Controllability of Periodically Switched Linear Systems with Delay in Control [1]

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Abstract: The controllability for switched linear systems with time-delay in control are first formulated and investigated. A sufficient and necessary condition for controllability of periodically switched linear systems is presented. Furthermore, it is proved that the controllability can be realized in $n+1$ periods at most. An example illustrates the above results. Some further results are also presented.

Keywords: switched linear systems, periodic, time-delay, controllability.

1 Introduction

Switched linear systems are an important class of hybrid dynamical systems which consist of a family of linear time-invariant systems and a switching law specifying the switching between them. In recent years, there has been increasing interest in the control problems of switched systems due to their significance both in theory and applications.

In the analysis and design of switched systems, controllability and reachability are two important issues that have been addressed in several references. Studies for the controllability, observability and stability for periodically switched linear systems can be found in [1], and some sufficient conditions and necessary conditions are given. In [3], a necessary and a sufficient condition are presented for reachability. [4] strictly defines controllability, reachability, controllable set and reachable set for general switched systems. The reachability of second-order switched linear systems is discussed in [5]. On the basis of [1], [6] presents a necessary and sufficient geometric condition for multiple-periodic controllability of periodically switched linear systems and points out that the controllability can be realized in n periods at most.

Time-delay phenomena are very common in practical systems, for instance, economic, biological and physiological systems. The controllability for linear time-invariant systems with time-delay in the control is studied in [2]. But for switched linear systems, almost all of the known results have not considered time-delay. In this paper, the controllability for switched linear systems with time-delay in control is first formulated and investigated. A sufficient and necessary condition for controllability of periodically switched linear systems is derived.

This paper is organized as follows. Section 2 formulates the problem and presents the preliminary results. Section 3 defines 1-periodic controllability and m-periodic controllability and presents the sufficient and necessary conditions. In section 4, an example is given to illustrate the results. Section 5 presents some further results. Section 6 concludes the whole paper.

2 Preliminaries

Consider a switched linear system with time-delay in the control function given by

$$\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t) + D_{r(t)}u(t-\tau) \tag{1}$$

where $x(t) \in \Re^n$ is the state, $u(t) \in \Re^p$ is the input, the piecewise constant scalar function $r(t) : \Re^+ \to \{1, 2, \cdots, N\}$ is the switching law, and $\{(A_i, B_i, D_i)|i = 1, \cdots, N\}$ is a finite family of system realizations. Moreover, $r(t) = i$ implies that (A_i, B_i, D_i) is chosen as the system realization at time t. $\tau > 0$ is the fixed time delay in control.

For system (1), a switching law is to specify when and to which system realization one should switch at each instant of time.

Definition 1 (Switching Sequence). *For system (1), a switching sequence π is a set with finite pairs*

$$\pi \overset{def}{=} \{(i_1, h_1), \cdots, (i_M, h_M)\} \tag{2}$$

where $M < \infty$ is the length of π, $i_m \in \{1, \cdots, N\}$ is the index of the m*th realization $(A_{i_m}, B_{i_m}, D_{i_m})$, and $h_m > \tau$ is the time interval of $(A_{i_m}, B_{i_m}, D_{i_m})$, for $m = 1, \cdots, M$.*

Given initial time t_0 and switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, an associated switching law $r(t)$ can be determined as

$$r(t) = i_m, \text{if } t \in [t_0 + \sum_{l=1}^{m-1} h_l, t_0 + \sum_{l=1}^{m} h_l) \tag{3}$$

for $m = 1, \cdots, M$.

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Remark 1. *Here we assume that $h_m > \tau$ in the definition of switching sequence in order to avoid unnecessary complexity.*

If $r(t)$ is restricted to be a periodic function, we get *periodically switched systems*. Without loss of generality, we just take the switching sequence $\pi = \{(1, h_1), \cdots, (n, h_n), \cdots, (N, h_N)\}$ as the period of system (1).

Now, we introduce some mathematical preliminaries as the basic tools for the discussion in the following parts of the paper.

Definition 2 (Column Space). *Given a matrix $B \in \Re^{n\times m}$, the column space $\mathcal{R}(B)$ is defined as*

$$\mathcal{R}(B) \overset{def}{=} \{By | y \in \Re^m\} \tag{4}$$

Definition 3 (Generalized Invariant Subspace). *Given matrices $A_1, \cdots, A_N \in \Re^{n\times n}$ and $B_1, \cdots, B_N \in \Re^{n\times p}$, the generalized invariant subspace $\langle A_1|B_1 + \cdots + A_N|B_N\rangle$ is defined as*

$$\langle A_1|B_1 + \cdots + A_N|B_N\rangle \overset{def}{=} \sum_{i=0}^{\infty} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N) \tag{5}$$

Especially,

$$\langle A_1|B_1\rangle \overset{def}{=} \sum\nolimits_{i=0}^{\infty} \mathcal{R}(A_1^i B_1) \tag{6}$$

Lemma 1. *Given matrices $A_1, \cdots, A_N \in \Re^{n\times n}$ and $B_1, \cdots, B_N \in \Re^{n\times p}$,*

$$\langle A_1|B_1 + \cdots + A_N|B_N\rangle = \sum_{i=0}^{Nn-1} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N) \tag{7}$$

Especially,

$$\langle A_1|B_1\rangle = \sum\nolimits_{i=0}^{n-1} \mathcal{R}(A_1^i B_1) \tag{8}$$

Proof. See Appendix A. □

Remark 2. *The subspaces $\langle A_1|B_1 + \cdots + A_N|B_N\rangle$ and $\langle A_1|B_1\rangle + \cdots + \langle A_N|B_N\rangle$ are quite different. It is easy to see that $\langle A_1|B_1 + \cdots + A_N|B_N\rangle \subseteq \langle A_1|B_1\rangle + \cdots + \langle A_N|B_N\rangle$.*

The following lemma is very basic, but it is the starting point to discuss the conditions for system controllability. For simplicity, let *p.c.* denote *piecewise continuous*.

Lemma 2. *Given matrices $A_1, \cdots, A_N \in \Re^{n\times n}$ and $B_1, \cdots, B_N \in \Re^{n\times p}$, for any $0 \le t_0 < t_f < +\infty$, we have*

$$\begin{aligned}&\{x | x = \sum_{m=1}^{N} \int_{t_0}^{t_f} \exp(A_m(t_f - s))B_m u(s)ds, \forall\ p.c.\ u\} \\ &= \langle A_1|B_1 + \cdots + A_N|B_N\rangle\end{aligned} \tag{9}$$

Especially,

$$\{x | x = \int_{t_0}^{t_f} e^{A_1(t_f - s)} B_1 u(s)ds, \forall\ p.c.\ u\} = \langle A_1|B_1\rangle \tag{10}$$

Proof. See Appendix B. □

Lemma 3. *Given matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$ and nonsingular matrix $P \in \Re^{p\times p}$, we have*

$$\langle A|BP\rangle = \langle A|B\rangle \tag{11}$$

Proof. See Appendix C. □

Lemma 4. *Given matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$, for any constant $h \in \Re$, we have*

$$\exp(Ah)\langle A|B\rangle = \langle A|B\rangle \tag{12}$$

$$\langle A|\exp(Ah)B\rangle = \langle A|B\rangle \tag{13}$$

Proof. See Appendix D. □

Lemma 5 (Separation Lemma). *Given matrices $A_1, A_2 \in \Re^{n\times n}$, $B_1, B_2 \in \Re^{n\times p}$, we have*

$$\langle A_1|B_1 + A_2|B_2\rangle + \langle A_2|B_2\rangle = \langle A_1|B_1\rangle + \langle A_2|B_2\rangle \tag{14}$$

Proof. See Appendix E. □

Remark 3. *The separation lemma can be extended to the multiple case, i.e.,*

$$\begin{aligned}&\langle A_1|B_1 + \cdots + A_N|B_N + C_1|D_1 + \cdots + C_M|D_M\rangle \\ &+\langle C_1|D_1 + \cdots + C_M|D_M\rangle \\ &= \langle A_1|B_1 + \cdots + A_N|B_N\rangle + \langle C_1|D_1 + \cdots + C_M|D_M\rangle\end{aligned} \tag{15}$$

Lemma 6. *Given matrix $A \in \Re^{n\times n}$, for almost all $T > 0$ and any linear subspace $\mathcal{W} \subseteq \Re^n$, we have*

$$\langle A|\mathcal{W}\rangle = \langle \exp(AT)|\mathcal{W}\rangle \tag{16}$$

Proof. See Appendix F. □

3 Main Results

In this section, we establish the controllability criteria for periodically switched linear systems.

3.1 1-periodic Controllability

Definition 4 (1-periodic Controllability). *System (1) is 1-periodic controllable if for any given pair of points x_0, x_f, and initial control function $u_0(t)$, $t \in [t_0 - \tau, t_0]$, there exists a piecewise continuous control function $u(t)$ which steers the state of the system from $x(t_0) = x_0$ to $x(t_N) = x_f$, where $t_N = t_0 + \sum_{l=1}^{N} h_l$.*

For system (1), given initial state x_0 and initial control input $u_0(t)$ on $[t_0 - \tau, t_0]$, let $t_m = t_0 + \sum_{l=1}^{m} h_l$, $m = 1, \cdots, N$, then the terminal state x_f can be expressed as follows

$$\begin{aligned}x_f &= \prod\nolimits_{i=N}^{1} \exp(A_i h_i)x_0 \\ &+ \sum_{i=1}^{N} \prod_{l=N}^{i+1} e^{A_l h_l} \int_{t_{i-1}}^{t_i} e^{A_i(t_i - s)}(B_i u(s) + D_i u(s - \tau))ds\end{aligned} \tag{17}$$

Since

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i-s)\right](B_iu(s)+D_iu(s-\tau))ds \\
&=\int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i-s)\right]B_iu(s)ds \\
&\quad+\int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i-s)\right]D_iu(s-\tau)ds \\
&=\int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i-s)\right]B_iu(s)ds \\
&\quad+\int_{t_{i-1}-\tau}^{t_i-\tau} \exp\left[A_i(t_i-s)\right]\exp(-A_i\tau)D_iu(s)ds \\
&=\int_{t_{i-1}-\tau}^{t_{i-1}} \exp\left[A_i(t_i-s)\right]\exp(-A_i\tau)D_iu(s)ds \\
&\quad+\int_{t_{i-1}}^{t_i-\tau} \exp\left[A_i(t_i-s)\right](B_i+\exp(-A_i\tau)D_i)u(s)ds \\
&\quad+\int_{t_i-\tau}^{t_i} \exp\left[A_i(t_i-s)\right]B_iu(s)ds \\
&=e^{A_ih_i}\int_{t_{i-1}-\tau}^{t_{i-1}} e^{A_i(t_{i-1}-s)}e^{-A_i\tau}D_iu(s)ds \\
&\quad+\int_{t_{i-1}}^{t_i-\tau} \exp\left[A_i(t_i-s)\right](B_i+\exp(-A_i\tau)D_i)u(s)ds \\
&\quad+\int_{t_i-\tau}^{t_i} \exp\left[A_i(t_i-s)\right]B_iu(s)ds
\end{aligned}$$

