Coalgebra and Supervisory Control with Partial Observations

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Abstract

Coalgebraic techniques are applied to the supervisory control of discrete-event systems with partial observations. Classical notions from concurrency theory are specialized to control theory. The concept of weak transitions enables the relational characterization of observability and gives rise to a coalgebraic formulation of the necessary and sufficient conditions for the existence of a supervisory control which achieves a considered legal language.

1 Introduction

Coalgebras are well suited for the study of automata and their various extensions, and more generally, for state transition (dynamical) systems. Discrete-event systems are often represented by automata viewed as a particular algebraic structure. However, it has been shown in [5] that they can be also viewed as deterministic partial automata (automata with partial transition function). These are coalgebras of a simple functor of the category of sets. Coalgebras are categorial duals of algebras (the corresponding functor operates from a given set rather than to a given set). The theory of universal coalgebra [6] has been developed in analogy with the corresponding theory of universal algebra. The notion of bisimulation relation is just the coalgebraic counterpart of congruence in algebra. Bisimulation has been used to formulate a proof principle called coinduction.

This paper aims at the development of control of discrete-event systems (DES) with partial observations using the coalgebraic approach. Being inspired by the theory of concurrency, we introduce the concept of weak transitions. This enables the definition of observational indistinguishability relations and also observability relations which correspond to the observability. The theorem stating the conditions for a given language to be achieved by the supervisory control can be formulated in this coalgebraic context using supervised product defined by coinduction.

2 Partial automata

In this section we formulate partial automata as coalgebras and relate these to the final coalgebra of partial automata, i.e. a partial automaton of partial languages. Let A be an arbitrary set (usually finite and referred to as alphabet or the set of inputs or events). The empty string will be denoted by ε. Denote by $1 = \{\emptyset\}$ the one element set and by $2 = \{0, 1\}$ the set of Booleans. A partial automaton is a pair $S = (S, \langle o, t\rangle)$, where S is a set of states, and a pair of functions $\langle o, t\rangle\ : S \to 2 \times (1 + S)^A$, consists of an output function $o:\ S \to 2$ and a transition function $t:\ S \to (1+S)^A$. The output function o indicates whether a state $s \in S$ is accepting (or terminating) : $o(s) = 1$, denoted by $s\downarrow$, or not: $o(s) = 0$, denoted by $s\uparrow$. The transition function t associates to each state s in S a function $t(s)\ :\ A \to (1+S)$. Set $1 + S$ is the disjoint union of S and $\{\emptyset\}$. The meaning of the state transition function is that $t(s)(a) = \emptyset$ iff $t(s)(a)$ is undefined, which means that there is no $a-$transition from state $s \in S$. Similarly, $t(s)(a) \in S$ means that $a-$transition from s is possible and we define in this case $t(s)(a) = s_a$, which is denoted mostly by $s \xrightarrow{a} s_a$. This notation can be extended by induction to arbitrary strings in A^*. Note that partial automata can be viewed as Moore automata with Boolean outputs.

A *homomorphism* between partial automata $S = (S, \langle o, t\rangle)$ and $S' = (S', \langle o', t'\rangle)$ is a function $f:\ S \to S'$ with, for all $s \in S$ and $a \in A$:

$$o'(f(s)) = o(s) \text{ and } s \xrightarrow{a} s_a \text{ iff } f(s) \xrightarrow{a} f(s)_a, \text{ in which case: } f(s)_a = f(s_a).$$

A *partial automaton* $S' = (S', \langle o', t'\rangle)$ is a *subautomaton* of $S = (S, \langle o, t\rangle)$ if $S' \subseteq S$ and the inclusion $i : S' \to S$ is a homomorphism.

A *bisimulation* between two partial automata $S = (S, \langle o, t\rangle)$ and $S' = (S', \langle o', t'\rangle)$ is a relation $R \subseteq S \times S'$ with, for all $s \in S$ and $s' \in S'$:

$$\text{if } \langle s, s'\rangle \in R \text{ then } \begin{cases} (i) & o(s) = o(s'), \\ (ii) & \forall a \in A\ :\ s \xrightarrow{a}\ \Rightarrow\ s' \xrightarrow{a} \text{ and } \langle s_a, s'_a\rangle \in R, \text{ and} \\ (iii) & \forall a \in A\ :\ s' \xrightarrow{a}\ \Rightarrow\ s \xrightarrow{a} \text{ and } \langle s_a, s'_a\rangle \in R. \end{cases}$$

We write $s \sim s'$ whenever there exists a bisimulation R with $\langle s, s'\rangle \in R$. This relation is the union of all bisimulations, i.e. the greatest bisimulation, also called bisimilarity.

