Functions of system and their perturbations

Alexey S. Tikhonov
Department of Informatics and Mathematics
Taurida National University
4, Yaltinskaya str.
Simferopol, Crimea 95007
Ukraine

Abstract

In this paper, we introduce a notion of function for a conservative linear discrete-time-invariant system. We show that functions of systems naturally appear for problems related to functional models, allow an informative definition of the transfer function, have interesting applications to perturbation theory, scattering theory, and problems related to factorizations of operator-valued functions.

0 Introduction

Functional calculuses for operators are well known. For example, if T_0 is a comletely nonunitary contraction acting in a separable Hilbert space H and φ is bounded, analytic on the unit disc $\mathbb{D}$ function, then, according to the S.-Nagy-Foias functional calculus [1], one can define the function $\varphi(T_0)$ of the operator T_0.

Now suppose that $\mathfrak{A}_0 = \begin{bmatrix} T_0 & N_0 \\ M_0 & L_0 \end{bmatrix} \in [H \oplus \mathfrak{N}_{in}, H \oplus \mathfrak{N}_{out}]$ is a simple unitary node [2] (this means that $\mathfrak{A}_0^*\mathfrak{A}_0 = I,\ \mathfrak{A}_0\mathfrak{A}_0^* = I$). The basic operator T_0 is a completely nonunitary contraction. The conservative linear discrete-time-invariant system (CLDTIS) [3]

$$\begin{array}{lll} x(n+1) & = T_0\, x(n) & + N_0\, \varphi_{in}(n) \\ \varphi_{out}(n) & = M_0\, x(n) & + L_0\, \varphi_{in}(n) \qquad (\, n = 0, 1, 2, \cdots) \end{array}$$

can be assigned to the node $\mathfrak{A}_0$. In the sequel, we consider systems with $\mathfrak{N}_{in} = \mathfrak{N}_{out} = \mathfrak{N}$.

In this context, it is reasonable to ask some questions. *What is a function of the system? What class of functions is admissible?*

In the case of linear-fractional functions $\varphi(z) = (z-a)/(1-\bar{a}z),\ |a| < 1$ the answer is clear. The function of $\mathfrak{A}_0$ is CLDTIS $\mathfrak{A}_a = \begin{bmatrix} T_a & N_a \\ M_a & L_a \end{bmatrix}$, where

$$\begin{array}{ll} T_a = (T_0 - a)(I - \bar{a}T_0)^{-1}, & N_a = (1-|a|^2)^{1/2}(I - \bar{a}T_0)^{-1}N_0, \\ M_a = (1-|a|^2)^{1/2}M_0(I - \bar{a}T_0)^{-1}, & L_a = L_0 + \bar{a}M_0(I - \bar{a}T_0)^{-1}N_0. \end{array}$$

Note that $T_a = \varphi(T_0),\ N_a = (\sqrt{\varphi'})(T_0)N_0,\ M_a = M_0(\sqrt{\varphi'})(T_0)$. In this connection we can define the tranfomation

$$(\, T_0,\ M_0,\ N_0\,) \ \mapsto\ (\, \varphi(T_0),\ M_0(\sqrt{\varphi'})(T_0),\ (\sqrt{\varphi'})(T_0)N_0\,).$$

We shall say that $(\varphi(T_0),\ M_0(\sqrt{\varphi'})(T_0),\ (\sqrt{\varphi'})(T_0)N_0)$ is the function of the short system $(T_0,\ M_0,\ N_0)$. The aim of this paper is to show that this definition is reasonable. We consider the class of all functions that are conformal maps of the unit disc onto domains bounded by $C^{2+\varepsilon}$-smooth, simple, closed curves. For this class of functions, we show that functions of system 1) naturally appear for problems related to functional models; 2) allow an informative definition of the transfer function; 3) have interesting applications to perturbation theory, scattering theory, and problems related to factorizations of operator-valued functions.