Then (17) can be rewritten as

$$\begin{aligned}
x_f &= \prod_{i=N}^{1} e^{A_ih_i}\left\{x_0+\int_{t_0-\tau}^{t_0} e^{A_1(t_0-s)}E_1u_0(s)ds\right\} \\
&+\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)\left\{\int_{t_{i-1}}^{t_i-\tau} e^{A_i(t_i-s)}F_iu(s)ds\right. \\
&\left.+\int_{t_i-\tau}^{t_i}\left(e^{A_i(t_i-s)}B_i+e^{A_{i+1}(t_i-s)}E_{i+1}\right)u(s)ds\right\} \\
&+\int_{t_{N-1}}^{t_N-\tau}\exp\left[A_N(t_N-s)\right](B_N+E_N)u(s)ds \\
&+\int_{t_N-\tau}^{t_N}\exp\left[A_N(t_N-s)\right]B_Nu(s)ds
\end{aligned} \tag{18}$$

where

$$E_i=\exp(-A_i\tau)D_i,\quad F_i=B_i+E_i,\quad i=1,\cdots,N. \tag{19}$$

We define the set

$$\begin{aligned}
\mathcal{V}_1=\Bigg\{x|x=&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}e^{A_lh_l}\left\{\int_{t_{i-1}}^{t_i-\tau}e^{A_i(t_i-s)}F_iu(s)ds\right. \\
&\left.+\int_{t_i-\tau}^{t_i}\left(e^{A_i(t_i-s)}B_i+e^{A_{i+1}(t_i-s)}E_{i+1}\right)u(s)ds\right\} \\
&+\int_{t_{N-1}}^{t_N-\tau}\exp\left[A_N(t_N-s)\right](B_N+E_N)u(s)ds \\
&+\int_{t_N-\tau}^{t_N}\exp\left[A_N(t_N-s)\right]B_Nu(s)ds,\forall\text{ p.c. }u\Bigg\}
\end{aligned} \tag{20}$$

Theorem 1. *System (1) is 1-periodic controllable if and only if* $\mathcal{V}_1=\Re^n$.

Proof. System (1) is 1-periodic controllable if and only if for any x_0, x_f and u_0, there exists $u(t)$ such that equation (18) holds. This is equivalent to $\mathcal{V}_1=\Re^n$. □

Corollary 1 (Sufficient). *System (1) is 1-periodic controllable if*

$$\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)\left\langle A_i|F_i\right\rangle+\left\langle A_N|F_N\right\rangle=\Re^n \tag{21}$$

Proof. Consider the set

$$\begin{aligned}
\mathcal{U}_1=\Bigg\{x|x=&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}e^{A_lh_l}\int_{t_{i-1}}^{t_i-\tau}e^{A_i(t_i-s)}F_iu(s)ds \\
&+\int_{t_{N-1}}^{t_N-\tau}e^{A_N(t_N-s)}F_Nu(s)ds,\forall\text{ p.c. }u\Bigg\}
\end{aligned} \tag{22}$$

It is easy to verify that $\mathcal{U}_1\subseteq\mathcal{V}_1$. In fact, we have

$$\begin{aligned}
\mathcal{U}_1=&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)\Bigg\{x|x= \\
&\int_{t_{i-1}}^{t_i-\tau}\exp\left[A_i(t_i-s)\right]F_iu(s)ds,\forall\text{ p.c. }u\Bigg\} \\
&+\Bigg\{x|x=\int_{t_{N-1}}^{t_N-\tau}\exp\left[A_N(t_N-s)\right]F_Nu(s)ds, \\
&\forall\text{ p.c. }u\Bigg\}
\end{aligned}$$

By Lemma 2, we have

$$\mathcal{U}_1=\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)\left\langle A_i|F_i\right\rangle+\left\langle A_N|F_N\right\rangle$$

Thus, $\mathcal{U}_1=\Re^n$ implies $\mathcal{V}_1=\Re^n$. □

Remark 4. *For system (1), let* $\tau=0$*, we get the following switched system without delay*

$$\dot{x}(t)=A_{r(t)}x(t)+F_{r(t)}u(t) \tag{23}$$

It is easy to verify that system (23) is controllable if and only if $\mathcal{U}_1=\Re^n$ *(For more details, see [1]). Thus, Corollary 1 means that the controllability of a switched system without delay implies the controllability of a switched system with delay.*

Corollary 2 (Sufficient and Necessary). *System (1) is 1-periodic controllable if and only if*

$$\begin{aligned}
&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}e^{A_lh_l}(\left\langle A_i|F_i\right\rangle+\left\langle A_i|B_i+A_{i+1}|E_{i+1}\right\rangle) \\
&+\left\langle A_N|[B_N,E_N]\right\rangle=\Re^n
\end{aligned} \tag{24}$$

Proof. From (20), by Lemma 2, we have

$$\begin{aligned}
\mathcal{V}_1=&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)\Bigg\{\{x|x= \\
&\int_{t_{i-1}}^{t_i-\tau}\exp\left[A_i(t_i-s)\right]F_iu(s)ds,\forall\text{ p.c. }u\} \\
&+\{x|x= \\
&\int_{t_i-\tau}^{t_i}\left(e^{A_i(t_i-s)}B_i+e^{A_{i+1}(t_i-s)}E_{i+1}\right)u(s)ds, \\
&\forall\text{ p.c. }u\}\Bigg\} \\
&+\{x|x=\int_{t_{N-1}}^{t_N-\tau}\exp\left[A_N(t_N-s)\right](B_N+E_N)u(s)ds, \\
&\forall\text{ p.c. }u\} \\
&+\{x|x=\int_{t_N-\tau}^{t_N}\exp\left[A_N(t_N-s)\right]B_Nu(s)ds,\forall\text{ p.c. }u\} \\
=&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)(\left\langle A_i|F_i\right\rangle+\left\langle A_i|B_i+A_{i+1}|E_{i+1}\right\rangle) \\
&+\left\langle A_N|(B_N+E_N)\right\rangle+\left\langle A_N|B_N\right\rangle
\end{aligned} \tag{25}$$

By Lemma 3 and Lemma 5, we have $\langle A_N|(B_N+E_N)\rangle+\langle A_N|B_N\rangle=\langle A_N|[B_N,E_N]\rangle$. By Theorem 1, the conclusion of Corollary 2 is obvious. □

3.2 Multiple-periodic Controllability

Definition 5 (m-periodic Controllability). *System (1) is m-periodic controllable if for any given pair of points x_0, x_f, and initial control function $u_0(t)$, $t \in [t_0 - \tau, t_0]$, there exists a piecewise continuous control function $u(t)$ which steers the state of the system from $x(t_0) = x_0$ to $x(t_{mN}) = x_f$, where $t_{mN} = t_0 + m\sum_{l=1}^{N} h_l$.*

Remark 5. *System (1) is said to be* multiple-periodic controllable *if there exists m such that the system is m-periodic controllable.*

Theorem 2. *System (1) is m-periodic controllable if*

$$\begin{aligned}\mathcal{U}_1 + \left(\prod_{i=N}^{1} \exp(A_i h_i)\right)\mathcal{U}_1 + \cdots \\ + \left(\prod_{i=N}^{1} \exp(A_i h_i)\right)^{m-1}\mathcal{U}_1 = \Re^n\end{aligned} \tag{26}$$

where $\mathcal{U}_1 = \sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)\,\langle A_i|F_i\rangle + \langle A_N|F_N\rangle$.

Proof. The proof proceeds in a way similar to that of Corollary 1 and is thus omitted here. □

Remark 6. *System (1) is m-periodic controllable, for any $m \geq n$ if*

$$\left\langle \prod_{i=N}^{1} \exp(A_i h_i) \middle| \mathcal{U}_1 \right\rangle = \Re^n \tag{27}$$

Theorem 3. *System (1) is m-periodic controllable if and only if*

$$\begin{aligned}\mathcal{V}_1 + \left(\prod_{i=N}^{1} \exp(A_i h_i)\right)\mathcal{Y}_1 + \cdots \\ + \left(\prod_{i=N}^{1} \exp(A_i h_i)\right)^{m-1}\mathcal{Y}_1 = \Re^n\end{aligned} \tag{28}$$

where $\mathcal{V}_1 = \sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle + \langle A_i|B_i + A_{i+1}|E_{i+1}\rangle) + \langle A_N|[B_N, E_N]\rangle$, *and* $\mathcal{Y}_1 = \sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle + \langle A_i|B_i + A_{i+1}|E_{i+1}\rangle) + \langle A_N|F_N\rangle + \langle A_N|B_N + A_1|E_1\rangle$.

Proof. The proof proceeds in a way similar to that of Corollary 2 and is thus omitted here. □

Remark 7. *System (1) is m-periodic controllable, for any $m \geq n + 1$ if and only if*

$$\mathcal{V}_1 + \left\langle \prod_{i=N}^{1} \exp(A_i h_i) \middle| \mathcal{Y}_1 \right\rangle = \Re^n \tag{29}$$

Thus, if system (1) is multiple-periodic controllable, then the controllability can be realized in $n+1$ periods at most.

Remark 8. *The above criteria are of geometric form. But it is easy to transfer them to algebraic form, i.e, by verifying certain matrices to be full rank or not.*

4 Example

In this section, we give an example to illustrate that a switched system with time-delay can be 1-periodic uncontrollable but multiple-periodic controllable.

Example 1. *Consider the following 5-dimensional switched system, with two realizations given by*

$$A_1 = \begin{bmatrix} 1&0&0&0&0\\0&2&0&0&0\\0&0&3&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix}, B_1 = \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}, D_1 = \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix};$$
$$A_2 = \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0 \end{bmatrix}, B_2 = \begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, D_2 = \begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} \tag{30}$$

Assume the time delay $\tau = 1$ and the switching sequence is $\{(1, 2), (2, 2)\}$.

By a simple calculation, we have

$$\mathcal{V}_1 = \text{span}\{\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\2\\1\end{bmatrix}\} \tag{31}$$

and

$$\begin{aligned}&\mathcal{U}_1 + \exp(2A_2)\exp(2A_1)\mathcal{U}_1 + (\exp(2A_2)\exp(2A_1))^2\mathcal{U}_1 \\ &= \text{span}\{\begin{bmatrix}e^2\\e^4\\e^6\\1\\0\end{bmatrix}, \begin{bmatrix}e^4\\e^8\\e^{12}\\1\\0\end{bmatrix}, \begin{bmatrix}e^6\\e^{12}\\e^{18}\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}\}\end{aligned} \tag{32}$$

It is obvious that $\dim(\mathcal{V}_1) = 3$ and $\dim(\mathcal{U}_1 + \exp(2A_2)\exp(2A_1)\mathcal{U}_1 + (\exp(2A_2)\exp(2A_1))^2\mathcal{U}_1) = 5$. Thus Example 1 is not 1-periodic controllable but 3-periodic controllable.

5 Some Further Results

Some further results will be presented in this section. The controllability, stability and stabilization of switched linear systems without time-delay can be referred to [8-24].