2.1 Final automaton of partial languages

The partial automaton of partial languages can be defined using the notion of input derivative. Below we define the partial automaton of partial languages over an alphabet (input set) A, denoted by $(\mathcal{L}, \langle o_{\mathcal{L}}, t_{\mathcal{L}}\rangle)$. More formally,

$$\mathcal{L} = \{(V, W) \mid V \subseteq W \subseteq A^*,\ W \neq \emptyset, \text{ and } W \text{ is prefix-closed.}\}$$

The state transition function $t_{\mathcal{L}}\ : \mathcal{L} \to (1 + \mathcal{L})^A$ is defined using the input derivatives. Recall that for any partial language $L = (L^1, L^2) \in \mathcal{L}$, $L_a = (L^1_a, L^2_a)$, where $L^i_a = \{w \in A^* \mid aw \in L^i\}$, $i = 1, 2$. If $a \notin L^2$ then L_a is undefined. Given any $L = (L^1, L^2) \in \mathcal{L}$, the

Moore automaton structure of $\mathcal{L}$ is given by:

$$o_{\mathcal{L}}(L) = \begin{cases} 1 & \text{if } \varepsilon \in L^1 \\ 0 & \text{if } \varepsilon \notin L^1 \end{cases}, \quad t_{\mathcal{L}}(L)(a) = \begin{cases} L_a & \text{if } L_a \text{ is defined} \\ \emptyset & \text{otherwise} \end{cases}.$$

Notice that if L_a is defined, then $L_a^1 \subseteq L_a^2$, $L_a^2 \neq \emptyset$, and L_a^2 is prefix-closed. The following notational conventions will be used: $L \downarrow$ iff $\varepsilon \in L^1$, $L \xrightarrow{w} L_w$ iff L_w is defined iff $w \in L^2$. Denote by $\bar{L}$ the prefix closure of L, whose definition is extended to partial languages componentwise. Recall from [5] that automaton $(\mathcal{L}, \langle o_{\mathcal{L}}, t_{\mathcal{L}} \rangle)$ is final among all partial automata: for any automaton $S = (S, \langle o, t \rangle)$ there exists a unique homomorphism $l : S \to \mathcal{L}$. For $s, s' \in S$, $s \sim s'$ iff $l(s) = l(s')$. Another characterization of finality of $\mathcal{L}$ is that it satisfies the principle of coinduction: for all K and L in $\mathcal{L}$, if $K \sim L$ then $K = L$. Recall yet that the unique homomorphism l given by finality of $\mathcal{L}$ maps a state $s \in S$ to the partial language $l(s) = (L_s^1, L_s^2) = (\{w \in A^* \mid s \xrightarrow{w} \text{ and } s_w \downarrow\}, \{w \in A^* \mid s \xrightarrow{w}\})$. The existence of unique behavior homomorphisms enables the so-called coinductive definitions. For instance, operations on languages (e.g. sum, concatenation, star, synchronized product or supervised product) can be defined by coinduction [5]. The coinductive definitions amount to defining an appropriate coalgebraic structure on the codomain of the defined operator. In the case of partial languages, transition and ouput functions must be defined.
Recall also that the simulation relation corresponds to (partial) language inclusion. We denote this componentwise inclusion simply by the ordinary inclusion relation. Some further notation from [5] is used, e.g. 'zero' (partial) language is denoted by 0, i.e. $0 = (\emptyset, \{\varepsilon\})$.

There is yet another important concept needed in this paper. Namely, given an (ordinary) language L, the suffix closure of L is defined by $\text{suffix}(L) = \{s \in A^* \mid \exists u \in A^* \text{ with } us \in L\}$. For partial languages, the suffix closure is defined in the same way as the prefix closure, i.e. componentwise. There is the following relation between the transition structure of L and its suffix closure operator.

