is an isomorphism from $\Sigma_{\mathrm{aff}}|U$ to $\tilde{\Sigma}_{\mathrm{aff}}|\tilde{U}$. Given an affine connection control system Σ_{aff}, one is often interested in what types of affine connection control systems are locally equivalent to Σ_{aff}. Indeed, this is one of the basic problems that we hope to be able to address with our approach.

3.3 Properties of ACCS morphisms

We first note that in [8] it is proved that if (ϕ, S, Λ) is an ACCS morphism then ϕ maps controlled trajectories to controlled trajectories, and that conversely, if a map $\phi\colon Q \to \tilde{Q}$ has this property, then it forms part of an ACCS morphism. Since we can think of ACCS as a subcategory of CAS, it follows that ACCS morphisms can be realised as CAS morphisms. This is easy to do, and the following result states the resulting correspondence.

Proposition 3.1 *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with $\Sigma = (TQ, \mathscr{F})$ and $\tilde{\Sigma} = (T\tilde{Q}, \tilde{\mathscr{F}})$ the corresponding control affine systems. If (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$, then $(\psi, \lambda_0, \Lambda')$ is a CAS morphism sending Σ to $\tilde{\Sigma}$ where $\psi = T\phi$, $\lambda_0^\alpha(v_q) = S^\alpha(v_q, v_q)$, and $\Lambda'(v_q) = \Lambda(q)$.*

The converse question here is not so clear. That is, if one has a CAS morphism $(\psi, \lambda_0, \Lambda')$ sending an object in ACCS $\subset$ CAS to another object in ACCS $\subset$ CAS, is it necessarily the case that $(\psi, \lambda_0, \Lambda')$ is derived from an ACCS morphism as described in Proposition 3.1? The following example answers the question in the negative.

Example 3.2 We take $Q = \tilde{Q} = \mathbb{R}^2$, ∇ and $\tilde{\nabla}$ to be the canonical flat connection on $\mathbb{R}^2$, and $Y_1 = \tilde{Y}_1 = (1, 0)$. Thus we have defined two identical single-input affine connection control systems, $\Sigma_{\mathrm{aff}} = (Q, \nabla, \{Y_1\})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \{\tilde{Y}_1\})$. One may then check that the triple $(\psi, \lambda_0, \Lambda')$ is a CAS morphism sending Σ to $\tilde{\Sigma}$ when ψ is defined by $\psi(x, y, u, v) = (x, y + v, u, v)$, $\lambda_0 = 0$, and $\Lambda' = 1$. However, we note that ψ is not a bundle mapping, and so in particular cannot be of the form $\psi = T\phi$ for some mapping $\phi\colon Q \to \tilde{Q}$. Therefore, there is no ACCS morphism that gives rise to the CAS morphism $(\psi, \lambda_0, \Lambda')$ in the manner described in Proposition 3.1. □

Thus ACCS morphisms are indeed a smaller class than are CAS morphisms. This is important because it tells us that the classification problem in ACCS will not be contained as a subset of the classification problem in CAS, since the former has more structure. Let us provide an instance of this additional structure.

Proposition 3.3 *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems. Suppose that (ϕ, S, Λ) is an ACCS morphism that maps Σ to $\tilde{\Sigma}_{\mathrm{aff}}$, and that $\Lambda(q) \in L(\mathbb{R}^m, \mathbb{R}^{\tilde{m}})$ is an epimorphism for each $q \in Q$ with right inverse denoted by $\Theta(q)$. On Q define an affine connection $\overline{\nabla}$ by $(\overline{\nabla}_X Y)_q = (\nabla_X Y)_q - S^\alpha(X(q), Y(q))\Theta^a_\alpha(q) Y_a(\phi(q))$. Then $\phi\colon Q \to \tilde{Q}$ is a totally geodesic mapping between $\overline{\nabla}$ and $\tilde{\nabla}$. Furthermore, there exists an ACCS isomorphism from Σ_{aff} to $\overline{\Sigma}_{\mathrm{aff}} = (Q, \overline{\nabla}, \mathscr{Y})$.*

3.4 Compositions and decompositions of ACCS morphisms

We now wish to determine conditions under which a morphism in ACCS can be written as a product of two simpler ACCS morphisms. If $\Sigma_{\mathrm{aff},1} = (Q^1, \nabla^1, \mathscr{Y}^1)$, $\Sigma_{\mathrm{aff},2} = (Q^2, \nabla^2, \mathscr{Y}^2)$, and $\Sigma_{\mathrm{aff},3} = (Q^3, \nabla^3, \mathscr{Y}^3)$ are affine connection control systems, and (ϕ_1, S_1, Λ_1) and (ϕ_2, S_2, Λ_2) are

ACCS morphisms sending $\Sigma_{\text{aff},1}$ to $\Sigma_{\text{aff},2}$ and $\Sigma_{\text{aff},2}$ to $\Sigma_{\text{aff},3}$, respectively, then one verifies that their composition is the ACCS morphism $(\phi_{21}, S_{21}, \Lambda_{21})$ defined by $\phi_{21} = \phi_2 \circ \phi_1$, $S^\sigma_{21}(X(q), Y(q)) = S^\sigma_2(T_q\phi_1(X(q)), T_q\phi_1(Y(q))) + S^\alpha_1(X(q), Y(q))(\Lambda_2)^\sigma_\alpha(\phi_1(q))$, and $(\Lambda_{21})^\sigma_a(q) = (\Lambda_1)^\sigma_\alpha(q)(\Lambda_2)^\alpha_a(\phi_1(q))$.