1 Functional model

Let $\mathcal{H}, \mathfrak{N}$ be separable Hilbert spaces, $\mathbb{T}$ be the unit circle. Consider maps $\pi_\pm \in [L^2(\mathbb{T},\mathfrak{N}),\mathcal{H}]$ such that the following conditions hold

$$(i)\ \pi_\pm^*\pi_\pm = I\,; \quad (ii)\ P_-\pi_-^*\pi_+P_+ = 0\,; \quad (iii)\ \operatorname{Ran}\pi_+ \vee \operatorname{Ran}\pi_- = \mathcal{H}\,.$$

Here P_+ is the orthoprojector on the vector Hardy space

$$H^2(\mathfrak{N}) = \{\, f \in L^2(\mathbb{T},\mathfrak{N}) : \int_0^{2\pi} f(e^{it})\, e^{-int}\, dt = 0,\ \ n<0\,\},$$

P_- is the orthoprojector on the orthogonal complement $H^2_-(\mathfrak{N}) = L^2(\mathbb{T},\mathfrak{N}) \ominus H^2(\mathfrak{N})$. Note that functions from $H^2(\mathfrak{N})$ and $H^2_-(\mathfrak{N})$ are boundary values of functions that are analytic in the interior and the exterior of the unit circle respectively. In the space $\mathcal{H}$ we define the subspace

$$\mathcal{K}_\Theta = \{\, f \in \mathcal{H} :\ P_+\pi_+^* f = 0,\ \ P_-\pi_-^* f = 0\,\}$$

and operators $\widehat{T} \in [\mathcal{K}_\Theta]$, $\widehat{M} \in [\mathcal{K}_\Theta,\mathfrak{N}]$, $\widehat{N} \in [\mathfrak{N},\mathcal{K}_\Theta]$, where

$$\widehat{T}f = Uf - \pi_+\widehat{M}f,\ \ \widehat{M}f = (\pi_+^*Uf)(\infty),\ \ \widehat{N}n = (I - \pi_+P_+\pi_+^*)\pi_-n,$$

Here $f \in \mathcal{K}_\Theta$, $n \in \mathfrak{N}$ and U is the unitary operator acting in $\mathcal{H}$ that is uniquely determined by conditions $U\pi_\pm = \pi_\pm z$, where z is the multiplication operator by independent variable. The construction, which is given above, is a functional model of S.-Nagy-Foias [1] for unitary nodes [2] in the coordinate-free form [4]. It can easily be checked that $\begin{bmatrix} \widehat{T} & \widehat{N} \\ \widehat{M} & (\pi_-^*\pi_+)(0) \end{bmatrix}$ is a simple unitary node. Conversely, for any simple unitary node there exists a model node that is unitarily equivalent to the given node.

The construction of S.-Nagy-Foias's functional model can easily be extended to the case of $C^{2+\varepsilon}$-smooth, simple, closed curve C. In the same way, we consider maps $\pi_\pm \in [L^2(C,\mathfrak{N}),\mathcal{H}]$ that satisfies the conditions (i), (ii), (iii). But here $P_\pm$ are (nonorthogonal) projectors on Smirnov spaces $E^2(G_\pm,\mathfrak{N})$, where $G_+ = Int(C)$, $G_- = Ext(C)$. Smirnov spaces are a natural generalization of Hardy spaces to the case of an arbitrary closed curve. A new problem arises now. We need to describe all such triples of operators $(\widehat{T},\ \widehat{M},\ \widehat{N})$. The following observation is the key for this problem.

Let $\varphi \in CM(G_{1+}, G_{2+})$, $\pi_{2\pm} = \pi_{1\pm} C_\varphi$. Then

$$\widehat{T}_2 = Z\varphi(\widehat{T}_1)Z^{-1}, \quad \widehat{M}_2 = \widehat{M}_1(\sqrt{\varphi'})(\widehat{T}_1)Z^{-1}, \quad \widehat{N}_2 = Z(\sqrt{\varphi'})(\widehat{T}_1)\widehat{N}_1,$$

where $(C_\varphi f(\cdot))(z) = \sqrt{\varphi'(z)}\, f(\varphi(z))$, $Z = (I - \pi_{2+}P_{2+}\pi_{2+}^*)(I - \pi_{2-}P_{2-}\pi_{2-}^*)|\mathcal{K}_{1\Theta},$. We write $\varphi \in CM(G_{1+}, G_{2+})$ if φ is a conformal map of $G_{1\pm}$ onto $G_{2\pm}$.

Thus, along with functional models, we consider their transformations $\pi_{2\pm} = \pi_{1\pm} C_\varphi,$. Then we can introduce the category of all fucntional models Mod. Objects in this category are pairs $\Pi = (\pi_+, \pi_-)$. Morphisms $\pi_{2\pm} = \pi_{1\pm} C_\varphi$ are determinded by conformal maps $\varphi \in CM(G_{1+}, G_{2+})$. We shall denote these morphisms by m_φ.

