Bounding the Solution Set of Uncertain Linear Equations: a Convex Relaxation Approach

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Abstract

In this paper, we discuss semidefinite relaxation techniques for computing minimal size ellipsoids that bound the solution set of a system of uncertain linear equations (ULE). The proposed technique is based on the combination of a quadratic embedding of the uncertainty, and the $\mathcal{S}$-procedure. The resulting bounding condition is expressed as a Linear Matrix Inequality (LMI) constraint on the ellipsoid parameters and the additional scaling variables. This formulation leads to a convex optimization problem that can be efficiently solved by means of interior point barrier methods.

1 Introduction

In this paper, we propose a technique for the determination of deterministic confidence bounds on the solutions of systems of linear equations, whose coefficients are imprecisely known. A similar problem arises for instance in the context of interval linear algebra, where we are given matrices $A \in \mathbb{R}^{n,n}$ and $y \in \mathbb{R}^n$, the elements of which are only known within intervals, and one seeks to compute intervals of confidence for the set of solutions, if any, to the equation $Ax = y$. Obtaining exact estimates on the confidence intervals for the elements of x in the above context is known to be an NP-hard problem, [8, 9].

Here, we consider a more general situation in which the data matrix $[A \;\; y]$ belongs to an uncertainty set $\mathcal{U}$ described by means of a linear fractional representation (LFR), and use semidefinite relaxation techniques [6] to determine readily computable minimal ellipsoidal bounds for the set of solutions. If desired, the bounding ellipsoid can then be projected along the coordinate axes to obtain intervals of confidence on the individual elements of x. Alternatively, the width of the uncertainty interval on an individual component of x can be directly used as the optimization criterion, in order to obtain tighter interval bounds. Besides, we discuss special situations in which semidefinite relaxations are lossless, and show how we can recover explicit solutions in these cases.

Semidefinite relaxation techniques for uncertain linear equations have been originally introduced by the authors in [1]. Similar techniques have also been applied to state estimation and filtering problems in the context of uncertain dynamical systems in [2, 5].

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1.1 Notation and preliminaries

For a square matrix X, $X \succ 0$ (resp. $X \succeq 0$) means X is symmetric, and positive-definite (resp. semidefinite). For a matrix $X \in \mathbb{R}^{n,m}$, $\mathcal{R}(X)$ denotes the space generated by the columns of X, and $\mathcal{N}(X)$ denotes the kernel of X. An orthogonal complement of X is denoted by $X_\perp$, which is a matrix containing by columns a basis for $\mathcal{N}(X)$, i.e. a matrix of maximal rank such that $XX_\perp = 0$. $X^\dagger$ denotes the (Moore-Penrose) pseudo-inverse of X.

Ellipsoids. Ellipsoids will be described as

$$\mathcal{E} = \{x : x = \hat{x} + Ez, \|z\| \leq 1\},$$

where $\hat{x} \in \mathbb{R}^n$ is the center, and $E \in \mathbb{R}^{n,m}$, $\mathrm{Rank}(E) = m \leq n$ is the *shape matrix* of the ellipsoid. This representation can handle all bounded ellipsoids, including "flat" ellipsoids, such as points or intervals. An alternative description involves the squared shape matrix $P = EE^T$

$$\mathcal{E}(P, \hat{x}) = \left\{ x : \begin{bmatrix} P & (x-\hat{x}) \\ (x-\hat{x})^T & 1 \end{bmatrix} \succeq 0 \right\}.$$

When $P \succ 0$, the previous expression is also equivalent to

$$\mathcal{E}(P, \hat{x}) = \left\{ x : (x-\hat{x})^T P^{-1} (x-\hat{x}) \leq 1 \right\}.$$

The "size" of an ellipsoid is a function of the squared shape matrix P, and will be denoted $f(P)$. Throughout this paper, $f(P)$ will be either $\mathrm{Tr}(P)$, which corresponds to the sum of squares of the semi-axes lengths, or $\log\det(P)$, which is related to the volume.

Uncertainty description. Structured uncertainty is described as follows: $\mathbf{\Delta}$ is a subspace of $\mathbb{R}^{n_p,n_q}$, called the *structure subspace* (for instance, the space of matrices with certain block-diagonal structure). Then, the uncertain matrix Δ is restricted to

$$\Delta \in \mathbf{\Delta}_1 \doteq \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\}.$$

Associated to the structure subspace, we introduce the *scaling subspace* $\mathcal{B}(\mathbf{\Delta})$

$$\mathcal{B}(\mathbf{\Delta}) = \left\{ (S, T, G) \ : \ \forall \Delta \in \mathbf{\Delta}, \ S\Delta = \Delta T, \ G\Delta = -\Delta^T G^T \right\}. \tag{1.1}$$

A structure that frequently arises in practice is the *independent block-diagonal* structure

$$\mathbf{\Delta} = \{\Delta : \ \Delta = \mathrm{diag}\,(\Delta_1, \ldots, \Delta_\ell), \ \Delta_i \in \mathbb{R}^{n_{pi},n_{qi}}\}. \tag{1.2}$$

For this structure, the scaling subspace is constituted of all triples S, T, G with $S = \mathrm{diag}\,(\lambda_1 I_{n_{p1}}, \ldots, \lambda_\ell I_{n_{p\ell}})$, $T = \mathrm{diag}\,(\lambda_1 I_{n_{q1}}, \ldots, \lambda_\ell I_{n_{q\ell}})$, $G = 0$. A particular case of this situation arises for $m = 1$, and it is denoted as the *unstructured uncertainty* case.

Independent scalar uncertainty parameters $\delta_1, \ldots, \delta_\ell$ with bounded magnitude $|\delta_i| \leq 1$ are represented in our framework via the structure subspace

$$\mathbf{\Delta} = \left\{ \Delta : \ \Delta = \mathrm{diag}\,(\delta_1 I_{n_{p1}}, \ldots, \delta_\ell I_{n_{p\ell}}), \ \delta_i \in \mathbb{R} \right\}, \tag{1.3}$$

and the corresponding scaling subspace constituted of all triples S, T, G with $S = T = \mathrm{diag}\,(S_1 \ldots, S_\ell)$, $S_i = S_i^T \in \mathbb{R}^{n_{pi},n_{pi}}$, $G = \mathrm{diag}\,(G_1, \ldots, G_\ell)$, $G_i = -G_i^T \in \mathbb{R}^{n_{pi},n_{pi}}$.

More general uncertainty structures, together with their corresponding scaling spaces, are detailed for instance in [6, 4].