Now let us define the special classes of ACCS morphisms one may consider. An ACCS morphism (ϕ, S, Λ) that maps $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a ***morphism over controls*** if $Q \subset \tilde{Q}$ and if $\phi\colon Q \to \tilde{Q}$ is the inclusion map. The category whose objects are affine connection control systems and whose morphisms are ACCS morphisms that are morphisms over controls we denote by CACCS. The idea is that a morphism over controls does essentially nothing to the system's states, and alters only the controls. Moreover, a morphism over controls is an algebraic operation since one only alters the controls by a map that is affine in control. It is CACCS morphisms that one will naturally use in practice when simplifying the equations. Often their application is given the name "partial feedback linearisation."

An ACCS morphism (ϕ, S, Λ) is a ***morphism over configurations*** if $S_q = 0$ and $\Lambda(q) = \text{id}_{\mathbb{R}^m}$ for each $q \in Q$. We denote by QACCS the category whose objects are affine connection control systems and whose morphisms are ACCS morphisms that are morphisms over configurations. The idea here is that one leaves the controls alone, and alters only the configuration spaces.

Let us give a list of results that concern the various types of morphisms and decompositions of given morphisms into products of the simpler types. The idea in all cases is that one can in many cases reduce the problem of finding a desired ACCS morphism into first one of a CACCS and a QACCS morphism, followed by the other.

Proposition 3.4 *A triple (ϕ, S, Λ) is a QACCS morphism mapping $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if the following two conditions hold:*

1. *$\phi\colon Q \to \tilde{Q}$ is a totally geodesic mapping between ∇ and $\tilde{\nabla}$;*
2. *each control vector field $\tilde{Y}_a$ on $\tilde{Q}$ is ϕ-related to the control vector field Y_a on Q.*

Proposition 3.5 *A control nondegenerate ACCS morphism (ϕ, S, Λ) is a composition of a CACCS isomorphism with a QACCS morphism.*

Proposition 3.6 *An ACCS isomorphism (ϕ, S, Λ) is a composition of a QACCS isomorphism with a CACCS isomorphism.*

Proposition 3.7 *Let $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems, and suppose that (ϕ, S, Λ) is a complete ACCS morphism and that $m = \tilde{m}$. Then for each $q \in Q$ there is a neighbourhood U of q and a neighbourhood $\tilde{U}$ of $\phi(q)$ so that the morphism $(\phi|U, S|U, \Lambda|U)$ from $\Sigma_{\text{aff}}|U$ to $\tilde{\Sigma}_{\text{aff}}|\tilde{U}$ is the composition of a CACCS isomorphism and a QACCS morphism.*

4 Future directions

The preceding discussion of the category ACCS and its morphisms is only cursory. There is some more that has been done, and much that needs to be done to fully exploit the way of thinking presented here. Let us outline some of this.

Restricted systems, etc. The idea of a "subsystem" or "restricted system" is that the controlled dynamics of a subsystem can be contained in that of the full system. The author has presented some of the basic results concerning subsystems and restrictions in ACCS [8]. Such systems will naturally arise, for example, in studies of controllability properties of affine connection control systems.

Factor systems The notion of a factor system comes up in any setting where there is a natural quotient operation. In mechanics, this happens frequently since a mechanical system often possesses a symmetry group which acts by affine transformations on the system's configuration space. At present, the bearing of such structure on a general affine connection is poorly understood. The control theoretic setting for factor systems may provide a suitable context in which to confront this gap in our understanding of affine differential geometry. Preliminary results are given in [8].

Local equivalence The problem of local equivalence in ACCS is likely to be a difficult one. However, the success of methods motivated by the category theory approach [5] gives some hope that significant progress may be possible. There are already some *ad hoc* approaches to this problem in the presence of additional structure. Indeed, the "kinetic shaping" techniques of [2] (see also [6]) may be seen as providing local equivalence to a desired form of the closed-loop system, remaining in the setting of the Levi-Civita connection. In coming to grips with the local equivalence problem, one will have to understand how the local invariants of an affine connection interact with the control vector fields $\mathscr{Y}$. As an instance of how the challenges here surpass those of local equivalence in CAS, we make the following observation. As a system in CAS, the input distribution of an ACCS system is involutive (vertically lifted vector fields always commute). However, in CAS, the classification problem for systems with involutive input distributions is known [5, Theorem 3.3]. However, none of the representatives of the possible equivalence classes in [5] are ACCS systems!

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