Observation 2.1. *For any (partial) language L:* $\text{suffix}(L) = \cup_{u \in L^2} L_u$.

Proof. It follows from the definition of $\text{suffix}(L)$ and L_u. □

3 Weak transitions.

In the following definition we introduce the notion of weak derivative (transition) for partial languages. It disregards unobservable transitions, which correspond to internal actions in the framework of process algebras. We make the standard assumptions from the control theory of partially observed DES [1]. Namely, $A = A_o \cup A_{uo}$ is composed of observable (A_o) and unobservable (A_{uo}) events with the natural projection $P : A^* \to A_o^*$. The effect of P is just to erase unobservable inputs. In particular, $P(\tau) = \varepsilon$ for $\tau \in A_{uo}^*$, $P(a) = a$ for $a \in A_o$, and P is catenative. Also, $A = A_c \cup A_{uc}$, where A_c stands for controllable and A_{uc} for uncontrollable events.

Definition 3.1. *(Weak transitions.) For $a \in A$ denote $L \overset{P(a)}{\Rightarrow}$ iff $\exists s \in A^* \ : \ P(s) = P(a)$ and $L \xrightarrow{s} L_s$. Denote in this case $L \overset{P(a)}{\Rightarrow} L_s$.*

Remark 3.1. *According to this notation for unobservable events $L \overset{\varepsilon}{\Rightarrow}$ is an abreviation for $\exists \tau \in A_{uo}^*$ such that $L \xrightarrow{\tau} L_\tau$. We admit $\tau = \varepsilon$, hence $L \overset{\varepsilon}{\Rightarrow}$ is always true. For $a \in A_o$ our notation means that there exist $\tau, \tau' \in A_{uo}^*$ such that $L \xrightarrow{\tau a \tau'} L_{\tau a \tau'}$. Remark that there may exist two or more such couples of unobservable strings. This definition can be extended to strings $s \in A^*$ in the following way:*
$L \overset{P(s)}{\Rightarrow}$ iff there exists $t \in A^* \ : \ P(s) = P(t)$ and $L \xrightarrow{t} L_t$. Denote in this case $L \overset{P(s)}{\Rightarrow} L_t$.

We have the following properties of weak transitions.

Lemma 3.1. *For all $L \in \mathcal{L}$ and $s \in A^*$ the following are equivalents:*

(1) $L \overset{P(s)}{\Rightarrow}$

(2) $\exists u \in L^2 \ : \ P(u) = P(s)$

(3) $P(s) \in P(L)^2$

(4) $P(L) \xrightarrow{P(s)}$

Proof. Obvious from the corresponding definitions. □

4 Observability relation

In supervisory control of DES with partial observations the observability of a (specification) language with respect to the plant and projection (to observable events) is a necessary condition for the existence of a supervisory control [4].

Definition 4.1. *(Observability.) A partial language K is said to be observable with respect to another partial language L (with $K \subseteq L$) and projection P if for all $s \in K^2$ and $a \in A_c$ the following implication holds true :*

$$sa \in L^2,\ s'a \in K^2, \text{ and } P(s) = P(s') \ \Rightarrow \ sa \in K^2.$$

Let K be a partial language. Denote $DK = \{K_u \mid u \in K^2\}$ and call it the set of language derivatives of K, i.e. it is the carrier set of subautomaton $\langle K \rangle$ of $\mathcal{L}$ generated by K. Therefore, DK is finite for regular languages K [5]. In order to characterize the observability property we first need to introduce the following auxiliary relation defined on $DK \times DL$. Note that any relation $R \subseteq (DK \times DL)^2$ can be endowed with the following transition structure: for $a \in A$ $(M, N) \xrightarrow{a} (M', N')$ iff $M \xrightarrow{a} M_a$ and $N \xrightarrow{a} N_a$ with $M' = M_a$ and $N' = N_a$. We write $(M, N) \overset{P(a)}{\Rightarrow} (M', N')$ iff $\exists s \in M^2 \cap N^2$: $P(s) = a$, $M' = M_s$, and $N' = N_s$.

