Stability Analysis of 2-D Dynamics in Roesser's Model

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Abstract

State-space stability of linear shift-invariant discrete two-dimensional (2-D) dynamics is considered. An approach to the making of Lyapunov functions for 2-D dynamics is presented. It produces quadratic forms involving finite cross-terms among local states. The scope of the stability analysis based upon the notion of parallel stability expands, and the accuracy can be improved by this approach.

1. Introduction

This paper attempts to develop a state-space method for analyzing the stability of discrete-time linear dynamics in the Roesser's two-dimensional (2-D) system models. Consider the following difference equation with two arguments:

$$\begin{pmatrix} x_1(i+1,\, j) \\ x_2(i,\, j+1) \end{pmatrix} = A \begin{pmatrix} x_1(i,\, j) \\ x_2(i,\, j) \end{pmatrix}, \tag{1.1}$$

where $x_1(i,\, j)$ is a real n_1-vector and $x_2(i,\, j)$ is a real n_2-vector, both are indexed by a pair of integers $(i,\, j)$. The matrix A is a real $n \times n$ matrix, $n = n_1 + n_2$, and it is assumed to be itemized as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{1.2}$$

the partition of which is created so as to be compatible with that of (1.1). For $t \in \mathbb{Z}$, let us define a bilateral sequence of n-vectors:

$$\mathcal{X}_t = \left\{ \begin{pmatrix} x_1(i,\, j) \\ x_2(i,\, j) \end{pmatrix} ;\ i + j = t \right\}. \tag{1.3}$$

We regard (1.1) as a dynamical equation that describes the evolution of $\mathcal{X}_t$, which is equivalent to the dynamics of a discrete-time linear 2-D system on the Roesser's model [1]. An alternative expression of this dynamics is given in the Fornasini-Marchesini model [2] as follows:

$$x(i,\, j) = \tilde{A}_1 x(i-1,\, j) + \tilde{A}_2 x(i,\, j-1), \tag{1.4}$$

where

$$x(i,\, j) = \begin{pmatrix} x_1(i,\, j) \\ x_2(i,\, j) \end{pmatrix}, \quad \tilde{A}_1 = \begin{pmatrix} A_{11} & A_{12} \\ O & O \end{pmatrix}, \quad \text{and} \quad \tilde{A}_2 = \begin{pmatrix} O & O \\ A_{21} & A_{22} \end{pmatrix}. \tag{1.5}$$

Introduce the standard square-norm of a bilateral sequence $\mathcal{X}_t = \{x(i, t-i)\,;\, i \in \mathbb{Z}\}$ by $\|\mathcal{X}_t\|_2 = \left(\sum_{i=-\infty}^{\infty}\|x(i, t-i)\|_2^2\right)^{1/2}$, where the notation $\|\cdot\|_2$ used in the right hand side denotes the Euclidian norm of the n-vector involved. We denote by $\mathfrak{X}^n$ the linear space of these square summable bilateral sequences. Define a linear operator $\mathscr{S}_A$ on $\mathfrak{X}^n$ so that we can express the dynamics (1.1) by

$$\mathcal{X}_{t+1} = \mathscr{S}_A \mathcal{X}_t\,. \tag{1.6}$$

Then the evolution of $\mathcal{X}_t$ in (1.1) is written as $\mathcal{X}_t = (\mathscr{S}_A)^t \mathcal{X}_0$. Let

$$\tilde{A}(z) = \begin{pmatrix} A_{11} & A_{12} \\ z\,A_{21} & z\,A_{22} \end{pmatrix}. \tag{1.7}$$