2 Uncertain Linear Equations

Let

$$[A(\Delta)\ \ y(\Delta)] = [A\ \ y] + L\Delta(I - D\Delta)^{-1}[R_A\ \ R_y], \tag{2.4}$$

where $A \in \mathbb{R}^{m,n}$, $y \in \mathbb{R}^m$, $L \in \mathbb{R}^{m,n_p}$, $R_A \in \mathbb{R}^{n_q,n}$, $R_y \in \mathbb{R}^{n_q}$, $D \in \mathbb{R}^{n_q,n_p}$, and $\Delta \in \mathbf{\Delta}_1 \subset \mathbb{R}^{n_p,n_q}$, and let this linear fractional representation (LFR) be well-posed over $\mathbf{\Delta}_1$, meaning that $\det(I - D\Delta) \neq 0$, $\forall \Delta \in \mathbf{\Delta}_1$; see Lemma A.1 for a readily checkable sufficient condition for well-posedness. The representation (2.4) includes as special cases, for instance, interval matrices and additive unstructured uncertainty. It also allows for representation of general rational matrix functions of a vector of uncertain parameters $\delta_1, \dots, \delta_\ell$, see [6, 4] for further details.

Consider the set $\mathcal{X}$ of all the possible solutions to the linear equations $A(\Delta)x = y(\Delta)$, i.e.

$$\mathcal{X} \doteq \{x : \ A(\Delta)x = y(\Delta), \text{ for some } \Delta \in \mathbf{\Delta}_1\}.$$

In the sequel, we provide conditions under which the set $\mathcal{X}$ is contained in a bounded ellipsoid $\mathcal{E}$. Then, we exploit these conditions to determine a minimal (in the sense of the selected size measure) ellipsoid containing the solution set $\mathcal{X}$. The key technique is explained below.

Consider the linear fractional description (2.4), then the equation $A(\Delta)x = y(\Delta)$ is rewritten as

$$Ax - y + L\Delta(I - D\Delta)^{-1}(R_A x - R_y) = 0,$$

which in turn can be expressed using a slack vector p in the form

$$Ax - y + Lp = 0 \tag{2.5}$$
$$R_A x + Dp - R_y = q \tag{2.6}$$
$$p = \Delta q. \tag{2.7}$$

Define

$$\Psi \doteq [A\ \ L\ \ y], \tag{2.8}$$
$$\xi \doteq [x^T\ \ p^T\ \ -1]^T, \tag{2.9}$$

then all vectors ξ compatible with (2.5) must be orthogonal to Ψ, and can be expressed as

$$\xi = \Psi_\perp \eta, \text{ with } \eta \doteq \begin{bmatrix} \nu \\ 1 \end{bmatrix}, \ \Psi_\perp \doteq \left[\begin{array}{c|c} \Psi_{\perp 1} & \psi_{\perp 2} \\ \hline 0 \cdots 0 & -1 \end{array}\right], \tag{2.10}$$

where $\Psi_{\perp 1}$ is an orthogonal complement of $[A\ L]$, and $\psi_{\perp 2}$ is any vector such that $[A\ L]\psi_{\perp 2} = y$. Notice that if no such $\psi_{\perp 2}$ exists, then the solution set $\mathcal{X}$ is clearly empty. All feasible ξ must therefore lie on the flat

$$\mathcal{F} \doteq \{\xi : \ \xi = \Psi_\perp \eta, \text{ with } \eta = \begin{bmatrix} \nu^T & 1 \end{bmatrix}^T\},$$

and the corresponding feasible x on the projection $\mathcal{F}_x \doteq \{x = [I_n\ \ 0_{n,n_p}\ \ 0_{n,1}]\xi : \ \ \xi \in \mathcal{F}\}$. The feasible ξ are further constrained by (2.6)–(2.7): By Lemma A.4, for any triple $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, the set of all pairs (q, p): $p = \Delta q$ for some $\Delta \in \mathbf{\Delta}_1$ is bounded by the set

$$\mathcal{Q}_{S,T,G} \doteq \left\{ \begin{bmatrix} q \\ p \end{bmatrix} : \ \begin{bmatrix} q \\ p \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \geq 0 \right\}. \tag{2.11}$$

Therefore, the set of ξ compatible with (2.6)–(2.7) is bounded by the set

$$\mathcal{H}_{S,T,G} \doteq \{\xi : \ \xi^T \Omega(S,T,G)\xi \geq 0\}, \tag{2.12}$$

where

$$\Omega(S,T,G) \doteq \Upsilon^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \Upsilon, \tag{2.13}$$

$$\Upsilon \doteq \begin{bmatrix} R_A & D & R_y \\ 0_{n_p,n} & I_{n_p} & 0_{n_p,1} \end{bmatrix}. \tag{2.14}$$

To conclude, the set of ξ compatible with all conditions (2.5)–(2.7) is bounded by the intersection $\mathcal{F} \cap \mathcal{H}_{S,T,G}$, and therefore $\mathcal{X} \subseteq \overline{\mathcal{X}}_{S,T,G}$, where $\overline{\mathcal{X}}_{S,T,G}$ is the projection

$$\overline{\mathcal{X}}_{S,T,G} = \left\{x = [I_n \ 0 \ 0]\Psi_\perp \eta : \ \eta^T \Psi_\perp^T \Omega(S,T,G)\Psi_\perp \eta \geq 0\right\}, \tag{2.15}$$

with η and $\Psi_\perp$ defined in (2.10).

To make the point clear, we remark that for any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, $\overline{\mathcal{X}}_{S,T,G}$ provides an outer approximation for the solution set $\mathcal{X}$. In particular, when $\mathbf{\Delta}$ is a full block (unstructured uncertainty) the embedding in Lemma A.4 is tight, and the approximation is exact, i.e. $\overline{\mathcal{X}}_{S,T,G} = \mathcal{X}$.

We now state the following theorem, which characterizes a bounding ellipsoid for the solution set $\mathcal{X}$.

Theorem 2.1. *If the there exist $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$ such that*

$$\left[\begin{array}{c|c} P & [I \ \ 0 \ \ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T [I \ \ 0 \ \ \hat{x}]^T & \Psi_\perp^T \left(\mathrm{diag}\,(0,0,1) - \Omega(S,T,G)\right)\Psi_\perp \end{array}\right] \succeq 0 \tag{2.16}$$

is feasible, then the ellipsoid $\mathcal{E}(P,\hat{x})$ contains the solution set $\mathcal{X}$.