Definition 4.2. *A binary relation* $Aux(K, L) \subseteq (DK \times DL)^2$ *is called an* observational indistinguishability relation *if the following two conditions hold :*

(i) $\langle (K, L), (K, L) \rangle \in Aux(K, L)$

(ii) *If* $\langle (M, N), (Q, R) \rangle \in Aux(K, L)$ *then* $\forall a \in A$: *if* $(M, N) \overset{P(a)}{\Rightarrow} (M', N')$ *and* $(Q, R) \overset{P(a)}{\Rightarrow} (Q', R') \;\Rightarrow\; \langle (M', N'), (Q', R') \rangle \in Aux(K, L)$

For $\langle (M, N), (Q, R) \rangle \in DK \times DL$ we write $(M, N) \approx^{K,L}_{Aux} (Q, R)$ whenever $\langle (M, N), (Q, R) \rangle \in Aux(K, L)$.

Lemma 4.1. *For given partial languages* K, L*:* $\langle (M, N), (Q, R) \rangle \in Aux(K, L)$ *iff there exist two strings* $s, s' \in K^2$ *such that* $P(s) = P(s')$ *and* $M = K_s$, $N = L_s$, $Q = K_{s'}$, *and* $R = L_{s'}$.

Proof. ($\Leftarrow$) Let $(M, N) \in DK \times DL$ and $(Q, R) \in DK \times DL$ and there exist two strings $s, s' \in K^2$ such that $P(s) = P(s')$, $M = K_s$, $N = L_s$, $Q = K_{s'}$, and $R = L_{s'}$. Let $s = s_1 \dots s_n$ and $s' = t_1 \dots t_m$. Let $P(s) = P(s') = a_1 \dots a_k$. Then $n \geq k$, $m \geq k$, and there exist two increasing sequences of integers (indices) $u_i \geq i$, $i = 1, \dots, k$ and $v_i \geq i$, $i = 1, \dots, k$ such that $a_i = s_{u_i} = t_{v_i}$. Since $s, s' \in K^2$, and all a_i are observable events we can write $(K, L) \overset{P(a_1)\dots P(a_n)}{\Longrightarrow} (M, N)$ and also $(K, L) \overset{P(a_1)\dots P(a_n)}{\Longrightarrow} (Q, R)$, whence by (ii) inductively applied $(M, N) \approx^{K,L}_{Aux} (Q, R)$.
($\Rightarrow$) Let $(M, N) \approx^{K}_{Aux} (Q, R)$. Then by the construction of $Aux(K, L)$ there exist $a_1, \dots, a_k \in A$ such that $(K, L) \overset{P(a_1)\dots P(a_k)}{\Longrightarrow} (M, N)$ and $(K, L) \overset{P(a_1)\dots P(a_k)}{\Longrightarrow} (Q, R)$. Therefore there exist two strings s, s' with the same projection with $M = K_s$, $N = L_s$, $Q = K_{s'}$, and $R = L_{s'}$. $\square$

Our aim is to provide a coalgebraic characterization of observability. Let us introduce the following relation called observability relation, in which the observational indistinguishability relation is used.

Definition 4.3. *(Observability relation.) Given two (partial) languages* K *and* L*, a binary relation* $O(K, L) \subseteq DK \times DL$ *is called an* observability relation *if for any* $\langle M, N \rangle \in O(K, L)$ *the following items hold:*

(i) $\forall a \in A : \; M \xrightarrow{a} \;\Rightarrow\; N \xrightarrow{a}$ *and* $\langle M_a, N_a \rangle \in O(K, L)$

(ii) $\forall a \in A_c : \; N \xrightarrow{a}$ *and* $(\exists M' \in DK, N' \in DL : (M', N') \approx^{K,L}_{Aux} (M, N)$ *and* $M' \xrightarrow{a}) \;\Rightarrow\; M \xrightarrow{a}$ *and* $\langle M_a, N_a \rangle \in O(K, L)$.