A fundamental stability notion is introduced from the following number:

$$\hat{\rho}(\mathscr{S}_A) = \max_{|z|=1} \rho\big(\tilde{A}(z)\big), \tag{1.8}$$

where the notation $\rho(\,\cdot\,)$ used in the right hand side denotes the spectral radius of the matrix involved. It can be shown that the condition $\hat{\rho}(\mathscr{S}_A) < 1$ is equivalent to saying that there exists a pair of numbers (L, γ) with $L \geq 1$ and $0 \leq \gamma < 1$ such that for any $\mathcal{X}_0 \in \mathfrak{X}^n$, $\|(\mathscr{S}_A)^t \mathcal{X}_0\|_2 \leq L\,\gamma^t\,\|\mathcal{X}_0\|_2$ for all $t \in \mathbb{N}$. The geometric progression law gives definite information, in theory, about the asymptotic property of $\mathscr{S}_A$. Indeed, by considering the scaling $\hat{\rho}(\mathscr{S}_A)^{-1}\mathscr{S}_A$, we know that the asymptotic convergency rate of $\mathscr{S}_A$ is $\hat{\rho}(\mathscr{S}_A)$. In many instances, however, this result is next to useless for talking about transient phenomena of $\mathscr{S}_A$ because the number L may be impossibly large. There is another approach to study the stability of 2-D dynamics. Consider the Lyapunov matrix inequality

$$P - A^{\mathsf{T}} P A > O\,. \tag{1.9}$$

It was shown in [3] that if (1.9) has a solution $P > O$ satisfying the block diagonal constraint

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \tag{1.10}$$

where the order of P_i is n_i for $i = 1, 2$, then the dynamics (1.1) is stable in the sense that there exists a number $0 \leq \gamma < 1$ such that the contractive relation

$$\|\mathscr{S}_A \mathcal{X}_t\|_{(P_1, P_2)} \leq \gamma \cdot \|\mathcal{X}_t\|_{(P_1, P_2)} \tag{1.11}$$

holds for any $\mathcal{X}_t \in \mathfrak{X}^n$, where

$$\|\mathcal{X}_t\|_{(P_1, P_2)} = \left(\sum_{i=-\infty}^{\infty}\big(x_1(i, t-i)^{\mathsf{T}} P_1 x_1(i, t-i) + x_2(i, t-i)^{\mathsf{T}} P_2 x_2(i, t-i)\big)\right)^{1/2}. \tag{1.12}$$

The fact is that the converse is also true. We use the following terminology [4].

Definition 1.1. A pair of matrices $(\tilde{A}_1, \tilde{A}_2)$ is said to be *parallel stable* if there exists a pair of positive definite matrices $(\tilde{P}_1, \tilde{P}_2)$ such that

$$\tilde{P}_1 : \tilde{P}_2 - \tilde{A}_1^{\mathsf{T}} \tilde{P}_1 \tilde{A}_1 - \tilde{A}_2^{\mathsf{T}} \tilde{P}_2 \tilde{A}_2 > O\,, \tag{1.13}$$

where $\tilde{P}_1 : \tilde{P}_2 = \left(\tilde{P}_1^{-1} + \tilde{P}_2^{-1}\right)^{-1}$.

Theorem 1.2 ([7]). *The following statements are equivalent to each other.*

(a) There exists a positive definite P satisfying the Lyapunov matrix inequality (1.9) and the block diagonal constraint (1.10);

(b) There exists a pair of positive definite matrix (P_1, P_2) and $0 \leq \gamma < 1$ such that the contractive relation (1.11) holds for any $\mathcal{X}_t \in \mathfrak{X}^n$;

(c) There exists a positive definite matrix P such that $P - \tilde{A}(z)^* P\, \tilde{A}(z) > O$ for any $|z| = 1$;

(d) $(\tilde{A}_1, \tilde{A}_2)$ is parallel stable.

It is easy to see that the conditions above imply the spectral condition $\hat{\rho}(\mathscr{S}_A) < 1$. But the converse is not true [5][6]. For the sake of later argument, we give the following.

The purpose of this paper is to widen the scope of application of simply formed quadratic Lyapunov functions. Our approach is based on our latest paper [6] in which the same approach is taken to enhance the stability analysis of 2-D systems in the Fornasini-Marchesini model. Since the Roesser's model gains wide usage in the study of digital image processing, it would be worthwhile to study it in (1.1).