Theorem 1.1. *The transformation* $\Pi \mapsto (\widehat{T},\ \widehat{M},\ \widehat{N})$ *determines a covariant functor* $\mathcal{F}_{ms}$ *from the category* Mod *to the category* Sys.

For the category Sys, we put

$$\begin{aligned} \mathrm{Ob}(Sys) \;=\; & \{ (T, M, N) : \exists\, \varphi \in CM(\mathbb{D}, G_+),\, \exists\, Z \in [H_0, H],\ Z^{-1} \in [H, H_0], \\ & \exists\ \mathfrak{A}_0 = \begin{bmatrix} T_0 & N_0 \\ M_0 & L_0 \end{bmatrix} \in [H_0 \oplus \mathfrak{N}],\ T_0\ c.n.u.,\ \mathfrak{A}_0^*\mathfrak{A}_0 = \mathfrak{A}_0\mathfrak{A}_0^* = I, \\ & T = Z\varphi(T_0)Z^{-1},\ M = M_0\sqrt{\varphi'}(T_0)Z^{-1},\ N = Z\sqrt{\varphi'}(T_0)N_0 \} \end{aligned}$$

(here c.n.u. = completely nonunitary). We shall say that $s_{\varphi,z}$ is morphism in the category Sys and write $s_{\varphi,z} \in \mathrm{Mor}(\alpha_1, \alpha_2)$ if

$$T_2 = Z\varphi(T_1)Z^{-1}, \quad M_2 = M_1(\sqrt{\varphi'})(T_1)Z^{-1}, \quad N_2 = Z(\sqrt{\varphi'})(T_1)N_1,$$

where $\alpha_1 = (T_1,\ M_1,\ N_1)$, $\alpha_2 = (T_2,\ M_2,\ N_2)$, $\varphi \in CM(G_{1+}, G_{2+})$.

Note that we use the language of categories with the aim of more compact formulations. One assertion with categories is a set of several elementary assertions. Some of them are technical exercises. For example, to verify that Mod and Sys are categories we need to check that the composition of morphisms is also morphism. In other words, we need to check that

$$m_{\varphi_{32}} \circ m_{\varphi_{21}} = m_{\varphi_{31}}, \qquad s_{\varphi_{32},z_{32}} \circ s_{\varphi_{21},z_{21}} = s_{\varphi_{31},z_{31}}$$

for $\varphi_{21} \in CM(G_{1+}, G_{2+})$, $\varphi_{32} \in CM(G_{2+}, G_{3+})$, $\varphi_{31} = \varphi_{32} \circ \varphi_{21}$, $Z_{31} = Z_{32}Z_{21}$.
But some of them is the core of the problem. The assertion

$$\mathcal{F}_{ms}(m_{32} \circ m_{21}) = \mathcal{F}_{ms}(m_{32}) \circ \mathcal{F}_{ms}(m_{21})$$

and the mentioned above observation are principal in this section. Using morphisms we reduce the problem to the case of the unit disc.

The following Theorem together with Theorem 1.1 give the complete answer on our question (to describe $\mathcal{F}_{ms}(\mathrm{Ob}(Mod))$).

Theorem 1.2. $\forall\,(T,M,N)\in \mathrm{Ob}(Sys)\ \ \exists\,\Pi\in\mathrm{Ob}(Mod)\ \ \exists\, Z\in[H,\mathcal{K}_\Theta]\ :$
$Z^{-1}\in[\mathcal{K}_\Theta,H],\ ZT=\hat{T}Z,\ M=\hat{M}Z,\ ZN=\hat{N}$ *, where* $(\hat{T},\hat{M},\hat{N})=\mathcal{F}_{ms}(\Pi)$.

This means that for any function of system there exists a linear-similar functional model.

2 Transfer functions

Let $\alpha=(T,M,N)\in\mathrm{Ob}(Sys)$. We shall consider the short transfer function

$$\Upsilon(z)=M(T-z)^{-1}N,\qquad z\in\rho(T).$$

As we shall show below, for a large enough class of systems, the transfer function $\Upsilon(z)$ carry the same information related to the system as the regular transfer function

$$W(z)=L_0+zM_0(I-zT_0)^{-1}N_0,\ \ 1/z\in\rho(T_0)$$

do for the system $\mathfrak{A}_0$. The subcategory under our interest is the category Sys_1 :

$$\mathrm{Ob}(Sys_1)=\{\,(T,M,N)\in\mathrm{Ob}(Sys):\ M,N\in\mathfrak{S}_2,\ \rho(T)\cap G_+\neq\emptyset\,\},$$

where $\mathfrak{S}_2$ is the Hilbert-Schmidt class, $\rho(T)$ is the set of all regular points for the operator T (complement to the spectrum $\sigma(T)$). For the category Sys_1 , the short transfer function $\Upsilon(z)$ possesses for a.e. $z\in C$ angular boundary values $\Upsilon_\pm(z)$ from the interior and the exterior of C respectively.