5.1 Controllability of Periodic-Type Systems

5.1.1 1-periodic Controllability

Theorem 4 (Sufficient and Necessary Condition). *System (1) is 1-periodic controllable if and only if*

$$\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)\langle A_i|[B_i, D_i]\rangle + \langle A_N|[B_N, D_N]\rangle = \Re^n \tag{33}$$

Proof. For (20), by Lemma 2, we have

$$\begin{aligned}
&\mathcal{V}_1 = \\
&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1} e^{A_l h_l}\Bigg\{\{x|x=\int_{t_{i-1}}^{t_i-\tau} e^{A_i(t_i-s)}F_i u(s)ds, \forall \text{ p.c. } u\} \\
&+\{x|x=\int_{t_i-\tau}^{t_i}\left(e^{A_i(t_i-s)}B_i+e^{A_{i+1}(t_i-s)}E_{i+1}\right)u(s)ds, \\
&\qquad \forall \text{ p.c. } u\}\Bigg\} \\
&+\{x|x=\int_{t_{N-1}}^{t_N-\tau} e^{A_N(t_N-s)}(B_N+E_N)u(s)ds, \forall \text{ p.c. } u\} \\
&+\{x|x=\int_{t_N-\tau}^{t_N} e^{A_N(t_N-s)}B_N u(s)ds, \forall \text{ p.c. } u\} \\
&=\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle+\langle A_i|B_i+A_{i+1}|E_{i+1}\rangle) \\
&+\langle A_N|(B_N+E_N)\rangle+\langle A_N|B_N\rangle
\end{aligned} \tag{34}$$

Next we will prove

$$\begin{aligned}
&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle+\langle A_i|B_i+A_{i+1}|E_{i+1}\rangle) \\
&+\langle A_N|(B_N+E_N)\rangle+\langle A_N|B_N\rangle \\
&\equiv \sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)\langle A_i|[B_i,D_i]\rangle+\langle A_N|[B_N,D_N]\rangle
\end{aligned} \tag{35}$$

We prove it by induction. For $N=1$, by Lemma 3 and Lemma 4, we have

$$\begin{aligned}
&\langle A_1|(B_1+E_1)\rangle+\langle A_1|B_1\rangle \\
&=\langle A_1|[B_1,E_1]\begin{bmatrix} I & 0 \\ I & I\end{bmatrix}\rangle \\
&=\langle A_1|[B_1,E_1]\rangle=\langle A_1|[B_1,D_1]\rangle
\end{aligned}$$

Hence, for $N=1$, (35)holds.

For $N=2$, by Lemma 5, we have

$$\begin{aligned}
&e^{A_2h_2}\Big(\langle A_1|B_1+E_1\rangle+\langle A_1|B_1+A_2|E_2\rangle\Big) \\
&+\langle A_2|B_2+E_2\rangle+\langle A_2|B_2\rangle \\
&=e^{A_2h_2}\Big(\langle A_1|B_1+E_1\rangle+\langle A_1|B_1+A_2|E_2\rangle+\langle A_2|E_2\rangle\Big) \\
&+\langle A_2|B_2\rangle \\
&=e^{A_2h_2}\Big(\langle A_1|B_1+E_1\rangle+\langle A_1|B_1\rangle+\langle A_2|E_2\rangle\Big) \\
&+\langle A_2|B_2\rangle \\
&=e^{A_2h_2}\Big(\langle A_1|[B_1,E_1]\rangle+\langle A_2|E_2\rangle\Big)+\langle A_2|B_2\rangle \\
&=e^{A_2h_2}\langle A_1|[B_1,D_1]\rangle+\langle A_2|[B_2,D_2]\rangle
\end{aligned}$$

Thus, for $N=2$, (35) holds.

Suppose for $N-1$, (35) holds, then we have

$$\begin{aligned}
&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle+\langle A_i|B_i+A_{i+1}|E_{i+1}\rangle) \\
&+\langle A_N|(B_N+E_N)\rangle+\langle A_N|B_N\rangle \\
=\;&\prod_{l=N-1}^{2}\exp(A_l h_l)(\langle A_1|F_1\rangle+\langle A_1|B_1+A_2|E_2\rangle) \\
&+\sum_{i=2}^{N-1}\prod_{l=N-1}^{i+1}\exp(A_l h_l)(\langle A_i|F_i\rangle+\langle A_i|B_i+A_{i+1}|E_{i+1}\rangle) \\
&+\langle A_N|(B_N+E_N)\rangle+\langle A_N|B_N\rangle \\
=\;&\prod_{l=N-1}^{2}\exp(A_l h_l)(\langle A_1|F_1\rangle+\langle A_1|B_1+A_2|E_2\rangle) \\
&+\sum_{i=2}^{N-1}\prod_{l=N-1}^{i+1}\exp(A_l h_l)\langle A_i|[B_i,D_i]\rangle+\langle A_N|[B_N,D_N]\rangle \\
=\;&\prod_{l=N-1}^{2}\exp(A_l h_l)(\langle A_1|F_1\rangle+\langle A_1|B_1+A_2|E_2\rangle+\langle A_2|E_2\rangle) \\
&+\sum_{i=2}^{N-1}\prod_{l=N-1}^{i+1}\exp(A_l h_l)\langle A_i|[B_i,D_i]\rangle+\langle A_N|[B_N,D_N]\rangle \\
=\;&\prod_{l=N-1}^{2}\exp(A_l h_l)\langle A_1|[B_1,D_1]\rangle \\
&+\sum_{i=2}^{N-1}\prod_{l=N-1}^{i+1}\exp(A_l h_l)\langle A_i|[B_i,D_i]\rangle+\langle A_N|[B_N,D_N]\rangle \\
=\;&\sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_l h_l)\langle A_i|[B_i,D_i]\rangle+\langle A_N|[B_N,D_N]\rangle
\end{aligned}$$

Thus, for N, (35) holds. Hence, for $1,2,\cdots,N$, (35) holds. The conclusion of Theorem 4 is obvious. □

5.1.2 Multiple-periodic Controllability

Theorem 5 (Sufficient and Necessary Condition). *System (1) is m-periodic controllable if and only if*

$$\mathcal{V}_1+(\prod_{i=N}^{1}\exp(A_ih_i))\mathcal{V}_1+\cdots+(\prod_{i=N}^{1}\exp(A_ih_i))^m\mathcal{V}_1^1=\Re^n \tag{36}$$

where $\mathcal{V}_1 = \sum_{i=1}^{N-1}\prod_{l=N}^{i+1}\exp(A_lh_l)(\langle A_i|[B_i,D_i]\rangle + \langle A_N|[B_N,D_N]\rangle$.

Proof. The proof is similar to that of Theorem 1. □

5.2 Controllability of Aperiodic-Type Systems

In this part, we discuss the controllability of aperiodic-type switched systems with time-delay in control. The results are similar to those of systems without time-delay in control.

Definition 6 (State Controllability). *For system (1), given initial state x_0 and initial control function $u_0(t)$, $t\in[t_0-\tau,t_0]$, the state x_f is said to be (x_0,u_0)-controllable, if there exist a switching sequence $\pi=\{(i_m,h_m)\}_{m=1}^M$ and a piecewise continuous function $u(t)$, $t\in[t_0,t_N]$ such that $x(t_0)=x_0$, $x(t_N)=x_f$, where $t_N=t_0+\sum_{l=1}^N h_l$.*

Definition 7 (System Controllability). *System (1)is (completely) controllable, if for any x_0 and initial control function $u_0(t)$, $t\in[t_0-\tau,t_0]$, any x_f is (x_0,u_0)-controllable.*

Now we introduce the definition of controllable set. Based on this concept, a geometric criterion for the controllability of switched systems is presented.

5.2.1 Controllable Set of Switching Sequence

Definition 8 (Controllable Set). *For system (1), given x_0, initial control function $u_0(t)$, $t \in [t_0-\tau, t_0]$ and switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, the set of all the states starting from x_0 and $u_0(t)$ through the switching sequence π is called the (x_0, u_0)-controllable set of the switching sequence π, denoted by $\mathcal{C}(x_0, u_0, \pi)$.*

Given initial state x_0 and initial control function $u_0(t)$, $t \in [t_0-\tau, t_0]$, the system state through the switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$ can be represented as

$$\begin{array}{l} x_f = \prod\limits_{l=N}^{1} e^{A_{i_l}h_l} \left\{ x_0 + \int\limits_{t_0-\tau}^{t_0} e^{A_{i_1}(t_0-s)} E_{i_1} u_0(s) ds \right\} \\ + \sum\limits_{m=1}^{M-1} \prod\limits_{l=M}^{m+1} e^{A_{i_l}h_l} \left\{ \int\limits_{t_{m-1}}^{t_m-\tau} e^{A_{i_m}(t_m-s)} F_{i_m} u(s) ds \right. \\ \left. + \int\limits_{t_m-\tau}^{t_m} (e^{A_{i_m}(t_{i_m}-s)} B_{i_m} + e^{A_{i_{m+1}}(t_m-s)} E_{i_{m+1}}) u(s) ds \right\} \\ + \int_{t_{M-1}}^{t_M-\tau} \exp\left[A_{i_M}(t_M-s)\right](B_{i_M}+E_{i_M}) u(s) ds \\ + \int_{t_M-\tau}^{t_M} \exp\left[A_{i_M}(t_M-s)\right] B_{i_M} u(s) ds \end{array} \tag{37}$$

where

$$\begin{array}{l} t_m = t_0 + \sum_{l=1}^{m} h_l, \quad E_{i_m} = \exp(-A_{i_m}h_m) D_{i_m}, \\ F_{i_m} = B_{i_m} + E_{i_m}, \quad m = 1, \cdots, M \end{array} \tag{38}$$

Then we can draw the following conclusion.

Theorem 6. *For system (1), given x_0, initial control function $u_0(t)$, $t \in [t_0-\tau, t_0]$ and switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, the (x_0, u_0)-controllable set $\mathcal{C}(x_0, u_0, \pi)$ of the switching sequence π is as follows:*

$$\begin{aligned} \mathcal{C}(x_0, u_0, \pi) &= \mathcal{I}n(x_0, u_0, \pi) \\ &\quad + \sum_{m=1}^{M-1} \prod_{l=M}^{m+1} e^{A_{i_l}h_l} \langle A_{i_m} | [B_{i_m}, D_{i_m}] \rangle \\ &\quad + \langle A_{i_M} | [B_{i_M}, D_{i_M}] \rangle \end{aligned} \tag{39}$$

where

$$\begin{array}{l} \mathcal{I}n(x_0, u_0, \pi) = \prod\limits_{l=N}^{1} e^{A_{i_l}h_l} \{ x_0 \\ \quad + \int\limits_{t_0-\tau}^{t_0} e^{A_{i_1}(t_0-s)} E_{i_1} u_0(s) ds \} \end{array} \tag{40}$$

In particular, for $x_0 = 0$, $u_0 = 0$, we have

$$\begin{array}{l} \mathcal{C}(0,0,\pi) = \sum\limits_{m=1}^{M-1} \prod\limits_{l=M}^{m+1} \exp(A_{i_l}h_l) \langle A_{i_m} | [B_{i_m}, D_{i_m}] \rangle \\ \quad + \langle A_{i_M} | [B_{i_M}, D_{i_M}] \rangle \end{array} \tag{41}$$

which is a linear subspace in $\Re^n$, denoted by $\mathcal{C}(\pi)$.