For $M \in DK$ and $N \in DL$ we write $M \approx_{O(K,L)} N$ whenever there exists an observability relation $O(K, L)$ on $DK \times DL$ such that $\langle M, N \rangle \in O(K, L)$. In order to check whether for a given pair of (partial) languages (K and L), K is observable with respect to L, it is sufficient to establish an observability relation $O(K, L)$ on $DK \times DL$ such that $\langle K, L \rangle \in O(K, L)$. Indeed, we have

Theorem 4.1. *A (partial) language* K *is observable with respect to* L *(with* $K \subseteq L$*) and* P *iff* $K \approx_{O(K,L)} L$.

Proof. ($\Rightarrow$) Let K be observable with respect to L. Denote

$$O_1(K,L) = \{\langle K_u, L_u\rangle \in DK \times DL \mid u \in K^2 \}.$$

Let us show that $O_1(K,L)$ is an observability relation.
Let $\langle M, N\rangle \in O_1(K,L)$. We can assume that $M = K_s$ and $N = L_s$ for $s \in K^2$. We must show that conditions (i) and (ii) are safisfied.
(i) Let $M \xrightarrow{a}$ for $a \in A$. Notice that $K \subseteq L$ implies that for any $u \in K^2$, $K_u \subseteq L_u$. In particular $N \xrightarrow{a}$, because $M = K_s \subseteq L_s = N$ and it follows from the definition of $O_1(K,L)$ that $\langle M_a, N_a\rangle \in O_1(K,L)$.
(ii) Let $N \xrightarrow{a}$ for $a \in A_c$ and $\exists (M', N') \approx^{K,L}_{Aux} (M,N) \quad : \quad M' \xrightarrow{a}$. Then by Lemma 4.1 there exist two strings $s', s'' \in K^2$ such that $P(s') = P(s'')$ and $M' = K_{s'}$, $N' = L_{s'}$, $M = K_{s''} (= K_s)$, and $N = L_{s''} (= L_s)$. Thus we have $s', s'' \in K^2$. Now $M' \xrightarrow{a}$ implies that $s'a \in K^2$. From $N \xrightarrow{a}$ and $N = L_{s''}$ follows $s''a \in L^2$. Now by application of the observability of K with respect to L and P we deduce $s''a \in K^2$, i.e. $a \in K^2_{s''} = M^2$. This means that $M \xrightarrow{a}$, which was to be proved. The rest follows from (i).
($\Leftarrow$) Let $K \approx_{O(K,L)} L$. Let us show that K is observable with respect to L and P. For this purpose, let $s \in K^2$ and $a \in A_c$ such that $s'a \in K^2$ and $sa \in L^2$ and $P(s) = P(s')$. Then $s \in K^2 \cap L^2$, i.e. $L \xrightarrow{s}$ and $K \xrightarrow{s}$, whence from (i) of definition 4.3 inductively applied $K_s \approx_{O(K,L)} L_s$. Since $K \subseteq L$ and $s'a \in K^2$, we have $s' \in L^2$, because K^2 is prefix-closed. According to Lemma 4.1 we have $(K_s, L_s) \approx^{K,L}_{Aux} (K_{s'}, L_{s'})$. Notice that $sa \in L^2$ means $L_s \xrightarrow{a}$, and similarly $s'a \in K^2$ means $K_{s'} \xrightarrow{a}$. By (ii) of the definition of observability relation we obtain that $K_s \xrightarrow{a}$, i.e. $sa \in K^2$. □

5 Coinductive definition of supervised product and partial bisimulation under partial observations.

In this section we give the definition of the supervised product of languages that describes the behavior of supervised DES under partial observations. Assume throughout this section that specification K and open loop partial language L ($K \subseteq L$) are given.

Definition 5.1. *(Supervised product.) Define the following binary operation on (partial) languages called supervised product under partial observations for all $M, N \in DK \times DL$:*

$$(M/^O_U N)_a = \begin{cases} M_a/^O_U N_a & \text{if } M \xrightarrow{a} \text{ and } N \xrightarrow{a} \\ (\cup_{\{(M',N')\approx^{K,L}_{Aux}(M,N)\}} M')_a/^O_U N_a & \text{if } M \not\xrightarrow{a} \text{ and } \exists (M',N') \approx^{K,L}_{Aux} (M,N) \ : \\ & M' \xrightarrow{a} \text{ and } N \xrightarrow{a} \text{ and } a \in A_c \\ 0/^O_U N_a & \text{if } M \not\xrightarrow{a} \text{ and } N \xrightarrow{a} \text{ and } a \in A_{uc} \\ \emptyset & \text{otherwise} \end{cases}$$

and $(M/^O_U N)\downarrow$ iff $N\downarrow$.