2. Main Results

We have shown that the form (1.12) does not have enough ability in dealing with the stability of (1.1). Taking account of the context that the dynamics (1.1) runs while exchanging data between adjacent local states, we will introduce the cross terms of local states into (1.12) as follows:

$$\begin{aligned} \|\mathcal{X}_t\|_{(P_1,P_2,R_1,R_2)} = \Big(&\sum_{i=-\infty}^{\infty} \big(x_1(i,t-i)^{\mathsf{T}} P_1 x_1(i,t-i) + x_2(i,t-i)^{\mathsf{T}} P_2 x_2(i,t-i) \big) \\ &+ 2 \sum_{i=-\infty}^{\infty} \big(x_1(i,t-i)^{\mathsf{T}} R_1 x_2(i,t-i) + x_2(i,t-i)^{\mathsf{T}} R_2 x_1(i+1,t-i-1) \big) \Big)^{1/2}, \end{aligned} \tag{2.14}$$

where (P_1, P_2, R_1, R_2) is a set of $n \times n$ matrices. The quartet (P_1, P_2, R_1, R_2) must be in such a condition as to make the form (2.14) a positive operator on $\mathfrak{X}^n$, i.e., $\|\mathcal{X}_t\|_{(P_1,P_2,R_1,R_2)} > 0$ for any nonzero $\mathcal{X}_t \in \mathfrak{X}^n$. The first requirement is that P_1 and P_2 are positive definite. Now, let us make from the matrix A the following expanded matrix:

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & O & O \\ O & O & A_{21} & A_{22} \\ O & O & A_{11} & A_{12} \\ A_{21} & A_{22} & O & O \end{pmatrix}. \tag{2.15}$$

Then we consider the associated Lyapunov matrix inequality

$$\boldsymbol{P} - \boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}\boldsymbol{A} > O\,. \tag{2.16}$$

We have the following.

Theorem 2.3. *If there exists a matrix $\boldsymbol{P} > O$ satisfying (2.16) under the constraint:*

$$\boldsymbol{P} = \begin{pmatrix} P_{11} & P_{12} & O & O \\ P_{12}^{\mathsf{T}} & P_{22} & P_{23} & O \\ O & P_{23}^{\mathsf{T}} & P_{33} & O \\ O & O & O & P_{44} \end{pmatrix}, \tag{2.17}$$

then the dynamics (1.1) is stable in the sense that by the use of the quartet (P_1, P_2, R_1, R_2) given by

$$P_1 = P_{11} + P_{33}, \quad P_2 = P_{22} + P_{44}, \quad R_1 = P_{12}, \ \textit{and} \quad R_2 = P_{23}, \tag{2.18}$$

the form (2.14) is a positive operator on $\mathfrak{X}^n$ and there exists a number $0 \le \gamma < 1$ such that the contractive relation $\|\mathscr{S}_A\,\mathcal{X}_t\|_{(P_1,P_2,R_1,R_2)} \le \gamma \cdot \|\mathcal{X}_t\|_{(P_1,P_2,R_1,R_2)}$ holds for any $\mathcal{X}_t \in \mathfrak{X}^n$.

3. Applications

Let $0 \le \delta \le 1$. A map $W : \mathbb{R}^n \to \mathbb{R}^n$, $(\xi_1, \ldots, \xi_n) \mapsto (\omega_1(\xi_1), \ldots, \omega_n(\xi_n))$, is called a δ-saturation arithmetic if it satisfies $(1-\delta)\xi_i \le \omega_i(\xi_i) \le \xi_i$ for $i = 1, \ldots, n$. Let Ω_δ denote the set of all δ-saturation arithmetic maps. Consider the following 2-D dynamics with saturation arithmetic:

$$\begin{pmatrix} y_1(i+1, j) \\ y_2(i, j+1) \end{pmatrix} = A\begin{pmatrix} x_1(i, j) \\ x_1(i, j) \end{pmatrix}, \quad \begin{pmatrix} x_1(i, j) \\ x_2(i, j) \end{pmatrix} = W_{i,j}\begin{pmatrix} y_1(i, j) \\ y_2(i, j) \end{pmatrix}, \quad W_{i,j} \in \Omega_\delta\,. \tag{3.19}$$