Proof. The proof is similar to that of Theorem 4. □

Some basic properties of $\mathcal{C}(\pi)$ are given below.

Definition 9 (Product of Switching Sequences). *Given two switching sequences $\pi_1 = \{(i_m, h_m)\}_{m=1}^{M}$ and $\pi_2 = \{(j_m, g_m)\}_{m=1}^{L}$, the product of the switching sequences π_1 and π_2 is defined as*

$$\begin{aligned} \pi_1 \wedge \pi_2 \overset{def}{=} \; &\{(i_1, h_1), \cdots, (i_M, h_M), \\ &(j_1, g_1), \cdots, (j_L, g_L)\} \end{aligned} \tag{42}$$

Since it is easy to prove that $(\pi_1 \wedge \pi_2) \wedge \pi_3 = \pi_1 \wedge (\pi_2 \wedge \pi_3)$, we denote it by $\pi_1 \wedge \pi_2 \wedge \pi_3$.

Definition 10 (Power of Switching Sequences). *Given a switching sequence π, the power of the switching sequence π is defined as*

$$\pi^{\wedge n} \overset{def}{=} \overbrace{\pi \wedge \cdots \wedge \pi}^{n \text{ times}} \tag{43}$$

Definition 11 (Exponential Matrix). *Given a switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, the exponential matrix of the switching sequence π is defined as*

$$\exp(\pi) \overset{def}{=} \prod_{m=M}^{1} \exp(A_{i_m}h_m) \tag{44}$$

Theorem 7. *Given switching sequences π_1 and π_2, we have*

$$\mathcal{C}(\pi_1 \wedge \pi_2) = \exp(\pi_2)\mathcal{C}(\pi_1) + \mathcal{C}(\pi_2) \tag{45}$$

Proof. It can be easily proved by the definitions of the product of switching sequences and the controllable set. □

Theorem 8. *Given a switching sequence π, we have*

$$\mathcal{C}(\pi^{\wedge n}) = \langle \exp(\pi) | \mathcal{C}(\pi) \rangle \tag{46}$$

Proof.

$$\begin{aligned} \mathcal{C}(\pi^{\wedge n}) &= \exp(\pi)\mathcal{C}(\pi^{\wedge(n-1)}) + \mathcal{C}(\pi) \\ &= [\exp(\pi)]^2 \mathcal{C}(\pi^{\wedge(n-2)}) + \exp(\pi)\mathcal{C}(\pi) + \mathcal{C}(\pi) \\ &= \cdots\cdots \\ &= \sum_{i=1}^{n} [\exp(\pi)]^{(i-1)} \mathcal{C}(\pi) \\ &= \langle \exp(\pi) | \mathcal{C}(\pi) \rangle \end{aligned}$$

□

Corollary 3. *Given a switching sequence π, we have*

$$\exp(\pi^{\wedge n})\mathcal{C}(\pi^{\wedge n}) = \mathcal{C}(\pi^{\wedge n}) \tag{47}$$

Proof. It's easy to verify it by the property of cyclic invariant subspaces. □

5.2.2 Geometric Criteria for Controllability

For system (1), a sequence of linear subspaces can be defined recursively as follows:

$$\begin{aligned}\mathcal{W}_1 &= \sum_{i=1}^{N} \langle A_i|[B_i, D_i]\rangle, \quad \mathcal{W}_2 = \sum_{i=1}^{N} \langle A_i|\mathcal{W}_1\rangle, \cdots, \\ \mathcal{W}_n &= \sum_{i=1}^{N} \langle A_i|\mathcal{W}_{n-1}\rangle\end{aligned} \tag{48}$$

It is easy to prove that for any switching sequence π, $\mathcal{C}(\pi) \subseteq \mathcal{W}_n$.

Theorem 9. *For system (1), there must exist a switching sequence π_b such that*

$$\mathcal{C}(\pi_b) = \mathcal{W}_n \tag{49}$$

Proof. By lemma 6, for every system realization (A_i, B_i, D_i), there must exist a constant $h_i > \tau$ such that for any linear subspace $\mathcal{W}$, $\langle A_i|\mathcal{W}\rangle = \langle \exp(A_i h_i)|\mathcal{W}\rangle$, $i = 1, \cdots, N$. Thus, the subspace sequence $\mathcal{W}_1, \cdots, \mathcal{W}_n$ can be redefined as

$$\begin{aligned}\mathcal{W}_1 &= \sum_{i=1}^{N}\langle A_i|[B_i, D_i]\rangle, \quad \mathcal{W}_2 = \sum_{i=1}^{N}\langle \exp(A_i h_i)|\mathcal{W}_1\rangle, \cdots, \\ \mathcal{W}_n &= \sum_{i=1}^{N}\langle \exp(A_i h_i)|\mathcal{W}_{n-1}\rangle\end{aligned} \tag{50}$$

Suppose $\dim(\mathcal{W}_n) = d$. By (50), there must exist d subspaces $\mathcal{V}_1, \cdots, \mathcal{V}_d$ such that

$$\mathcal{W}_n = \sum_{m=1}^{d} \mathcal{V}_m$$

where each subspace has the form as follows:

$$\prod_{m=1}^{M} \exp(A_{i_m} h_{i_m})\langle A_j|[B_j, D_j]\rangle \tag{51}$$

where $M < \infty$, $i_1, \cdots, i_M, j \in \{1, \cdots, N\}$.

Consider the subspace of form (51), we can choose the switching sequence

$$\pi = \{(j, 1), (i_M, h_{i_M}), \cdots, (i_1, h_{i_1})\} \tag{52}$$

such that

$$\prod_{m=1}^{M} \exp(A_{i_m} h_{i_m})\langle A_j|[B_j, D_j]\rangle \subseteq \mathcal{C}(\pi)$$

So we can choose the switching sequence $\pi_1, \cdots, \pi_d$ such that $\mathcal{V}_m \subseteq \mathcal{C}(\pi_m)$, for $m = 1, \cdots, d$. Then we have

$$\mathcal{W}_n = \sum_{m=1}^{d} \mathcal{C}(\pi_m) \tag{53}$$

Now we construct the switching sequence π_b.

First, if $\mathcal{C}(\pi_1^{\wedge n}) = \mathcal{W}_n$, we can get $\pi_b = \pi_1^{\wedge n}$. If not, there must exist a switching sequence $k \in \{2, \cdots, d\}$, (without loss of generality, let $k = 2$) such that

$$\mathcal{C}(\pi_2) \nsubseteq \mathcal{C}(\pi_1^{\wedge n})$$

Since

$$\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n}) = \exp(\pi_1^{\wedge n})\mathcal{C}(\pi_2) + \mathcal{C}(\pi_1^{\wedge n})$$

By (47), we get

$$\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n}) = \exp(\pi_1^{\wedge n})(\mathcal{C}(\pi_2) + \mathcal{C}(\pi_1^{\wedge n}))$$

then

$$\begin{aligned}\dim(\mathcal{C}(\pi_2 \wedge \pi_1^{\wedge n})) &= \dim(\mathcal{C}(\pi_2) + \mathcal{C}(\pi_1^{\wedge n})) \\ &\geq \dim(\mathcal{C}(\pi_1^{\wedge n})) + 1 \\ &= 2\end{aligned}$$

Thus, we can construct the switching sequence as follows

$$\begin{aligned}&\overline{\pi}_1 = \pi_1 \\ &\overline{\pi}_2 = \pi_2 \wedge \overline{\pi}_1^{\wedge n} \\ &\cdots \\ &\overline{\pi}_d = \pi_d \wedge (\overline{\pi}_{d-1})^{\wedge n}\end{aligned}$$

and let

$$\pi_b = \overline{\pi}_d$$

It's obvious that

$$\dim(\mathcal{C}(\pi_b)) \geq d$$

Hence, $\mathcal{C}(\pi_b) = \mathcal{W}_n$. □

Corollary 4 (Sufficient and Necessary Condition). *System (1)is controllable if and only if*

$$\mathcal{W}_n = \Re^n \tag{54}$$

Proof. By Theorem 9, there must exist a switching sequence $\pi_b = \{(i_m, h_m)\}_{m=1}^{M}$ such that $\mathcal{C}(\pi_b) = \mathcal{W}_n = \Re^n$. Given any initial state x_0, any initial input u_0, and any terminal state x_f, considering the state

$$x_f - \prod_{l=N}^{1} \exp(A_{i_l} h_l) \left\{x_0 + \int_{t_0-\tau}^{t_0} \exp\left[A_{i_1}(t_0 - s)\right]E_{i_1} u_0(s)ds\right\} \in \mathcal{C}(\pi_b),$$

there must exist an input function $u(t)$ such that

$$\begin{aligned}x_f - \prod_{l=N}^{1} \exp(A_{i_l} h_l) &\left\{x_0 + \int_{t_0-\tau}^{t_0} \exp\left[A_{i_1}(t_0 - s)\right]E_{i_1} u_0(s)ds\right\} = \\ \sum_{m=1}^{M-1} \prod_{l=M}^{m+1} \exp(A_{i_l} h_l) &\left\{\int_{t_{m-1}}^{t_m-\tau} \exp\left[A_{i_m}(t_m - s)\right]F_{i_m} u(s)ds \right. \\ &\left. + \int_{t_m-\tau}^{t_m} \left(\exp\left[A_{i_m}(t_{i_m} - s)\right]B_{i_m} + \exp\left[A_{i_{m+1}}(t_m - s)\right]E_{i_{m+1}}\right) u(s)ds\right\} \\ + \int_{t_{M-1}}^{t_M-\tau} \exp\left[A_{i_M}(t_M - s)\right](B_{i_M} + E_{i_M})u(s)ds &+ \int_{t_M-\tau}^{t_M} \exp\left[A_{i_M}(t_M - s)\right]B_{i_M} u(s)ds\end{aligned}$$

where

$$t_m = t_0 + \sum_{i=1}^{m} h_i, \quad E_{i_m} = \exp(-A_{i_m} h_m) D_{i_m}, \quad F_{i_m} = B_{i_m} + E_{i_m}, \quad m = 1, \cdots, M$$

That is,

$$\begin{aligned} x_f \; = \; & \prod_{l=N}^{1} \exp(A_{i_l} h_l) \left\{ x_0 + \int_{t_0-\tau}^{t_0} \exp\left[A_{i_1}(t_0 - s)\right] E_{i_1} u_0(s) ds \right\} \\ & + \sum_{m=1}^{M-1} \prod_{l=M}^{m+1} \exp(A_{i_l} h_l) \left\{ \int_{t_{m-1}}^{t_m-\tau} \exp\left[A_{i_m}(t_m - s)\right] F_{i_m} u(s) ds \right. \\ & \qquad \left. + \int_{t_m-\tau}^{t_m} \left(\exp\left[A_{i_m}(t_{i_m} - s)\right] B_{i_m} + \exp\left[A_{i_{m+1}}(t_m - s)\right] E_{i_{m+1}} \right) u(s) ds \right\} \\ & + \int_{t_{M-1}}^{t_M-\tau} \exp\left[A_{i_M}(t_M - s)\right](B_{i_M} + E_{i_M}) u(s) ds + \int_{t_M-\tau}^{t_M} \exp\left[A_{i_M}(t_M - s)\right] B_{i_M} u(s) ds \end{aligned}$$

By the definition of controllability, system (1) must be controllable. □

5.3 Controllability of switched systems with multiple time-delay

Consider the switched linear system with multiple time-delays in control function given by

$$\dot{x}(t) = A_{r(t)} x(t) + B_{r(t)} u(t) + \sum_{k=1}^{K} D_{r(t),k} u(t - \tau_k) \quad (55)$$

where $x(t), u(t)$ and $r(t)$ are defined as before, and $\{(A_i, B_i, D_{i,1}, \cdots, D_{i,K}) | i = 1, \cdots, N\}$ is a finite family of system realizations. Moreover, $r(t) = i$ implies $(A_i, B_i, D_{i,1}, \cdots, D_{i,K})$ is chosen as the system realization at time t. $K < \infty$ is the number of time-delays of the system. $0 < \tau_1 < \cdots < \tau_K$ are K fixed time-delays.