Our next problem is to describe short transfer functions for the category Sys_1. The following fact plays the key role in the solution of this problem.

Let $\alpha_1,\ \alpha_2\in\mathrm{Ob}(Sys_1),\ s_{\varphi,z}\in\mathrm{Mor}(\alpha_1,\ \alpha_2)$. Then for a.e. $z\in C_2$ we have

$$(\Upsilon_{2+}-\Upsilon_{2-})(z)=((\Upsilon_{1+}-\Upsilon_{1-})\circ\varphi^{-1})(z),\quad \Upsilon_{2-}(z)=(P_{2-}(\Upsilon_{1-}\circ\varphi^{-1}))(z).$$

Note also that in the case $G_+=\mathbb{D}$ the short transfer function can be expressed in terms of the regular transfer fanction

$$\Upsilon=\Phi(W)\ \ :\qquad \Upsilon(1/z)=W(0)-W(z),\quad 1/z\in\rho(T).$$

Conversely, the operator $L_0=W(0)$ can be expressed (up to unessential isometric part) in terms of the short transfer function

$$L_0=U|L_0|,\quad |L_0|=\chi(\Upsilon(0)^*\Upsilon(0)),\quad U=\Upsilon(0)(\,|L_0|-|L_0|^{-1})^{-1},$$

where $\chi(z)=(1/\sqrt{2})\sqrt{2+z-\sqrt{z^2+4z}}$. There is no loss of generality in assuming that there exists L_0^{-1}. Thus we can consider the inverse transformation $W=\Psi^{-1}(\Upsilon)$.

Define the transformation $\Upsilon_2 = \Phi(\Upsilon_1, \varphi)$ by formulas

$$\Upsilon_{2-} = P_{2-}\Upsilon_{1-} \circ \varphi^{-1}, \qquad \Upsilon_{2+} = \Upsilon_{2-} + (\Upsilon_{1+} - \Upsilon_{1-}) \circ \varphi^{-1},$$

where $\varphi \in CM(G_{1+}, G_{2+})$. In this connection, we would like to underline here that the map $\Upsilon_{1-} \mapsto P_{2-}\Upsilon_{1-} \circ \varphi^{-1}$ is analogue of the well-known Faber transform in complex analysis.

Now we can define (formally independently of the category Sys_1) the category of all short transfer functions Tfn_1:

$$\begin{aligned}&\mathrm{Ob}(Tfn_1) = \{\, \Upsilon : \exists\, \varphi \in CM(\mathbb{D}, G_+) : \ \ \Psi^{-1}(\Phi(\Upsilon, \varphi^{-1})) \in S_1(\mathbb{D}, [\mathfrak{N}])\,\},\\ &t_\varphi \in \mathrm{Mor}(\Upsilon_1, \Upsilon_2) \quad if \quad \Upsilon_2 = \Phi(\Upsilon_1, \varphi), \quad \varphi \in CM(G_{1+}, G_{2+}).\end{aligned}$$

Here $S_1(\mathbb{D}, [\mathfrak{N}])$ is the class of all analytic on $\mathbb{D}$ operator-valued functions $W(z)$ such that

$$\sup_{|z|<1} ||W(z)|| \le 1, \qquad \exists\, W(0)^{-1}, \qquad I - W(0)^* W(0) \in \mathfrak{S}_1,$$

where $\mathfrak{S}_1$ is the trace class.

Finally, we define the covariant functor $\mathcal{F}_{st}$, which acts from the category Sys_1 to the category Tfn_1 :

$$\mathcal{F}_{st}(T,\ M,\ N) = M(T-z)^{-1}N, \qquad \mathcal{F}_{st}(s_{\varphi,z}) = t_\varphi.$$

Theorem 2.1. *1) Sys_1 and Tfn_1 are categories; 2) $\mathcal{F}_{st}$ is a covariant functor; 3) $\mathcal{F}_{st}(\mathrm{Ob}(Sys_1)) = \mathrm{Ob}(Tfn_1)$.*

Note that a short transfer function $\Upsilon(z)$ determines the function of system $(T,\ M,\ N)$ up to similarity.