Remark 5.1. *1. According to Observation 2.1, $DL \subseteq \mathrm{Pwr}(\mathrm{suffix}(L))$ and since $K \subseteq L$ also $DK \subseteq \mathrm{Pwr}(\mathrm{suffix}(L))$.*
2. It follows from the definition of supervised product that $M \subseteq M/_U^O N \subseteq N$. Both inclusions can be verified by construction of the corresponding simulation relations. Indeed, $M \xrightarrow{a} \ \Rightarrow (M/_U^O N) \xrightarrow{a} \ \Rightarrow \ N \xrightarrow{a}$. As a consequence we conclude that the range of supervised product is again $\mathrm{Pwr}(\mathrm{suffix}(L))$. Therefore, the supervised product can be also viewed as a (partial) binary operation on $\mathrm{Pwr}(\mathrm{suffix}(L))$.

Now we proceed in the same way as in the case of full observations. Let us define the following relation called partial bisimulation under partial observations.

Definition 5.2. *(Partial bisimulation.) A binary relation $R(K,L) \subseteq DK \times DL$ is called a* partial bisimulation under partial observations *if for all $\langle M, N\rangle \in R(K,L)$:*

(i) $o(M) = o(N)$ *($M\downarrow$ iff $N\downarrow$)*

(ii) $\forall a \in A: \ M \xrightarrow{a} \ \Rightarrow \ N \xrightarrow{a}$ *and* $\langle M_a, N_a\rangle \in R(K,L)$

(iii) $\forall u \in A_{uc}: \ N \xrightarrow{u} \ \Rightarrow \ M \xrightarrow{u}$ *and* $\langle M_u, N_u\rangle \in R(K,L)$

(iv) $\forall a \in A_c: \quad N \xrightarrow{a}$ *and* $(\exists (M', N') \approx_{Aux}^{K,L} (M,N) \ : \ M' \xrightarrow{a}) \ \Rightarrow \ M \xrightarrow{a}$ *and* $\langle M_a, N_a\rangle \in R(K,L)$.

For $M \in DK$ and $N \in DL$ we write $M \approx_U^{O(K,L)} N$ whenever there exists a partial bisimulation under partial observations $R(K,L)$ such that $\langle M, N\rangle \in R(K,L)$. This relation is called partial bisimilarity under partial observations.

Remark 5.2. *Notice that (i) relates the marking components of the languages involved and (ii) corresponds to the language simulation (inclusion), while (iii) to the controllability and (iv) to the observability condition. Observe also that the second statements on the righthand sides of implications (iii) and (iv) follow from the corresponding first statements and (ii).*

Now we are ready to formulate the main theorem, which gives a coalgebraic formulation of the controllability and observability theorem in supervisory control of DES with partial observations.

Theorem 5.1. $K \approx_U^{O(K,L)} L$ *iff* $K = K/_U^O L$.

Proof. ($\Rightarrow$) Let $K \approx_U^{O(K,L)} L$. Define

$$R(K,L) = \{\langle M, (M/_U^O N)\rangle \ \mid M \in DK, \ N \in DL \text{ and } M \approx_U^{O(K,L)} N\}.$$

According to the coinduction proof principle it is sufficient to prove that $R(K,L)$ is a bisimulation, because then $K \sim_U^{O(K,L)} L$ implies that $\langle K, L\rangle \in R(K,L)$ hence $K = (K/_U^O L)$. Let $\langle M, (M/_U^O N)\rangle \in R(K,L)$.
(i) $M\downarrow$ iff $N\downarrow$ (because $M \approx_U^{O(K,L)} N$) iff $(M/_U^O N)\downarrow$.