Remark 9. *Similarly, we can describe the switching law by introducing the switching sequence. In order to avoid unnecessary complexity, we suppose the holding time $h_m > \tau_K$.*

In this part, we will discuss the controllability of system (55). The 1-periodic controllability, the m-periodic controllability of periodic-type systems, and the controllability of aperiodic-type systems are investigated, respectively. All the sufficient and necessary criteria are presented in geometric form.

5.3.1 Controllability of Periodic-Type Systems

Without loss of generality, we still choose the switching sequence $\pi = \{(1, h_1), \cdots, (N, h_N)\}$ as the period of system (55).

Theorem 10 (Sufficient and Necessary condition). *System (55) is 1-periodic controllable if and only if*

$$\sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \langle A_i | [B_i, D_{i,1}, \cdots, D_{i,K}] \rangle + \langle A_N | [B_N, D_{N,1}, \cdots, D_{N,K}] \rangle = \Re^n \quad (56)$$

Proof. We just give a proof for $K = 2$. Whereas for $K > 2$, the proof is similar. For system (55), given initial state x_0 and initial control function $u_0(t)$, $t \in [t_0 - \tau, t_0]$, let $t_m = t_0 + \sum_{l=1}^{m} h_l$, $m = 1, \cdots, N$, then the state x_f can be represented as :

$$\begin{aligned} x_f = \; & \prod_{i=N}^{1} \exp(A_i h_i) x_0 \\ & + \sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i - s)\right](B_i u(s) + D_{i,1} u(s - \tau_1) + D_{i,2} u(s - \tau_2)) ds \\ & + \int_{t_{N-1}}^{t_N} \exp\left[A_N(t_N - s)\right](B_N u(s) + D_{N,1} u(s - \tau_1) + D_{N,2} u(s - \tau_2)) ds \end{aligned} \quad (57)$$

Since

$$\begin{aligned} & \int_{t_{i-1}}^{t_i} \exp\left[A_i(t_i - s)\right](D_{i,1} u(s - \tau_1) + D_{i,2} u(s - \tau_2)) ds \\ = \; & \int_{t_{i-1}-\tau_1}^{t_i-\tau_1} \exp\left[A_i(t_i - s)\right] \exp(-A_i \tau_1) D_{i,1} u(s) ds + \int_{t_{i-1}-\tau_2}^{t_i-\tau_2} \exp\left[A_i(t_i - s)\right] \exp(-A_i \tau_2) D_{i,2} u(s) ds \\ = \; & \int_{t_{i-1}-\tau_2}^{t_{i-1}-\tau_1} \exp\left[A_i(t_i - s)\right] \exp(-A_i \tau_2) D_{i,2} u(s) ds \\ & + \int_{t_{i-1}-\tau_1}^{t_{i-1}} \exp\left[A_i(t_i - s)\right] \left[\exp(-A_i \tau_1) D_{i,1} + \exp(-A_i \tau_2) D_{i,2} \right] u(s) ds \\ & + \int_{t_{i-1}}^{t_i-\tau_2} \exp\left[A_i(t_i - s)\right] \left[B_i + \exp(-A_i \tau_1) D_{i,1} + \exp(-A_i \tau_2) D_{i,2} \right] u(s) ds \\ & + \int_{t_i-\tau_2}^{t_i-\tau_1} \exp\left[A_i(t_i - s)\right] \left[B_i + \exp(-A_i \tau_1) D_{i,1} \right] u(s) ds \\ & + \int_{t_i-\tau_1}^{t_i} \exp\left[A_i(t_i - s)\right] B_i u(s) ds \\ = \; & \exp(A_i h_i) \int_{t_{i-1}-\tau_2}^{t_{i-1}-\tau_1} \exp\left[A_i(t_{i-1} - s)\right] \exp(-A_i \tau_2) D_{i,2} u(s) ds \\ & + \exp(A_i h_i) \int_{t_{i-1}-\tau_1}^{t_{i-1}} \exp\left[A_i(t_{i-1} - s)\right] \left[\exp(-A_i \tau_1) D_{i,1} + \exp(-A_i \tau_2) D_{i,2} \right] u(s) ds \\ & + \int_{t_{i-1}}^{t_i-\tau_2} \exp\left[A_i(t_i - s)\right] \left[B_i + \exp(-A_i \tau_1) D_{i,1} + \exp(-A_i \tau_2) D_{i,2} \right] u(s) ds \\ & + \int_{t_i-\tau_2}^{t_i-\tau_1} \exp\left[A_i(t_i - s)\right] \left[B_i + \exp(-A_i \tau_1) D_{i,1} \right] u(s) ds \\ & + \int_{t_i-\tau_1}^{t_i} \exp\left[A_i(t_i - s)\right] B_i u(s) ds \end{aligned} \quad (58)$$

Thus, (57) can be rewritten as

$$
\begin{aligned}
x_f &= \prod_{i=N}^{1} \exp(A_i h_i) \Big\{ x_0 + \int_{t_0-\tau_2}^{t_0-\tau_1} \exp\left[A_1(t_0-s)\right] \exp(-A_1\tau_2) D_{1,1} u_0(s) ds \\
&\qquad + \int_{t_0-\tau_2}^{t_0-\tau_1} \exp\left[A_1(t_0-s)\right] \exp(-A_1\tau_2) D_{1,1} u_0(s) ds \Big\} \\
&+ \sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \Big\{ \int_{t_{i-1}}^{t_i-\tau_2} e^{[A_i(t_i-s)]} \Big[B_i + e^{(-A_i\tau_1)} D_{i,1} + e^{(-A_i\tau_2)} D_{i,2} \Big] u(s) ds \\
&+ \int_{t_i-\tau_2}^{t_i-\tau_1} \Big\{ e^{[A_i(t_i-s)]} \Big[B_i + e^{(-A_i\tau_1)} D_{i,1} \Big] + e^{[A_{i+1}(t_i-s)]} e^{(-A_{i+1}\tau_2)} D_{i+1,2} \Big\} u(s) ds \\
&+ \int_{t_i-\tau_2}^{t_i} \Big\{ e^{[A_i(t_i-s)]} B_i + e^{[A_{i+1}(t_i-s)]} \Big[e^{(-A_{i+1}\tau_1)} D_{i+1,1} + e^{(-A_{i+1}\tau_2)} D_{i+1,2} \Big] \Big\} u(s) ds \Big\} \\
&+ \int_{t_{N-1}}^{t_N-\tau_2} \exp\left[A_N(t_N-s)\right] \left[B_N + \exp(-A_N\tau_1) D_{N,1} + \exp(-A_N\tau_2) D_{N,2} \right] u(s) ds \\
&+ \int_{t_N-\tau_2}^{t_N-\tau_1} \exp\left[A_N(t_N-s)\right] \left[B_N + \exp(-A_N\tau_1) D_{N,1} \right] u(s) ds \\
&+ \int_{t_N-\tau_1}^{t_N} \exp\left[A_N(t_N-s)\right] B_N u(s) ds
\end{aligned} \tag{59}
$$

Then, system is 1-periodic controllable if and only if the following linear space is the entire space.

$$
\mathcal{V}_1 = \{x | x = f(u), p.c. \quad u\} \tag{60}
$$

where

$$
\begin{aligned}
f(u) &= \sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \Big\{ \int_{t_{i-1}}^{t_i-\tau_2} e^{[A_i(t_i-s)]} \Big[B_i + \exp(-A_i\tau_1) D_{i,1} + \exp(-A_i\tau_2) D_{i,2} \Big] u(s) ds \\
&+ \int_{t_i-\tau_2}^{t_i-\tau_1} \Big\{ e^{[A_i(t_i-s)]} \Big[B_i + \exp(-A_i\tau_1) D_{i,1} \Big] + e^{[A_{i+1}(t_i-s)]} \exp(-A_{i+1}\tau_2) D_{i+1,2} \Big\} u(s) ds \\
&+ \int_{t_i-\tau_2}^{t_i} \Big\{ e^{[A_i(t_i-s)]} B_i + e^{[A_{i+1}(t_i-s)]} \Big[e^{(-A_{i+1}\tau_1)} D_{i+1,1} + e^{(-A_{i+1}\tau_2)} D_{i+1,2} \Big] \Big\} u(s) ds \Big\} \\
&+ \int_{t_{N-1}}^{t_N-\tau_2} \exp\left[A_N(t_N-s)\right] \left[B_N + \exp(-A_N\tau_1) D_{N,1} + \exp(-A_N\tau_2) D_{N,2} \right] u(s) ds \\
&+ \int_{t_N-\tau_2}^{t_N-\tau_1} \exp\left[A_N(t_N-s)\right] \left[B_N + \exp(-A_N\tau_1) D_{N,1} \right] u(s) ds \\
&+ \int_{t_N-\tau_1}^{t_N} \exp\left[A_N(t_N-s)\right] B_N u(s) ds
\end{aligned} \tag{61}
$$

By Lemma 2, we have

$$
\begin{aligned}
\mathcal{V}_1 &= \sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \Big\{ \langle A_i | (B_i + \exp(-A_i\tau_1) D_{i,1} + \exp(-A_i\tau_2) D_{i,2}) \rangle \\
&\qquad + \langle A_i | B_i + A_i | \exp(-A_i\tau_1) D_{i,1} + A_{i+1} | \exp(-A_{i+1}\tau_2) D_{i+1,2} \rangle \\
&\qquad + \langle A_i | B_i + A_{i+1} | \exp(-A_{i+1}\tau_1) D_{i+1,1} + A_{i+1} | \exp(-A_{i+1}\tau_2) D_{i+1,2} \rangle \Big\} \\
&+ \langle A_N | B_N + \exp(-A_N\tau_1) D_{N,1} + \exp(-A_N\tau_2) D_{N,2} \rangle \\
&+ \langle A_N | B_N + \exp(-A_N\tau_1) D_{N,1} \rangle \\
&+ \langle A_N | B_N \rangle
\end{aligned} \tag{62}
$$

By Lemma 4 and 5, it's easy to prove that

$$
\begin{aligned}
\mathcal{V}_1 &= \sum_{i=1}^{N-1} \prod_{l=N}^{i+1} \exp(A_l h_l) \langle A_i | [B_i, \exp(-A_i\tau_1) D_{i,1}, \exp(-A_i\tau_2) D_{i,2}] \rangle \\
&+ \langle A_N | [B_N, \exp(-A_N\tau_1) D_{N,1}, \exp(-A_N\tau_2) D_{N,2}] \rangle
\end{aligned} \tag{63}
$$

By Lemma 3, the conclusion of the theorem is proved. □

Theorem 11 (Sufficient and Necessary Condition). *System(55) is m-periodic controllable if and only if*

$$\mathcal{V}_1 + (\prod_{i=N}^{1} \exp(A_i h_i))\mathcal{V}_1 + \cdots + (\prod_{i=N}^{1} \exp(A_i h_i))^{m-1}\mathcal{V}_1 = \Re^n \tag{64}$$

where $\mathcal{V}_1$ is defined as (63).