3 Applications to perturbation theory

In this section we consider some interesting applications. To that end we need to use some genereralization of functions for systems. We extend the category Sys to the category Sys^{ext}:

$$\begin{aligned}\mathrm{Ob}(Sys^{ext}) \;=\; \{\, (T,\ M,\ N) : \exists\, \eta, \psi \in A^{1+\varepsilon}(G_+),\ \ 1/\eta, 1/\psi \in A^{1+\varepsilon}(G_+),\\ (T,\ M\,\eta(T),\ \psi(T)\,N) \in \mathrm{Ob}(Sys)\,\},\end{aligned}$$

where $A^{1+\varepsilon}(G)$ is the class of analytic on the domain G functions such that their derivative is Hölder continuous for some $\varepsilon > 0$. For such functions of systems, we can construct the generalized functional model. The following nontrivial fact is the base for applications of this generalized model and functions of systems to perturbation theory.

Let U be a normal operator (i.e., $U^*U = UU^*$), C be a $C^{4+\varepsilon}$-smooth, simple, closed curve, $\sigma(U) \subset C,\ S - U \in \mathfrak{S}_1,\ \rho(S) \cap G_+ \neq \emptyset$. Then there exist a generalized functional model

$\Pi=(\pi_+,\pi_-)$ and operators $Z\in[H,\mathcal{K}_\Theta]$, $\varkappa\in[\mathfrak{N}]$ such that $S=Z^{-1}(\widehat{T}+\widehat{N}\varkappa\widehat{M})Z$, where $\widehat{M},\ \widehat{N}\in\mathfrak{S}_2$ and $(\widehat{T},\ \widehat{M},\ \widehat{N})=\mathcal{F}_{ms}(\Pi)$.

Extending S.-Nagy-Foias-Naboko functional model to our case, we establish the duality of spectral components for such operators S. Without entering into details we present here

Theorem 3.1. *Let* $(T,\ M,\ N)\in\mathrm{Ob}(Sys^{ext})$, $M,N\in\mathfrak{S}_2$, $S=T+N\varkappa M$. *Then*
1) $\mathrm{clos}\,(\widetilde{N}_+(S,M)\bigcap\widetilde{N}_-(S,M))\oplus M(S^*,N^*)=H$;
2) $D_\pm(S,M)\oplus\ \mathrm{clos}\,(\widetilde{N}_\mp(S^*,N^*)\bigcap M(S^*,N^*))=H$;
3) $N_\pm(S,M)\oplus\ \mathrm{clos}\,(\widetilde{D}_\mp(S^*,N^*)\bigcap M(S^*,N^*))=H$.

Here $M(S,M)$, $N_\pm(S,M)$, $D_\pm(S,M)$ are spectral components for the operator S:

$$\begin{aligned}
\widetilde{M}(S,M)\ &=\ \{\,f\in H:\ \gamma_+(f)=\gamma_-(f)\,\}, & M(S,M)&=\mathrm{clos}\,\widetilde{M}(S,M)\,;\\
\widetilde{N}_\pm(S,M)\ &=\ \{\,f\in H:\ \gamma_\pm(f)\in E^2(G_\pm,\mathfrak{N})\}, & N_\pm(S,M)&=\mathrm{clos}\,\widetilde{N}_\pm(S,M)\,;\\
\widetilde{D}_\pm(S,M)\ &=\ \{\,f\in H:\ \gamma_\pm(f)\in D^2(G_\pm,\mathfrak{N})\}, & D_\pm(S,M)&=\mathrm{clos}\,\widetilde{D}_\pm(S,M)\,,
\end{aligned}$$

where $\gamma_\pm(f)(z)$, $z\in C$ are angular boundary values of $M(S-z)^{-1}f$ from $G_\pm$ respectively and $D^2(G,\mathfrak{N})=\{\,u:u(z)=(1/\delta(z))g(z),\ g\in E^2(G,\mathfrak{N}),\ \delta\ \textit{is outer}\}$.

The duality of spectral components plays important role for non-selfajoint scattering theory [5, 6] and for extreme factorizations of J-contractive-valued functions (J-inner-outer, A-regular-singular, and other [7, 8, 9, 10]). These applications were the main stimulus for the author to develop the presented above notion of function for system.

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