Solving the convex optimization problem in the variables $P, \hat{x}, S, T, G$

$$\textit{minimize } f(P) \textit{ subject to} \tag{2.17}$$

$$(S,T,G) \in \mathcal{B}(\mathbf{\Delta}), \ S \succeq 0, T \succeq 0, (2.16) \tag{2.18}$$

yields an outer ellipsoidal approximation of $\mathcal{X}$, that is optimal in the sense of the sufficient condition (2.16).

Proof of Theorem 2.1.

1. For any triple $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, the condition $\mathcal{E}(P,\hat{x}) \supseteq \overline{\mathcal{X}}_{S,T,G}$ obviously implies $\mathcal{E}(P,\hat{x}) \supseteq \mathcal{X}$.

2. The family of ellipsoids $\mathcal{E}(P,\hat{x})$ that lie in $\mathcal{F}_x$ satisfy the flatness condition $(I - P^\dagger P)(x - \hat{x}) = 0$, $\forall x \in \mathcal{F}_x$, which can be expressed using the notation introduced previously, as $(I - P^\dagger P)[I_n \ 0 \ \hat{x}]\Psi_\perp \eta = 0$, $\forall \eta$, i.e.

$$(I - P^\dagger P)[I_n \ 0 \ \hat{x}]\Psi_\perp = 0. \tag{2.19}$$

3. An ellipsoid $\mathcal{E}(P, \hat{x}) \subset \mathcal{F}_x$ contains the point $x = [I_n\ 0\ 0]\Psi_\perp \eta \in \mathcal{F}_x$ if and only if (notice that $x - \hat{x} = [I_n\ 0\ \hat{x}]\Psi_\perp \eta$)
$$\left[\begin{array}{cc} P & [I_n\ 0\ \hat{x}]\Psi_\perp \eta \\ * & 1 \end{array}\right] \succeq 0. \tag{2.20}$$
Using Lemma A.5, this is rewritten as
$$\begin{aligned} 1 - \eta^T \Psi_\perp^T [I_n\ 0\ \hat{x}]^T P^\dagger [I_n\ 0\ \hat{x}]\Psi_\perp \eta \ &\geq \ 0 &(2.21)\\ (I - P^\dagger P)[I_n\ 0\ \hat{x}]\Psi_\perp \eta \ &= \ 0. &(2.22)\end{aligned}$$
Since (2.19) holds for all ellipsoids that lie entirely in $\mathcal{F}_x$, condition (2.22) is always satisfied, therefore the ellipsoid $\mathcal{E}(P, \hat{x}) \subset \mathcal{F}_x$ contains the point $x = [I_n\ 0\ 0]\Psi_\perp \eta \in \mathcal{F}_x$ if and only if (2.21) is satisfied.

4. The ellipsoid $\mathcal{E}(P, \hat{x})$ lies in $\mathcal{F}_x$ and contains $\overline{\mathcal{X}}_{S,T,G}$ if and only if (2.19) holds, and (2.21) is satisfied for all η such that $\eta^T \Psi_\perp^T \Omega(S, T, G)\Psi_\perp \eta \geq 0$. By the $\mathcal{S}$-procedure and homogenization (see Lemma A.2 and Lemma A.3), the above happens if (2.19) holds, and there exist $\tau \geq 0$ such that
$$\Psi_\perp^T \left(\mathrm{diag}\,(0, 0, 1) - [I\ \ 0\ \ \hat{x}]^T P^{-1} [I\ \ 0\ \ \hat{x}]\right) \Psi_\perp \succeq \tau \Psi_\perp^T \Omega(S, T, G)\Psi_\perp.$$
Using the Schur complement rule, the two previous conditions are written in the equivalent matrix inequality form as
$$\left[\begin{array}{c|c} P & [I\ \ 0\ \ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T [I\ \ 0\ \ \hat{x}]^T & \Psi_\perp^T \left(\mathrm{diag}\,(0, 0, 1) - \tau \Omega(S, T, G)\right) \Psi_\perp \end{array}\right] \succeq 0. \tag{2.23}$$
Further, from Lemma A.3, we have that (2.23) is also a necessary condition for the inclusion, if there exist η_0: $\eta_0^T \Psi_\perp^T \Omega(S, T, G)\Psi_\perp \eta_0 > 0$.

In synthesis, if there exist $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$, such that (2.23) is satisfied (notice that τ can be absorbed in the S, T, G variables and then eliminated from the condition), then the ellipsoid $\mathcal{E}(P, \hat{x})$ lies in $\mathcal{F}_x$ and contains $\mathcal{X}$. Moreover, if there exist η_0: $\eta_0^T \Psi_\perp^T \Omega(S, T, G)\Psi_\perp \eta_0 > 0$, (2.23) is also necessary for an ellipsoid $\mathcal{E}(P, \hat{x}) \subset \mathcal{F}_x$ to include $\overline{\mathcal{X}}_{S,T,G}$.

Based on the condition (2.23), we can then minimize a (convex) size measure $f(P)$ of the bounding ellipsoid, which results in the statement (2.17) of the theorem. Notice that this optimization problem is a semidefinite program (SDP), if $f(P) = \mathrm{Tr}(P)$, and a MAXDET problem, if $f(P) = \log\det(P)$. In both cases the problem can be efficiently solved in polynomial-time by interior point methods for convex programming, [10, 11]. $\square$

In the case of unstructured uncertainty, the condition expressed in the above theorem becomes necessary and sufficient, as detailed in the following corollary.

Corollary 2.1. *Let $\mathbf{\Delta} = \mathbb{R}^{n_p, n_q}$, and assume there exists η_0 such that*
$$\eta_0^T \Psi_\perp^T \Upsilon^T \left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right] \Upsilon \Psi_\perp \eta_0 > 0. \tag{2.24}$$

Then the ellipsoid $\mathcal{E}(P, \hat{x})$ lies in $\mathcal{F}_x$ and contains the solution set $\mathcal{X}$ if and only if there exists $\tau \geq 0$ such that

$$\left[\begin{array}{c|c} P & [I\ \ 0\ \ \hat{x}]\Psi_\perp \\ \hline \Psi_\perp^T[I\ \ 0\ \ \hat{x}]^T & \Psi_\perp^T \left(\mathrm{diag}\,(0,0,1) - \tau\Upsilon^T \left[\begin{array}{cc} I & 0 \\ 0 & -I \end{array}\right] \Upsilon\right) \Psi_\perp \end{array}\right] \succeq 0. \tag{2.25}$$

Minimizing the ellipsoid size $f(P)$ under the above constraint then yields the *optimal ellipsoid containing $\mathcal{X}$.*

2.1 Decoupled ellipsoid equations

In this section, we build upon the LMI condition given in Theorem 2.1 and derive decoupled conditions for the optimal ellipsoid, in terms of its shape matrix P and center $\hat{x}$ separately. These decoupled conditions yield further insight into the problem and permit to obtain explicit results in the case of unstructured uncertainty. A first result is stated in the following corollary.