Remark 10. *For $m \geq n$, system (55) is m-periodic controllable if and only if*

$$\left\langle \prod_{i=N}^{1} \exp(A_i h_i) \middle| \mathcal{V}_1 \right\rangle = \Re^n \tag{65}$$

5.3.2 Controllability of Aperiodic-Type Systems

Similar to the discussion in the single time-delay case, we can easily extend the results to the multiple time-delay case. In the following, we present the corresponding results without proof.

Theorem 12. *For system (55), given initial state x_0, initial control function $u_0(t)$, $t \in [t_0 - \tau, t_0]$ and switching sequence $\pi = \{(i_m, h_m)\}_{m=1}^{M}$, the (x_0, u_0)-controllable set $\mathcal{C}(x_0, u_0, \pi)$ of the switching sequence π is*

$$\begin{aligned}\mathcal{C}(x_0, u_0, \pi) = \mathcal{I}n(x_0, u_0, \pi) + \sum_{m=1}^{M-1} \prod_{l=M}^{m+1} e^{(A_{i_l} h_l)} \langle A_{i_m} | [B_{i_m}, D_{i_m,1}, \cdots, D_{i_m,K}] \rangle \\ + \langle A_{i_M} | [B_{i_M}, D_{i_M}, \cdots, D_{i_M,K}] \rangle\end{aligned} \tag{66}$$

where

$$\begin{aligned}\mathcal{I}n(x_0, u_0, \pi) &= \prod_{l=N}^{1} \exp(A_{i_l} h_l) \Bigg\{ x_0 + \int_{t_0-\tau_K}^{t_0-\tau_{K-1}} \exp\left[A_{i_1}(t_0 - s)\right] \exp(-A_{i_1}\tau_K) D_{i_1,K} u_0(s) ds \\ &+ \int_{t_0-\tau_{K-1}}^{t_0-\tau_{K-2}} \exp\left[A_{i_1}(t_0 - s)\right] \Big(\exp(-A_{i_1}\tau_K) D_{i_1,K} + \exp(-A_{i_1}\tau_{K-1}) D_{i_1,K-1} \Big) u_0(s) ds \\ &+ \cdots \\ &+ \int_{t_0-\tau_1}^{t_0} \exp\left[A_{i_1}(t_0 - s)\right] \Big(\exp(-A_{i_1}\tau_K) D_{i_1,K} + \cdots + \exp(-A_{i_1}\tau_1) D_{i_1,1} \Big) u_0(s) ds \Bigg\}\end{aligned} \tag{67}$$

In particular, for $x_0 = 0$ and $u_0 = 0$, we get

$$\mathcal{C}(0, 0, \pi) = \sum_{m=1}^{M-1} \prod_{l=M}^{m+1} \exp(A_{i_l} h_l) \langle A_{i_m} | [B_{i_m}, D_{i_m,1}, \cdots, D_{i_m,K}] \rangle + \langle A_{i_M} | [B_{i_M}, D_{i_M}, \cdots, D_{i_M,K}] \rangle \tag{68}$$

which is a linear space in $\Re^n$, denoted as $\mathcal{C}(\pi)$.

For system (55), a sequence of linear subspaces is defined recursively as follows:

$$\mathcal{W}_1 = \sum_{i=1}^{N} \langle A_i | [B_i, D_{i,1}, \cdots, D_{i,K}] \rangle, \quad \mathcal{W}_2 = \sum_{i=1}^{N} \langle A_i | \mathcal{W}_1 \rangle, \quad \cdots, \quad \mathcal{W}_n = \sum_{i=1}^{N} \langle A_i | \mathcal{W}_{n-1} \rangle \tag{69}$$

Theorem 13. *For system (55), there must exist a switching sequence π_b, such that*

$$\mathcal{C}(\pi_b) = \mathcal{W}_n \tag{70}$$

Corollary 5 (Sufficient and Necessary Condition). *System (55) is controllable if and only if*

$$\mathcal{W}_n = \Re^n \tag{71}$$

5.4 Controllability of Switched Systems with Distinct Time-delays

The above criteria for controllability, from single time-delay case to multiple time-delays case, and from periodic-type systems to aperiodic-type systems, are all derived on the assumption that the time-delays of every system realization are consistent. A common characteristic of all of the above results is that the criteria are independent of the size of time-delays. Next, we discuss controllability criteria for switched linear systems with distinct time-delays.

The switched linear system with single distinct time-delay can be described as follows

$$\dot{x}(t) = A_{r(t)}x(t) + B_{r(t)}u(t) + D_{r(t)}u(t-\tau_{r(t)}) \quad (72)$$

where $x(t)$, $u(t)$, $r(t)$ are defined as before. $\{(A_i, B_i, D_i, \tau_i)|i = 1, \cdots, N\}$ is a finite family of system realizations. Moreover, $r(t) = i$ implies (A_i, B_i, D_i, τ_i) is chosen as the system realization at time t. $\tau_i > 0$ is the fixed time-delay, for $i = 1, \cdots, N$.

Theorem 14. *For system (72), for any switching sequence* $\pi = \{(i_m, h_m)\}_{m=1}^{M}$*, if* $h_m > \tau_{i_m}$*,* $m = 1, \cdots, M$*, then its controllable set is*

$$\mathcal{C}(\pi) = \sum_{m=1}^{M-1}\prod_{l=M}^{m+1} e^{A_{i_l}h_l}\langle A_{i_m}|[B_{i_m}, D_{i_m}]\rangle + \langle A_{i_M}|[B_{i_M}, D_{i_M}]\rangle \quad (73)$$

Proof. We just give the proof for the switching sequence $\pi = \{(1, h_1), (2, h_2)\}$. The process can be easily extended to more general switching sequence. For the switching sequence $\pi = \{(1, h_1), (2, h_2)\}$, we have

$$\begin{aligned}\mathcal{C}(\pi) = \Big\{x|x \;=\; & \exp(A_2h_2)\int_{t_0}^{t_1}\exp[A_1(t_1-s)](B_1u(s) + D_1u(s-\tau))ds \\ & + \int_{t_1}^{t_2}\exp[A_2(t_2-s)](B_2u(s) + D_2u(s-\tau))ds, \forall\ p.c.\ u\Big\}\end{aligned} \quad (74)$$

We just consider two cases: 1) $\tau_1 > \tau_2$ and 2) $\tau_1 < \tau_2$. For $\tau_1 = \tau_2$, it has already been discussed. For $\tau_1 > \tau_2$ (see Fig. 1), we can divide the integral interval into five parts, i.e,

$$\begin{aligned}\mathcal{C}(\pi) &= e^{(A_2h_2)}\Big\{x|x = \int_{t_0}^{t_1-\tau_1}\exp[A_1(t_1-s)](B_1 + \exp[-A_1\tau_1]D_1)u(s)ds, \forall\ \text{p.c.}\ u\Big\} \\ &\quad + e^{(A_2h_2)}\Big\{x|x = \int_{t_1-\tau_1}^{t_1-\tau_2}\exp[A_1(t_1-s)]B_1u(s)ds, \forall\ \text{p.c.}\ u\Big\} \\ &\quad + e^{(A_2h_2)}\Big\{x|x = \int_{t_1-\tau_2}^{t_1}\Big\{e^{[A_1(t_1-s)]}B_1e^{[A_2(t_1-s)]}e^{[-A_2\tau_2]}D_2\Big\}u(s)ds, \forall\ \text{p.c.}\ u\Big\} \\ &\quad + \Big\{x|x = \int_{t_1}^{t_2-\tau_2}\exp[A_2(t_2-s)](B_2 + \exp[-A_2(\tau_2)]D_2u(s))ds, \forall\ \text{p.c.}\ u\Big\} \\ &\quad + \Big\{x|x = \int_{t_2-\tau_2}^{t_2}\exp[A_2(t_2-s)]B_2u(s)ds, \forall\ \text{p.c.}\ u\Big\} \\ &= \exp(A_2h_2)\Big\{\langle A_1|B_1 + \exp[-A_1\tau_1]D_1\rangle + \langle A_1|B_1\rangle + \langle A_1|B_1 + A_2|\exp[-A_2\tau_2]D_2\rangle\Big\} \\ &\quad + \langle A_2|B_2 + \exp[-A_2\tau_2]D_2\rangle + \langle A_2|B_2\rangle \\ &= \exp(A_2h_2)\langle A_1|[B_1, D_1]\rangle + \langle A_2|[B_2, D_2]\rangle\end{aligned} \quad (75)$$

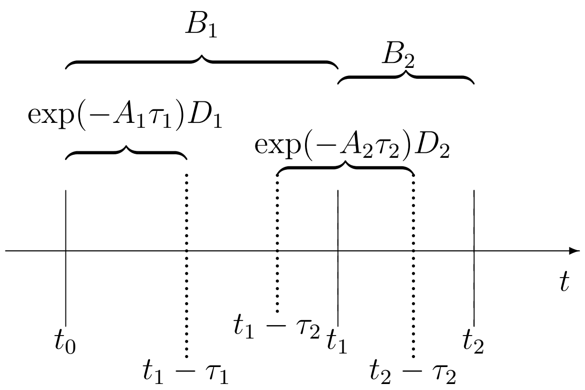

Fig. 1 If $\tau_1 > \tau_2$

For $\tau_1 < \tau_2$, there are two cases: $t_0 < t_1 - \tau_2$ and $t_0 \geq t_1 - \tau_2$. For $t_0 < t_1 - \tau_2$ (see Fig 2), we can divide the integral interval into five parts, i.e,

$$
\begin{aligned}
\mathcal{C}(\pi) &= e^{(A_2h_2)}\Big\{x|x=\int_{t_0}^{t_1-\tau_2}\exp[A_1(t_1-s)](B_1+\exp[-A_1\tau_1]D_1)u(s)ds, \forall \text{ p.c. } u\Big\}\\
&\quad + e^{(A_2h_2)}\Big\{x|x=\int_{t_1-\tau_2}^{t_1-\tau_1}\Big\{\exp[A_1(t_1-s)](B_1+\exp[-A_1\tau_1]D_1)\\
&\qquad + \exp[A_2(t_1-s)]\exp[-A_2\tau_2]D_2\Big\}u(s)ds, \forall \text{ p.c. } u\Big\}\\
&\quad + e^{(A_2h_2)}\Big\{x|x=\int_{t_1-\tau_1}^{t_1}\Big\{e^{[A_1(t_1-s)]}B_1+e^{[A_2(t_1-s)]}e^{[-A_2\tau_2]}D_2\Big\}u(s)ds, \forall \text{ p.c. } u\Big\}\\
&\quad + \Big\{x|x=\int_{t_1}^{t_2-\tau_2}\exp[A_2(t_2-s)](B_2+\exp[-A_2(\tau_2)]D_2u(s))ds, \forall \text{ p.c. } u\Big\}\\
&\quad + \Big\{x|x=\int_{t_2-\tau_2}^{t_2}\exp[A_2(t_2-s)]B_2u(s)ds, \forall \text{ p.c. } u\Big\}\\
&= \exp(A_2h_2)\Big\{\Big\langle A_1|(B_1+\exp[-A_1\tau_1]D_1)\Big\rangle + \Big\langle A_1|(B_1+\exp[-A_1\tau_1]D_1)\\
&\qquad + A_2|\exp[-A_2\tau_2]D_2\Big\rangle + \Big\langle A_1|B_1+A_2|\exp[-A_2\tau_2]D_2\Big\rangle\Big\}\\
&\quad + \langle A_2|B_2+\exp[-A_2\tau_2]D_2\rangle + \langle A_2|B_2\rangle\\
&= \exp(A_2h_2)\langle A_1|[B_1,D_1]\rangle + \langle A_2|[B_2,D_2]\rangle
\end{aligned} \tag{76}
$$