(ii) If $M \xrightarrow{a}$ for $a \in A$ then by (ii) of definition 5.2 $N \xrightarrow{a}$ and $M_a \approx_U^{O(K,L)} N_a$. Thus, $(M/_U^O N) \xrightarrow{a}$, $(M/_U^O N)_a = (M_a/_U^O N_a)$, and $\langle M_a, (M/_U^O N)_a \rangle \in R(K,L)$.
(iii) If $(M/_U^O N) \xrightarrow{a}$, then according to the (coinductive) definition of the supervised product we have three possibilites : either $M \xrightarrow{a}$ and $N \xrightarrow{a}$ or $M \not\xrightarrow{a}$ and $\exists (M', N') \approx_{Aux}^{K,L} (M,N)$: $M' \xrightarrow{a}$ and $N \xrightarrow{a}$ and $a \in A_c$ or finally $M \not\xrightarrow{a}$ and $N \xrightarrow{a}$ and $a \in A_{uc}$. Notice however that the second case is contradicted by (iv) of definition 5.2 and also the third case is impossible due to (iii) of the same definition. Hence only the first possibility can occur, which brings us back to the previous case (ii).
($\Leftarrow$) Let us show that the following relation is a partial bisimulation under partial observations. Define

$$T(K,L) = \{\langle M, N \rangle \mid M \in DK,\ N \in DL \text{ and } M = (M/_U^O N)\}.$$

Let $\langle M, N \rangle \in T(K,L)$.
(i) $M \downarrow$ iff $(M/_U^O N) \downarrow$ (from the definition of $T(K,L)$) iff $N \downarrow$ (from definition 5.1).
(ii) If $M \xrightarrow{a}$ for $a \in A$ then $(M/_U^O N) \xrightarrow{a}$ and clearly (from the coinductive definition of supervised product) $N \xrightarrow{a}$. Also $M_a = (M/_U^O N)_a = (M_a/_U^O N_a)$, whence $\langle M_a, N_a \rangle \in T(K,L)$.
(iii) If $N \xrightarrow{u}$ for $u \in A_{uc}$ then $(M/_U^O N) \xrightarrow{u}$ according to the definition of supervised product. Thus $M \xrightarrow{u}$ as well. Furthermore, $M_u = (M/_U^O N)_u = (M_u/_U^O N_u)$, which means $\langle M_u, N_u \rangle \in T(K,L)$.
(iv) If $N \xrightarrow{a}$ for $a \in A_c$ and ($\exists (M', N') \approx_{Aux}^{K,L} (M,N)$: $M' \xrightarrow{a}$) then from the definition of supervised product (the second case occurs) $(M/_U^O N) \xrightarrow{a}$, i.e. $M \xrightarrow{a}$, which was to be shown. $\square$

Finally, similarly as in the case of full observations, there is the following characterization of partial bisimilarity.

Corollary 5.1. *$K \approx_U^{O(K,L)} L$ iff ($K \subseteq L$, $K^2 A_{uc} \cap L^2 \subseteq K^2$, $K \approx_{O(K,L)} L$, and $K^1 = K^2 \cap L^1$).*

Proof. It is quite analogous to the full observations case. In particular, notice that partial bisimulation under partial observations implies partial bisimulation as it has been first introduced in [5]. Thus, it is sufficient to consider only the additional property of observability, which appears in both sides of the claimed equivalence. $\square$

The concepts developed in this paper lead to new algorithms for supervisory control with partial observations presented in [3]. In that paper the concept of normality of a language with respect to a plant is also captured by relations. These are introduced on finite automata representations in order to make the computations feasible. The approach is inspired by the work of Cho and Marcus [2], where algebraic characterizations using the concept of invariant relations have been presented. The main advantage of the coalgebraic approach is that the formulations using relations provide a canonical way how to check different properties of languages (like controllability, observability, and normality). Since all these relations are in fact different weaker forms of bisimulation, we can proceed in the same way as for checking the bisimilarity [5].

6 Conclusion.

Coalgebraic techniques have been applied to the supervisory control of DES. This constitutes an extension of the usual algebraic approach and contributes to a better understanding of supervisory control of partially observed DES. However, the coalgebraic approach provides more then just an insight to the well known algebraic theory. It offers new characterizations of basic properties (e.g. observability and normality), which give rise to new efficient algorithms for the synthesis of optimal normal and controllable approximations [3].

References

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