Corollary 2.2. *Let*

$$\begin{aligned} Q(S,T,G) = \left[\begin{array}{c|c} Q_{11} & q_{12} \\ \hline q_{12}^T & 1 - q_{22} \end{array}\right] &\doteq \Psi_\perp^T \left(\mathrm{diag}\,(0,0,1) - \Omega(S,T,G)\right) \Psi_\perp, && (2.26) \\ B &\doteq [I_n\ 0]\Psi_{\perp 1}. && (2.27) \end{aligned}$$

Consider the optimization problem in the variables $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$

$$\begin{aligned} &\textit{minimize } f(BQ_{11}^\dagger B^T) \textit{ subject to:} && (2.28) \\ &S \succeq 0, T \succeq 0, && (2.29) \\ &Q(S,T,G) \succeq 0, && (2.30) \\ &(I - Q_{11}^\dagger Q_{11})B^T = 0. && (2.31) \end{aligned}$$

If the above problem is feasible, then there exist a bounded ellipsoid that contains $\mathcal{X}$. In this case, calling $S_{opt}, T_{opt}, G_{opt}$ the optimal values of the problem variables, the ellipsoid $\mathcal{E}(P_{opt}, \hat{x}_{opt})$ with

$$\begin{aligned} P_{opt} &= BQ_{11}^\dagger(S_{opt}, T_{opt}, G_{opt})B^T && (2.32) \\ \hat{x}_{opt} &= [I_n\ 0]\psi_{\perp 2} - BQ_{11}^\dagger(S_{opt}, T_{opt}, G_{opt})Q_{12} && (2.33) \end{aligned}$$

is an outer ellipsoidal approximation of $\mathcal{X}$, that is optimal in the sense of the sufficient condition (2.16). This solution is equivalent to the one provided by Theorem 2.1.

Proof. With the position (2.10), let us define the following partitions

$$\begin{aligned} \left[\begin{array}{c|c} [I_n\ 0] & \hat{x} \end{array}\right] \Psi_\perp &\doteq \left[\begin{array}{c|c} B & z \end{array}\right] && (2.34) \\ \Psi_\perp^T \left(\mathrm{diag}\,(0,0,1) - \Omega(S,T,G)\right) \Psi_\perp &\doteq Q = \left[\begin{array}{c|c} Q_{11} & q_{12} \\ \hline q_{12}^T & 1 - q_{22} \end{array}\right], && (2.35) \end{aligned}$$

where $B = [I_n\ 0]\Psi_{\perp 1}$, $z = [I_n\ 0]\psi_{\perp 2} - \hat{x}$. Notice that $\Psi_\perp^T \mathrm{diag}\,(0,0,1)\Psi_\perp = \mathrm{diag}\,(0,0,1)$. Then, condition (2.16) is equivalent to the following condition, obtained by simple reordering of the blocks (dependence on S, T, G is sometimes omitted to avoid clutter)

$$\begin{bmatrix} P & z & B \\ z^T & 1 - q_{22} & q_{12}^T \\ B^T & q_{12} & Q_{11} \end{bmatrix} \succeq 0. \tag{2.36}$$

Now, by Lemma A.6 the above is feasible for some P, z if and only if

$$Q(S,T,G) \succeq 0, \ \begin{bmatrix} P & B \\ B^T & Q_{11} \end{bmatrix} \succeq 0 \tag{2.37}$$

is feasible for some P. Therefore problem (2.17) is equivalent to

$$\min_{S,T,G} \min_{P} f(P) \text{ subject to (2.37)}, \tag{2.38}$$

$$(S,T,G) \in \mathcal{B}(\mathbf{\Delta}),\ S \succeq 0, T \succeq 0, \tag{2.39}$$

which, by Corollary A.1, is equivalent to

$$\min_{S,T,G} f(\bar{X}(S,T,G)) \text{ subject to} \tag{2.40}$$

$$(S,T,G) \in \mathcal{B}(\mathbf{\Delta}),\ S \succeq 0, T \succeq 0, \tag{2.41}$$

$$Q(S,T,G) \succeq 0, \tag{2.42}$$

$$(I - Q_{11}^\dagger Q_{11})B^T = 0, \tag{2.43}$$

where $\bar{X}(S,T,G) = BQ_{11}^\dagger(S,T,G)B^T$.

If $S_{opt}, T_{opt}, G_{opt}$ are the optimal values of the above optimization problem, then (again by Corollary A.1) the optimal ellipsoid is given by

$$P_{opt} = BQ_{11}^\dagger(S_{opt}, T_{opt}, G_{opt})B^T \tag{2.44}$$

$$z_{opt} = BQ_{11}^\dagger(S_{opt}, T_{opt}, G_{opt})Q_{12}. \tag{2.45}$$

From the latter we then retrieve the ellipsoid center as

$$\hat{x}_{opt} = [I_n\ 0]\psi_{\perp 2} - z_{opt}. \tag{2.46}$$

Remark 2.1 (Boundedness). From Corollary 2.2 we immediately obtain a readily checkable sufficient condition for the solution set of uncertain linear equations to be bounded: If there exist $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$ such that (2.29)–(2.31) are satisfied, then the solution set $\mathcal{X}$ is bounded. These conditions become also necessary, under the hypotheses of Corollary 2.1.

Remark 2.2 (Emptiness and uniqueness). A preliminary analysis of (2.5) through (2.10) shows that a necessary condition in order to have (at least) a solution is that $y \in \mathcal{R}([A\ L])$. Notice also that if $\mathcal{N}([A\ L])$ is empty, then the uncertain linear equations may have at most one solution. In this case, the solution of the optimization problems in Theorem 2.1 and Corollary 2.2 would yield an ellipsoid reduced to a point, i.e. $P_{opt} = 0$.

Without need to solve any optimization problem, we may therefore conclude that:

$$\begin{array}{rcl} \text{if } y \notin \mathcal{R}([A\ L]) & \Rightarrow & \mathcal{X} \text{ is empty;} \\ \text{if } \mathcal{N}([A\ L]) = 0 & \Rightarrow & \mathcal{X} \text{ is either empty or reduced to a point.} \end{array}$$

In the latter case, if $y \notin \mathcal{R}([A\ L]$ then $\mathcal{X}$ is certainly empty, otherwise the only candidate solution is of the form $\hat{x} = [I_n\ 0]\psi_{\perp 2}$, with $\psi_{\perp 2} \doteq [\hat{x}^T\ \hat{p}^T]^T$. To check if this is actually a solution, we can in some cases proceed by direct inspection. For instance, let $\hat{q} = R_A\hat{x} + D\hat{p} - R_y$, then in the case of unstructured uncertainty $\hat{x}$ is the unique solution if and only if $\hat{p}^T\hat{p} \le \hat{q}^T\hat{q}$.