For $t_0 \geq t_1-\tau_2$ (see Fig 3), we can divide the integral interval into four parts, i.e,

$$
\begin{aligned}
\mathcal{C}(\pi) &= \exp(A_2h_2)\Big\{x|x=\int_{t_0}^{t_1-\tau_1}\Big\{\exp[A_1(t_1-s)](B_1+\exp[-A_1\tau_1]D_1)\\
&\qquad + \exp[A_2(t_1-s)]\exp[-A_2\tau_2]D_2\Big\}u(s)ds, \forall\ u\Big\}\\
&\quad + e^{(A_2h_2)}\Big\{x|x=\int_{t_1-\tau_1}^{t_1}\Big\{e^{[A_1(t_1-s)]}B_1+e^{[A_2(t_1-s)]}e^{[-A_2\tau_2]}D_2\Big\}u(s)ds, \forall\ u\Big\}\\
&\quad + \Big\{x|x=\int_{t_1}^{t_2-\tau_2}\exp[A_2(t_2-s)](B_2+\exp[-A_2(\tau_2)]D_2u(s))ds, \forall\ u\Big\}\\
&\quad + \Big\{x|x=\int_{t_2-\tau_2}^{t_2}\exp[A_2(t_2-s)]B_2u(s)ds, \forall\ u\Big\}\\
&= \exp(A_2h_2)\Big\{\langle A_1|(B_1+\exp[-A_1\tau_1]D_1)+A_2|\exp[-A_2\tau_2]D_2\rangle\\
&\quad + \langle A_1|B_1+A_2|\exp[-A_2\tau_2]D_2\rangle\Big\}\\
&\quad + \langle A_2|B_2+\exp[-A_2\tau_2]D_2\rangle + \langle A_2|B_2\rangle\\
&= \exp(A_2h_2)\langle A_1|[B_1,D_1]\rangle + \langle A_2|[B_2,D_2]\rangle
\end{aligned} \tag{77}
$$

Thus, for every case, we get the same result (73). □

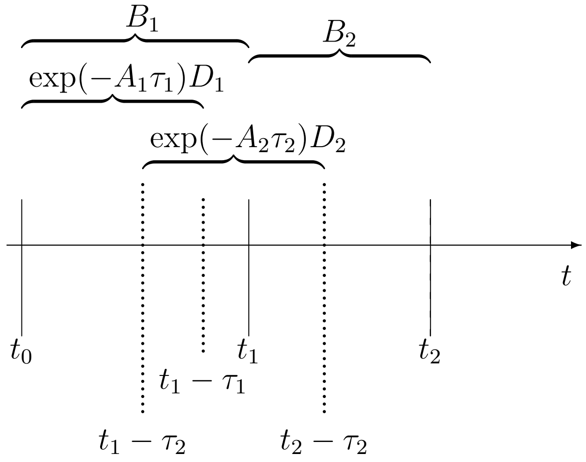

Fig. 2 If $\tau_1 < \tau_2$, $t_0 < t_1 - \tau_2$

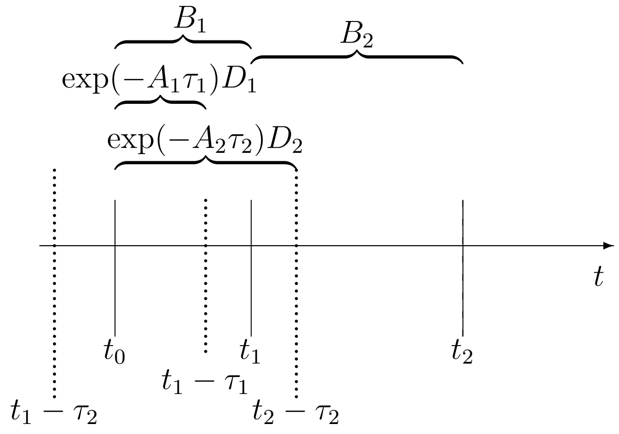

Fig. 3 If $\tau_1 < \tau_2$, $t_0 \geq t_1 - \tau_2$

Corollary 6. *System (72)and system (55)have the same controllability, i.e.,*

1)For periodic-type case, system(72) is 1-periodic controllable (m-periodic controllable) if and only if system (55) is 1-periodic controllable(m-periodic controllable).

2)For aperiodic-type case, system (72) is controllable if and only if system (55) is controllable .

Remark 11. *For system with multiple time-delays, we have similar conclusions, i.e, controllability is independent of the size of time-delays.*

6 Conclusion

The controllability for switched linear systems with time-delay in control has been first formulated and investigated. A sufficient and necessary condition for controllability of periodically switched linear systems has been presented. Furthermore, it is proved that the controllability can be realized in $n+1$ periods at most. An example illustrate the above results. We have also presented some further results in Section 5.

Appendix A

Proof of Lemma 1. For $m = 1, \cdots, N$, we have

$$A_m^n = \sum_{i=0}^{n-1} \alpha_{m,i} A_m^i \tag{78}$$

Denote $f_m(x) = x^n - \sum_{i=0}^{n-1} \alpha_{m,i} x^i$, $f(x) = \prod_{m=1}^{N} f_m(x)$. Suppose that $f(x) = x^{Nn} - \sum_{i=0}^{Nn-1} \alpha_i x^i$, then for $m = 1, \cdots, N$, we have

$$A_m^{Nn} = \sum_{i=0}^{Nn-1} \alpha_i A_m^i \tag{79}$$

This implies that for any $m > Nn - 1$

$$\mathcal{R}(A_1^m B_1 + \cdots + A_N^m B_N) \subseteq \sum_{i=0}^{Nn-1} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N) \tag{80}$$

Thus, $\langle A_1 | B_1 + \cdots + A_N | B_N \rangle = \sum_{i=0}^{Nn-1} \mathcal{R}(A_1^i B_1 + \cdots + A_N^i B_N)$. □

Appendix B

Proof of Lemma 2. First, we have

$$\begin{aligned}
& \{x | x = \sum_{i=1}^{N} \int_{t_0}^{t_f} \exp[A_i(t_f - s)] B_i u(s) ds, \forall \text{ p.c. } u\} \\
= \; & \{x | x = \sum_{i=1}^{N} \int_{t_0}^{t_f} \sum_{m=0}^{\infty} \frac{[A_i(t_f - s)]^m}{m!} B_i u(s) ds, \forall \text{ p.c. } u\} \\
= \; & \{x | x = \sum_{m=0}^{\infty} \int_{t_0}^{t_f} \frac{(t_f - s)^m}{m!} u(s) ds \sum_{i=1}^{N} A_i^m B_i, \forall \text{ p.c. } u\} \\
\subseteq \; & \sum_{m=0}^{\infty} \mathcal{R}\left(\sum_{i=1}^{N} (A_i)^m B_i \right) \\
= \; & \langle A_1 | B_1 + \cdots + A_N | B_N \rangle
\end{aligned} \tag{81}$$

Secondly, let $h = t_f - t_0$, then we have

$$\begin{aligned}
& \{x | x = \textstyle\sum_{i=1}^{N} \int_{t_0}^{t_f} \exp[A_i(t_f - s)] B_i u(s) ds, \\
& \quad \forall \text{ p.c. } u\} \\
= \; & \{x | x = \textstyle\sum_{i=1}^{N} \int_0^h \exp[A_i(h - t)] B_i u(t) dt, \\
& \quad \forall \text{ p.c. } u\}
\end{aligned}$$

Now we prove

$$\begin{aligned}
& \{x | x = \textstyle\sum_{i=1}^{N} \int_0^h \exp[A_i(h - t)] B_i u(t) dt, \forall \text{ p.c. } u\} \\
& \supseteq \langle A_1 | B_1 + \cdots + A_N | B_N \rangle
\end{aligned} \tag{82}$$

Consider the matrix

$$W = \int_0^h \Big\{ \sum_{i=1}^{N} e^{A_i(h-s)} B_i \Big\} \Big\{ \sum_{i=1}^{N} e^{A_i(h-s)} B_i \Big\}^T ds \tag{83}$$

Since $W^T = W$ and it is positive semi-definite, we have

$$\mathcal{R}(W) = \mathcal{N}(W)_{\perp}$$

where $\mathcal{N}(W)$ is the null space of matrix, which is defined as

$$\mathcal{N}(W) = \{x | Wx = 0\} \tag{84}$$

and "$\mathcal{V}_{\perp}$" denotes the orthogonal complement space of the subspace $\mathcal{V}$.