2.2 Special case: unstructured uncertainty

In the unstructured uncertainty case, we have $S = \lambda I, T = \lambda I, G = 0$. The matrices $Q_{11}(\lambda), q_{12}(\lambda), q_{22}(\lambda)$ are linear in λ, and it is convenient to express them as $Q_{11}(\lambda) = \lambda\bar{Q}_{11}$, $q_{12}(\lambda) = \lambda\bar{q}_{12}$, $q_{22}(\lambda) = \lambda\bar{q}_{22}$, with

$$\begin{aligned} \bar{Q}_{11} &= \Psi_{\perp 1}^T([0\ I]^T[0\ I] - [R_A\ D]^T[R_A\ D])\Psi_{\perp 1}, && (2.47) \\ \bar{q}_{12} &= \Psi_{\perp 1}^T[R_A\ D]^T R_y + \Psi_{\perp 1}^T([0\ I]^T[0\ I] - [R_A\ D]^T[R_A\ D])\psi_{\perp 2}, && (2.48) \\ \bar{q}_{22} &= R_y^T R_y - 2\psi_{\perp 2}^T[R_A\ D]^T R_y - \psi_{\perp 2}^T([0\ I]^T[0\ I] - [R_A\ D]^T[R_A\ D])\psi_{\perp 2}. && (2.49) \end{aligned}$$

The optimal ellipsoid containing the solution set is in this case computable in closed form, as detailed in the following corollary, whose proof is omitted for brevity.

Corollary 2.3. *Let $\mathbf{\Delta} = \mathbb{R}^{n_p, n_q}$, $B \doteq [I_n\ 0]\Psi_{\perp 1}$, and assume that $y \in \mathcal{R}([A\ L])$ (if this condition is not satisfied, the solution set is empty).*

Then, the solution set $\mathcal{X}$ is bounded if

$$\begin{aligned} \bar{Q}_{11} &\succeq 0, && (2.50) \\ (I - \bar{Q}_{11}^\dagger\bar{Q}_{11})B &= 0, && (2.51) \\ (I - \bar{Q}_{11}^\dagger\bar{Q}_{11})\bar{q}_{12} &= 0. && (2.52) \end{aligned}$$

The above conditions are also necessary, if there exists η_0 such that

$$\eta_0^T \begin{bmatrix} -\bar{Q}_{11} & \bar{q}_{12} \\ \bar{q}_{12}^T & \bar{q}_{22} \end{bmatrix} \eta_0 > 0. \tag{2.53}$$

When (2.50)–(2.52) are satisfied, the optimal ellipsoid containing $\mathcal{X}$ is given by

$$\begin{aligned} P_{opt} &= \frac{1}{\lambda_{opt}} B\bar{Q}_{11}^\dagger B^T && (2.54) \\ \hat{x}_{opt} &= [I_n\ 0]\psi_{\perp 2} - B\bar{Q}_{11}^\dagger\bar{q}_{12}, && (2.55) \end{aligned}$$

with

$$\frac{1}{\lambda_{opt}} = \max\{\bar{q}_{22} + \bar{q}_{12}^T\bar{Q}_{11}^\dagger\bar{q}_{12}, 0\}.$$

When $P_{opt} = 0$ then the solution set contains at most one point. In particular, if $\bar{q}_{22} \ge 0$, then $\mathcal{X} = \{[I_n\ 0]\psi_{\perp 2}\}$, otherwise $\mathcal{X}$ is empty.

2.2.1 Additive uncertainty

As a special case of the unstructured uncertainty situation above, we consider a classical problem in linear algebra, where the data A, y are affected by additive uncertainty

$$[A(\Delta)\ y(\Delta)] = [A\ y] + L\Delta[R_A\ R_y],$$

with $L = \rho I_m$, $\rho > 0$, $[R_A\ R_y] = I_{n+1}$, $\Delta \in \mathbb{R}^{m,n+1}$, $\|\Delta\| \leq 1$. In this case, we may choose the orthogonal complements as

$$\Psi_{\perp 1} = \begin{bmatrix} \rho I_n \\ -A \end{bmatrix};\ \psi_{\perp 2} = \begin{bmatrix} 0 \\ y/\rho \end{bmatrix},$$

and therefore $\bar{Q}_{11} = A^T A - \rho^2 I$, $\bar{q}_{12} = -A^T y/\rho$, $\bar{q}_{22} = 1 - y^T y/\rho^2$. Condition (2.53) is then satisfied if and only if $\rho^2 > \lambda_{\min}\{[A\ y]^T[A\ y]\}$. In this case, the solution set is bounded if and only if $\bar{Q}_{11} \succ 0$, i.e. for $\rho^2 < \lambda_{\min}\{A^T A\}$. On the other hand, if $\rho^2 < \lambda_{\min}\{[A\ y]^T[A\ y]\}$ then $\rho^2 < \lambda_{\min}\{A^T A\}$ and $\bar{q}_{22} < 0$,[2] therefore the solution set is empty. Lastly, we consider the situation when $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\}$. There are two possibilities: i) $\lambda_{\min}\{[A\ y]^T[A\ y]\} < \lambda_{\min}\{A^T A\}$, then $\mathcal{X}$ is bounded (and in particular, $\mathcal{X}$ is empty if $\bar{q}_{22} < 0$ and it is a singleton if $\bar{q}_{22} = 0$); ii) $\lambda_{\min}\{[A\ y]^T[A\ y]\} = \lambda_{\min}\{A^T A\}$, then $\mathcal{X}$ is empty if $\bar{q}_{22} < 0$, and it is unbounded if $\bar{q}_{22} = 0$. We may resume these results as follows.

- If $\lambda_{\min}\{[A\ y]^T[A\ y]\} < \rho^2 < \lambda_{\min}\{A^T A\}$, then the optimal bounding ellipsoid for $\mathcal{X}$ is given by

$$\begin{aligned} P_{opt} &= \alpha(A^T A - \rho^2 I)^{-1} \\ \hat{x}_{opt} &= (A^T A - \rho^2 I)^{-1} A^T y, \end{aligned}$$

  with $\alpha \doteq \rho^2 - y^T y + y^T A(A^T A - \rho^2 I)^{-1} A^T y$.

- If $\rho^2 < \lambda_{\min}\{[A\ y]^T[A\ y]\}$, then the solution set is empty.

- If $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\} < \lambda_{\min}\{A^T A\}$, then the solution set is empty if $\rho^2 < y^T y$, and it is the singleton $\mathcal{X} = \{\hat{x}_{opt}\}$ if $\rho^2 = y^T y$.

- If $\rho^2 = \lambda_{\min}\{[A\ y]^T[A\ y]\} = \lambda_{\min}\{A^T A\}$, then the solution set is empty if $\rho^2 < y^T y$, and it is unbounded if $\rho^2 = y^T y$.

- If $\rho^2 \geq \lambda_{\min}\{A^T A\}$, then the solution set is unbounded (with the exception of the previous particular case).

We notice that the center of the above ellipsoid is closely related to the well-known Total Least Squares solution to the uncertain equations, see [4] for further details.

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[2] This is since $\lambda_{\min}\{[A\ y]^T[A\ y]\} \leq \lambda_{\min}\{A^T A\}$, and $\begin{bmatrix} -\bar{Q}_{11} & \bar{q}_{12} \\ \bar{q}_{12}^T & \bar{q}_{22} \end{bmatrix} \preceq 0$ if and only if $\rho^2 \leq \lambda_{\min}\{[A\ y]^T[A\ y]\}$.

3 Numerical Examples

Consider the data

$$A(\Delta) = I_2 + 0.2\delta_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 0.5\delta_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \;\; y = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

with $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. Here, the matrix $A(\Delta)$ is the identity, plus two additive perturbations. The uncertain data can be expressed in LFR format as

$$[A(\Delta) \mid y(\Delta)] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] + L\Delta[R_A \mid R_y], \tag{3.56}$$

$$L = \begin{bmatrix} 0.2 & 0 & 0 & 0.5 \\ 0 & -0.2 & -0.5 & 0 \end{bmatrix}, \;\; R_A = [I_2 \;\; I_2]^T, \;\; R_y = 0,$$

$\Delta = \mathrm{diag}\,(\delta_1 I_2, \delta_2 I_2)$, with $|\delta_1| \leq 1$, $|\delta_2| \leq 1$. The scaling subspace is in this case given by the set of triples (S, T, G) with $S = T = \mathrm{diag}\,(S_1, S_2)$, with $S_1, S_2 \in \mathbb{R}^{2,2}$ symmetric, and $G = \mathrm{diag}\,(G_1, G_2)$, with $G_1, G_2 \in \mathbb{R}^{2,2}$ skew-symmetric.

To have an approximate idea of the shape of the solution set $\mathcal{X}$, we randomly generated a number of samples of δ_1, δ_2, and solved the corresponding linear equations. The points obtained are shown in Figure 1 (Notice that the solution set of this ULE is *not* convex), together with the optimal bounding ellipsoid, determined by the solution of the convex problem in Theorem 2.1, having parameters

$$\hat{x} = \begin{bmatrix} 0.859 \\ 0.859 \end{bmatrix}; \;\; P = \begin{bmatrix} 0.462 & -0.246 \\ -0.246 & 0.462 \end{bmatrix}.$$

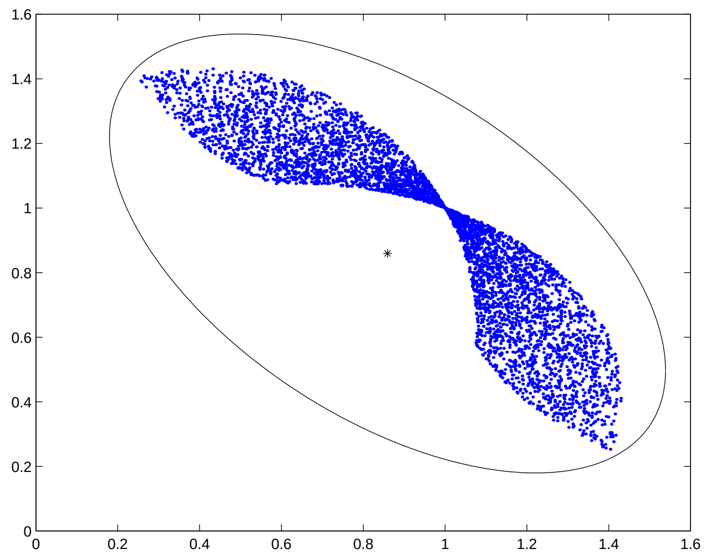

Figure 1: Solution set and bounding ellipsoid for the ULE resulting from the data in (3.56), and structured uncertainty $\Delta = \mathrm{diag}\,(\delta_1 I_2, \delta_2 I_2)$.

As a second example, consider again the LFR (3.56), but assume now that the uncertainty matrix Δ is unstructured, i.e. $\Delta \in \mathbb{R}^{4,4}, \|\Delta\| \leq 1$. In this case, applying the results of Corollary 2.3, we obtain

$$\hat{x} = \begin{bmatrix} 2.381 \\ 2.381 \end{bmatrix}; \quad P = \begin{bmatrix} 6.576 & 0 \\ 0 & 6.576 \end{bmatrix}.$$

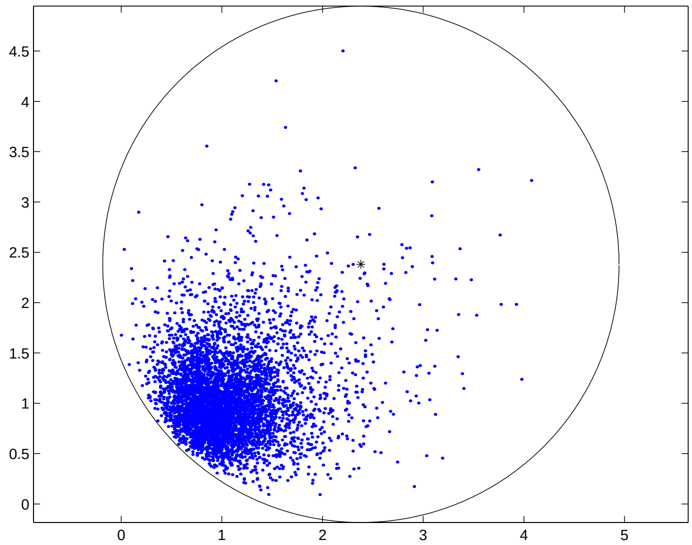

Figure 2: Solution set and bounding ellipsoid for the ULE resulting from the data in (3.56), and unstructured uncertainty $\Delta \in \mathbb{R}^{4,4}, \|\Delta\| \leq 1$.