Furthermore,

$$\begin{aligned}
& y \in \mathcal{N}(W) \\
\Leftrightarrow \; & y^T W y = 0 \\
\Leftrightarrow \; & \int_0^h y^T \Big\{ \sum_{i=1}^{N} e^{A_i(h-s)} B_i \Big\} \Big\{ \sum_{i=1}^{N} e^{A_i(h-s)} B_i \Big\}^T y ds = 0 \\
\Leftrightarrow \; & \Big\{ \sum_{i=1}^{N} e^{A_i(h-s)} B_i \Big\}^T y = 0, \quad 0 \leq s \leq h
\end{aligned} \tag{85}$$

By(85), we know that any order derivations of $\left\{\sum_{i=1}^N \exp[A_i(h-s)]B_i\right\}^T y$ to s should be 0 at $s=h$, i.e.,

$$\begin{array}{l}\left\{\sum_{i=1}^N B_i^T\right\}y=0,\ \left\{\sum_{i=1}^N B_i^T A_i^T\right\}y=0,\\ \cdots,\left\{\sum_{i=1}^N B_i^T\left(A_i^T\right)^{Nn-1}\right\}y=0\end{array} \tag{86}$$

It follows that

$$\begin{aligned} y &\in \mathcal{N}\left(\sum_{i=1}^N B_i^T\right)\bigcap\mathcal{N}\left(\sum_{i=1}^N B_i^T A_i^T\right)\\ &\quad\bigcap\cdots\bigcap\mathcal{N}\left(\sum_{i=1}^N B_i^T\left(A_i^T\right)^{Nn-1}\right)\\ &= \left[\mathcal{R}\left(\sum_{i=1}^N B_i\right)+\mathcal{R}\left(\sum_{i=1}^N A_iB_i\right)\right.\\ &\quad\left.+\cdots+\mathcal{R}\left(\sum_{i=1}^N (A_i)^{Nn-1}B_i\right)\right]_{\perp}\\ &= \langle A_1|B_1+\cdots+A_N|B_N\rangle_{\perp}\end{aligned}$$

Conversely, it can be proved that if $y\in\langle A_1|B_1+\cdots+A_N|B_N\rangle_{\perp}$, then (85)holds, i.e., $y\in\mathcal{N}(W)$. Thus, we have

$$\mathcal{N}(W)=\langle A_1|B_1+\cdots+A_N|B_N\rangle_{\perp}$$

or equivalently

$$\mathcal{R}(W)=\langle A_1|B_1+\cdots+A_N|B_N\rangle \tag{87}$$

For any $x\in\langle A_1|B_1+\cdots+A_N|B_N\rangle$, by (87), there exists z such that $x=Wz$. Now let

$$u(s)=\left\{\sum_{i=1}^N B_i^T\left(\exp[A_i(h-s)]\right)^T\right\}z,\quad s\in[0,h]$$

then

$$\begin{aligned} x &= Wz\\ &= \int_0^h\left\{\sum_{i=1}^N e^{A_i(h-s)}B_i\right\}\left\{\sum_{i=1}^N B_i^T e^{A_i^T(h-s)}\right\}zds\\ &= \int_0^h\left\{\sum_{i=1}^N e^{A_i(h-s)}B_i\right\}u(s)ds\\ &= \sum_{i=1}^N\int_0^h e^{A_i(h-s)}B_iu(s)ds\end{aligned}$$

It implies (82). Based on (81) and (82), we can see that the lemma holds. □

Appendix C

Proof of Lemma 3. For any nonsingular matrix $P\in\Re^{p\times p}$, we have

$$\mathcal{R}(BP)=\{BPy|y\in\Re^p\}=\{Bz|z=Py,y\in\Re^p\}$$

It follows that $\mathcal{R}(BP)\subseteq\mathcal{R}(B)$. On the other hand, $\mathcal{R}(B)=\mathcal{R}[(BP)P^{-1}]\subseteq\mathcal{R}(BP)$. Then we have $\mathcal{R}(BP)=\mathcal{R}(B)$. Therefore,

$$\begin{aligned}\langle A|BP\rangle &= \mathcal{R}(BP)+\mathcal{R}(ABP)+\cdots+\mathcal{R}(A^{n-1}BP)\\ &= \mathcal{R}(B)+\mathcal{R}(AB)+\cdots+\mathcal{R}(A^{n-1}B)\\ &= \langle A|B\rangle\end{aligned}$$

□

Appendix D

Proof of Lemma 4. It's easy to prove that $\exp(Ah)\langle A|B\rangle\subseteq\langle A|B\rangle$. Since $\exp(Ah)$ is nonsingular, we have

$$\dim(\exp(Ah)\langle A|B\rangle)=\dim(\langle A|B\rangle)$$

Hence, (12) holds. For any positive integer m, we have

$$\mathcal{R}(A^m\exp(Ah)B)=\mathcal{R}(\exp(Ah)A^mB)=\exp(Ah)\mathcal{R}(A^mB)$$

Thus,

$$\begin{aligned}\langle A|\exp(Ah)B\rangle &= \sum_{m=0}^{n-1}\mathcal{R}(A^m\exp(Ah)B)\\ &= \exp(Ah)\sum_{m=0}^{n-1}\mathcal{R}(A^mB)\\ &= \exp(Ah)\langle A|B\rangle\\ &= \langle A|B\rangle\end{aligned}$$

□

Appendix E

Proof of Lemma 5. For any positive integer m, we have

$$\begin{aligned}&\mathcal{R}(A_1^mB_1+A_2^mB_2)+\mathcal{R}(A_2^mB_2)\\ &=\mathcal{R}([A_1^mB_1+A_2^mB_2,A_2^mB_2])\\ &=\mathcal{R}([A_1^mB_1,A_2^mB_2]\begin{bmatrix} I & 0\\ I & I\end{bmatrix})\\ &=\mathcal{R}([A_1^mB_1,A_2^mB_2])\end{aligned}$$

Thus,

$$\begin{aligned}&\langle A_1|B_1+A_2|B_2\rangle+\langle A_2|B_2\rangle\\ &=\sum_{m=0}^{2n-1}\left(\mathcal{R}(A_1^mB_1+A_2^mB_2)+\mathcal{R}(A_2^mB_2)\right)\\ &=\sum_{m=0}^{2n-1}\left(\mathcal{R}(A_1^mB_1)+\mathcal{R}(A_2^mB_2)\right)\\ &=\langle A_1|B_1\rangle+\langle A_2|B_2\rangle\end{aligned}$$

□

Appendix F

Proof of Lemma 6. Suppose $A^n=\sum_{i=1}^n\alpha_iA^{i-1}$, $\exp(At)=\sum_{i=1}^n\lambda_i(t)A^{i-1}$.

First, considering the derivation of $\exp(At)$ to t, we have

$$\begin{aligned} d\exp(At)/dt &= A\exp(At)\\ &= A\left(\sum_{i=1}^n\lambda_i(t)A^{i-1}\right)\\ &= \sum_{i=1}^n\lambda_i(t)A^i\end{aligned}$$

It follows that

$$\sum_{i=1}^n\lambda_i(t)A^i=\sum_{i=1}^n\dot{\lambda}_i(t)A^{i-1}$$

Since $A^n=\sum_{i=1}^n\alpha_iA^{i-1}$, let

$$\left\{\begin{array}{lcl}\dot{\lambda}_1(t) &=& \alpha_1\lambda_n(t)\\ \dot{\lambda}_2(t) &=& \lambda_1(t)+\alpha_2\lambda_n(t)\\ &\vdots&\\ \dot{\lambda}_n(t) &=& \lambda_{n-1}(t)+\alpha_n\lambda_n(t)\end{array}\right. \tag{88}$$

and

$$\begin{bmatrix}\lambda_1(0)\\ \lambda_2(0)\\ \vdots\\ \lambda_n(0)\end{bmatrix}=\begin{bmatrix}1\\ 0\\ \vdots\\ 0\end{bmatrix}. \tag{89}$$

Equation (88) can be rewritten in matrix form

$$\begin{bmatrix}\dot{\lambda}_1(t)\\ \dot{\lambda}_2(t)\\ \vdots\\ \dot{\lambda}_n(t)\end{bmatrix}=\vec{A}\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \vdots\\ \lambda_n(t)\end{bmatrix} \tag{90}$$

where

$$\vec{A}=\begin{bmatrix}0 & \cdots & 0 & \alpha_1\\ 1 & & & \alpha_2\\ & \ddots & & \vdots\\ & & 1 & \alpha_n\end{bmatrix} \tag{91}$$

Then the solution of the initial value problem (89) and (90) is

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \\ \vdots \\ \lambda_n(t) \end{bmatrix} = exp\left(\vec{A}t\right) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{92}$$

Secondly, consider the linear system

$$\dot{y}(t) = \vec{A}y(t) + \vec{b}v(t) \tag{93}$$

where $\vec{A}$ is defined as (91), and $\vec{b} = [1, 0, \cdots, 0]^T$. Obviously, system (93) is completely controllable. According to the discretization theory of continuous systems, for almost all $T > 0$, the following discrete system

$$y(k+1) = Hy(k) + Dv(k) \tag{94}$$

is complete controllable, where

$$H = \exp(\vec{A}T), \qquad D = \left(\int_0^T \exp(\vec{A}t)dt\right)\vec{b}. \tag{95}$$

Thus, the matrix $[D, HD, \cdots, H^{n-1}D]$ is nonsingular. Notice that the matrix

$$\begin{array}{l} [D, HD, \cdots, H^{n-1}D] \\ = \left(\int_0^T \exp(\vec{A}t)dt\right)[\vec{b}, H\vec{b}, \cdots, H^{n-1}\vec{b}] \end{array} \tag{96}$$

and the matrix $\int_0^T \exp(\vec{A}t)dt$ are both nonsingular. So the matrix $[\vec{b}, H\vec{b}, \cdots, H^{n-1}\vec{b}]$ is also nonsingular.

Thirdly, consider the following equation

$$(\exp(AT))^m = \sum_{i=1}^{n} \lambda_i(mT)A^{i-1} \tag{97}$$

for $m = 0, 1, \cdots, n-1$, where T is taken as above. Equation (97) can be rewritten as

$$\begin{bmatrix} I \\ \exp(AT) \\ \vdots \\ (\exp(AT))^{n-1} \end{bmatrix} = (\Xi \otimes I) \cdot \begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} \tag{98}$$

where

$$\Xi = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \lambda_1(T) & \lambda_2(T) & \cdots & \lambda_n(T) \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1((n-1)T) & \lambda_2((n-1)T) & \cdots & \lambda_n((n-1)T) \end{bmatrix} \tag{99}$$

Here $\otimes$ denotes the Kronecker-product of matrices.

By (92), it's easy to prove that

$$\Xi = [\vec{b}, H\vec{b}, \cdots, H^{n-1}\vec{b}]^T \tag{100}$$

which shows that Ξ is non-singular. By the definition of Kronecker-product, $(\Xi \otimes I)^{-1} = \Xi^{-1} \otimes I$. Thus, we have

$$\begin{bmatrix} I \\ A \\ \vdots \\ A^{n-1} \end{bmatrix} = (\Xi^{-1} \otimes I) \cdot \begin{bmatrix} I \\ \exp(AT) \\ \vdots \\ (\exp(AT))^{n-1} \end{bmatrix} \tag{101}$$

Denote $\Xi^{-1} = [\xi_{ij}]_{n\times n}$, we have

$$A^{i-1} = \sum_{j=1}^{n} \xi_{ij}(\exp(AT))^{j-1}. \tag{102}$$

for $i = 1, \cdots, n$.

Finally, for any linear space $\mathcal{W} \subseteq \Re^n$, we have

$$A^{i-1}\mathcal{W} = \sum_{j=1}^{n} \xi_{ij}(\exp(AT))^{j-1}\mathcal{W} \subseteq \langle \exp(AT)|\mathcal{W}\rangle \tag{103}$$

for $i = 1, \cdots, n$.

Hence, we have $\langle A|\mathcal{W}\rangle \subseteq \langle \exp(AT)|\mathcal{W}\rangle$. Since $\langle A|\mathcal{W}\rangle \supseteq \langle \exp(AT)|\mathcal{W}\rangle$, $\langle A|\mathcal{W}\rangle = \langle \exp(AT)|\mathcal{W}\rangle$ holds. □

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