This ellipsoid is depicted in Figure 2, together with $5,000$ randomly generated solutions in $\mathcal{X}$. We remark that in this case (unstructured uncertainty) the solution set indeed coincides with the bounding ellipsoid computed by means of Corollary 2.3. This fact would become apparent if more solution points (corresponding to random uncertainty samples of Δ) are plotted.

Remark. The "concentration" of the solutions that we observe in Figure 2 may suggest an alternative probabilistic approach for bounding the solution set. In fact, we could seek for a minimal ellipsoid that contains not *all* the possible solutions, but only a given percentage of them. The result would of course depend upon the underlying probability distribution that it is assumed on Δ. This approach is out of the scope of this paper, but seems interesting for further research.

A Appendix

Lemma A.1 (Well-posedness). *The LFR* $M(\Delta) = M + L\Delta(I - D\Delta)^{-1}R$ *is well-posed over* $\mathbf{\Delta}_1$ *if and only if* $\det(I - D\Delta) \neq 0$ *for all* $\Delta \in \mathbf{\Delta}_1$. *A sufficient condition for well-posedness is: there exist a triple* $(S, T, G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succ 0$, $T \succ 0$ *such that*

$$\begin{bmatrix} D \\ I \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} D \\ I \end{bmatrix} \preceq 0.$$

The above condition is also necessary in the unstructured case, i.e. when $\boldsymbol{\Delta} = \mathbb{R}^{n_p,n_q}$.

This lemma was first derived in the context of μ-analysis in [3]. A proof of the results in the form given here may be found in [6].

Lemma A.2 (Homogenization). *Let* $T = T^T$. *The following two conditions are equivalent.*

$$\text{(a)} \quad \begin{bmatrix} \xi \\ 1 \end{bmatrix}^T \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} \geq 0 \textit{ for all } \xi;$$

$$\text{(b)} \quad \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \succeq 0.$$

Proof. *The implication from (b) to (a) is trivial. We show that (a) implies (b) by contradiction. Suppose* $\exists\, \bar{\xi}, \alpha$ *such that* $\begin{bmatrix} \bar{\xi}^T & \alpha \end{bmatrix} \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \bar{\xi}^T & \alpha \end{bmatrix}^T < 0$. *Then, if* $\alpha \neq 0$, *dividing both sides by* α^2, *we get* $\begin{bmatrix} \frac{1}{\alpha}\bar{\xi}^T & 1 \end{bmatrix} \begin{bmatrix} T & u \\ u^T & v \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha}\bar{\xi}^T & 1 \end{bmatrix}^T < 0$, *which clearly contradicts the hypothesis (a). On the other hand,* $\alpha = 0$ *would imply that* $\bar{\xi}^T T \bar{\xi} < 0$. *Choosing then* $\xi = \beta\bar{\xi}$ *and substituting in (a) we have*

$$\beta^2(\bar{\xi}^T T \bar{\xi}) + 2\beta u^T \bar{\xi} + v, \tag{A.57}$$

which is a concave parabola in β, *since* $\bar{\xi}^T T \bar{\xi} < 0$. *Therefore, there will exist a value of* β *such that (A.57) is negative, which contradicts the hypothesis.* $\square$

Lemma A.3 ($\mathcal{S}$-procedure). *Let* $F_0(\xi), F_1(\xi), \ldots, F_p(\xi)$ *be quadratic forms in the variable* $\xi \in \mathbb{R}^n$

$$F_i(\xi) = \xi^T T_i \xi + 2u_i^T \xi + v_i, \; i = 0, \ldots, p,$$

with $T_i = T_i^T$. *Then, the implication*

$$F_1(\xi) \geq 0, \ldots, F_p(\xi) \geq 0 \;\Rightarrow\; F_0(\xi) \geq 0 \tag{A.58}$$

holds if there exist $\tau_1, \ldots, \tau_p \geq 0$ *such that*

$$F_0(\xi) - \sum_{i=1}^{p} \tau_i F_i(\xi) \geq 0, \; \forall \xi. \tag{A.59}$$

When $p = 1$, *condition (A.59) is also necessary for (A.58), provided there exist some* ξ_0 *such that* $F_1(\xi_0) > 0$. *Notice also that, by homogenization, condition (A.59) is equivalent to*

$$\exists \tau_1, \ldots, \tau_p \geq 0 \textit{ such that } \begin{bmatrix} T_0 & u_0 \\ u_0^T & v_0 \end{bmatrix} - \sum_{i=1}^{p} \tau_i \begin{bmatrix} T_i & u_i \\ u_i^T & v_i \end{bmatrix} \succeq 0. \tag{A.60}$$

Lemma A.4 (Quadratic embedding). *Let* $\mathcal{Q} \doteq \left\{ \begin{bmatrix} q^T & p^T \end{bmatrix}^T : \, p = \Delta q \textit{ for some } \Delta \in \boldsymbol{\Delta}_1 \right\}$, *and* $\mathcal{B}(\boldsymbol{\Delta}) = \left\{ (S, T, G) \, : \, \forall \Delta \in \boldsymbol{\Delta}, \; S\Delta = \Delta T, \; G\Delta = -\Delta^T G^T \right\}$. *For any triple* $(S, T, G) \in \mathcal{B}(\boldsymbol{\Delta})$, $S \succeq 0$, $T \succeq 0$, *define the set*

$$\mathcal{Q}_{S,T,G} \doteq \left\{ \begin{bmatrix} q \\ p \end{bmatrix} : \begin{bmatrix} q \\ p \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \geq 0 \right\}. \tag{A.61}$$

Then $\mathcal{Q} \subseteq \mathcal{Q}_{S,T,G}$.

In the case of unstructured uncertainty $\mathbf{\Delta} \equiv \mathbb{R}^{n_p,n_q}$ we have in particular that $\mathcal{Q} \equiv \mathcal{Q}_{S,T,G}$, for $S = \lambda I_{n_p}$, $T = \lambda I_{n_q}$, $\lambda \in \mathbb{R}$, and $G = 0$, $\lambda > 0$.

Proof. Let $[q^T \; p^T]^T \in \mathcal{Q}$, then for any $(S,T,G) \in \mathcal{B}(\mathbf{\Delta})$, $S \succeq 0$, $T \succeq 0$ we have that $q^T G p = q^T G \Delta q = 0$, by the skew-symmetry of $G\Delta$. In addition, we have

$$\begin{aligned} q^T T q - p^T S p &= q^T (T - \Delta^T S \Delta) q \\ &= q^T (T - \Delta^T \Delta T) q \succeq 0. \end{aligned}$$

In the above, we have used the fact that, since $S\Delta = \Delta T$, the matrix $\Delta^T \Delta T$ is symmetric, then T commutes in the product with $\Delta^T \Delta$, and therefore these two matrices are simultaneously diagonalizable ([7], Corollary 4.5.18), i.e. we may write the factorizations $T = V J_T V^T$, $\Delta^T \Delta = V J_\Delta V^T$, where J_T, J_Δ are diagonal, and V is orthogonal. It then follows that the eigenvalues of $T - \Delta^T \Delta T$ are the diagonal terms of $(I - J_\Delta) J_T$, which are non-negative, if $T \succeq 0$. The previous conditions are written compactly as

$$\begin{bmatrix} q \\ p \end{bmatrix}^T \begin{bmatrix} T & G \\ G^T & -S \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \geq 0, \tag{A.62}$$

which proves the first part of the lemma.

To prove the second part of the lemma, we consider the case of unstructured uncertainty, i.e. $\mathbf{\Delta} = \mathbb{R}^{n_p,n_q}$ (only one full block). In this case the set $\mathcal{B}(\mathbf{\Delta})$ reduces to the set of triples (S,T,G), with $S = \lambda I_{n_p}$, $T = \lambda I_{n_q}$, $\lambda \in \mathbb{R}$, and $G = 0$. Clearly, $p = \Delta q$ for some $\Delta : \; \|\Delta\| \leq 1$ if and only if $p^T p \leq q^T q$, which is equivalent to (A.62), for any $\lambda > 0$. □

Lemma A.5 (Schur complements). *The condition*

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \succeq 0$$

is equivalent to

$$D \succeq 0, \; A - B D^\dagger B^T \succeq 0, \; (I - D^\dagger D) B^T = 0$$

and also to

$$A \succeq 0, \; D - B^T A^\dagger B \succeq 0, \; (I - A^\dagger A) B = 0,$$

where $A^\dagger$, $D^\dagger$ denote the Moore-Penrose pseudoinverse of A and D, respectively. Notice that the condition $(I - A^\dagger A)B = 0$ means that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Similarly, the condition $(I - D^\dagger D)B^T = 0$ means that $\mathcal{N}(D) \subseteq \mathcal{N}(B)$ or, equivalently, that $\mathcal{R}(B^T) \subseteq \mathcal{R}(D)$.

Lemma A.6 (Block elimination). *Let $Q_{11} = Q_{11}^T$, $Q_{22} = Q_{22}^T$. There exist matrices $X = X^T$ and Z such that*

$$\begin{bmatrix} X & Z & B \\ Z^T & Q_{11} & Q_{12} \\ B^T & Q_{12}^T & Q_{22} \end{bmatrix} \succeq 0 \tag{A.63}$$

if and only if

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \succeq 0, \ \textit{and} \tag{A.64}$$

$$\exists X = X^T : \begin{bmatrix} X & B \\ B^T & Q_{22} \end{bmatrix} \succeq 0. \tag{A.65}$$

Proof. The implication from (A.63) to (A.64)–(A.65) is straightforward, since if a matrix is positive semi-definite, so are all principal sub-matrices. The converse is proved below.

By Lemma A.5, (A.63) is equivalent to

$$Q_{22} \succeq 0 \tag{A.66}$$

$$\begin{bmatrix} X & Z \\ Z^T & Q_{11} \end{bmatrix} - \begin{bmatrix} B \\ Q_{12} \end{bmatrix} Q_{22}^{\dagger} \begin{bmatrix} B \\ Q_{12} \end{bmatrix}^T \succeq 0 \tag{A.67}$$

$$(I - Q_{22}^{\dagger} Q_{22}) \begin{bmatrix} B \\ Q_{12} \end{bmatrix}^T = 0. \tag{A.68}$$

Clearly, (A.64) implies $(I - Q_{22}^{\dagger} Q_{22}) Q_{12}^T = 0$, and (A.65) implies $(I - Q_{22}^{\dagger} Q_{22}) B^T = 0$, therefore (A.64)–(A.65) imply (A.68). Define now

$$\bar{X} \doteq B Q_{22}^{\dagger} B^T \tag{A.69}$$

$$\bar{Z} \doteq B Q_{22}^{\dagger} Q_{12}^T \tag{A.70}$$

$$\bar{Q}_{11} \doteq Q_{11} - Q_{12} Q_{22}^{\dagger} Q_{12}^T, \tag{A.71}$$

then (A.67) writes

$$\begin{bmatrix} X - \bar{X} & (Z - \bar{Z}) \\ (Z - \bar{Z})^T & \bar{Q}_{11} \end{bmatrix} \succeq 0. \tag{A.72}$$

From (A.64) it follows that $\bar{Q}_{11} \succeq 0$, therefore (A.72) is feasible for $X = \bar{X}$, $Z = \bar{Z}$, which concludes the proof. □

Corollary A.1 (Decoupling). *Let all symbols be defined as in Lemma A.6, and let*

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \succeq 0.$$

Then the problem

$$\min_{X,Z} f(X) \ \textit{subject to (A.63)} \tag{A.73}$$

is equivalent to

$$\min_{X} f(X) \ \textit{subject to (A.65)}. \tag{A.74}$$

Moreover, if problem (A.74) is feasible and $f(\cdot)$ is either the trace function $f(X) = \mathrm{Tr}(X)$, or the log-det function $f(X) = \log\det(X)$, then problem (A.73) has a unique optimal solution given by

$$\bar{X} \doteq B Q_{22}^{\dagger} B^T \tag{A.75}$$

$$\bar{Z} \doteq B Q_{22}^{\dagger} Q_{12}^T \tag{A.76}$$

Proof. When (A.64) holds, we know from Lemma A.6 that (A.63) is feasible if and only if (A.65) is feasible, which immediately proves the equivalence between problems (A.73) and (A.74).

If the latter is feasible, then (A.63) is also feasible, and therefore (A.72) holds (with the symbols defined in (A.69)–(A.71)), which means that

$$\begin{array}{l} X \succeq \bar{X} + (Z - \bar{Z})\bar{Q}_{11}^{\dagger}(Z - \bar{Z})^T \\ (I - \bar{Q}_{11}^{\dagger}\bar{Q}_{11})(Z - \bar{Z})^T = 0. \end{array}$$

Now, since the function $f(\cdot)$ is concave on the cone of positive-semidefinite matrices (both in the case of trace and log-determinant) the minimum of $f(X)$ is achieved for $X = \bar{X}$, $Z = \bar{Z}$. $\square$

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