Let $\mathscr{W}_t$ denote the pointwize application of $W_{i,t-i} \in \Omega_\delta$ on $\mathfrak{X}^n$. Then the dynamics is written by

$$\mathcal{X}_{t+1} = \mathscr{W}_{t+1}\,\mathscr{S}_A\,\mathcal{X}_t\,. \tag{3.20}$$

We will discuss the condition by which the stability is ensured against any realization of saturation arithmetic. Such advantageous property does not necessarily follow from the stability of (1.1), as can be easily imagined from the well-known arguments about limit-cycles in the theory of finite dimensional dynamics. We associate a measure $\eta(P)$ with an $n \times n$ positive definite matrix P as follows. Let $\mathfrak{K}_q = \{(k_1, k_2, \ldots, k_q) \in \mathbb{Z}^q\,;\, 1 \le k_1 < k_2 < \cdots < k_q \le n\}$ for each $1 \le q \le n-1$, and let $\mathfrak{K} = \bigcup_{q=1}^{n-1} \mathfrak{K}_q$. The cardinality of $\mathfrak{K}$ is $2^n - 2$. With each $\mathfrak{k} \in \mathfrak{K}$, we associate three submatrices of $P = (\pi_{k,l})_{k,l=1,\ldots,n}$ those defined by $P_{\mathfrak{k}} = (\pi_{k,l})_{k,\,l\in\mathfrak{k}}$, $Q_{\mathfrak{k}} = (\pi_{k,l})_{k,\,l\in\mathfrak{k}^{\mathrm{c}}}$, and $R_{\mathfrak{k}} = (\pi_{k,l})_{k\in\mathfrak{k}^{\mathrm{c}},\,l\in\mathfrak{k}}$, where $\mathfrak{k}^{\mathrm{c}}$ denotes the complement of $\mathfrak{k}$ in $\{1, 2, \ldots, n\}$. The matrices $P_{\mathfrak{k}}$, $Q_{\mathfrak{k}}$, and $R_{\mathfrak{k}}$ are of order $q \times q$, $(n-q)\times(n-q)$, and $(n-q)\times q$ for $\mathfrak{k} \in \mathfrak{K}_q$, and we have $P_{\mathfrak{k}} > O$ and $Q_{\mathfrak{k}} > O$. For each $\mathfrak{k} \in \mathfrak{K}$, let us compute

$$\eta_{\mathfrak{k}}(P) = \left(\lambda_{\max}\left(P_{\mathfrak{k}}\,(P_{\mathfrak{k}} - R_{\mathfrak{k}}^{\mathsf{T}}\,Q_{\mathfrak{k}}^{-1}R_{\mathfrak{k}})^{-1}\right)\right)^{1/2}. \tag{3.21}$$

Notice that $O \le R_{\mathfrak{k}}^{\mathsf{T}} Q_{\mathfrak{k}}^{-1} R_{\mathfrak{k}} < P_{\mathfrak{k}}$ for any $\mathfrak{k} \in \mathfrak{K}$ since P is positive definite. Hence the values $\eta_{\mathfrak{k}}(P)$'s are well defined. A positive number $\eta(P)$ associated with a positive definite matrix P is then defined by

$$\eta(P) = \max_{\mathfrak{k} \in \mathfrak{K}} \eta_{\mathfrak{k}}(P), \tag{3.22}$$

where $\eta_{\mathfrak{k}}(P)$'s are those computed in (3.21) for $\mathfrak{k} \in \mathfrak{K}$. The number of $\eta_{\mathfrak{k}}$ in (3.22) is reduced by half to $2^{n-1} - 1$ because $\eta_{\mathfrak{k}}(P) = \eta_{\mathfrak{k}^c}(P)$. For $P = \left(\begin{smallmatrix} \alpha & \gamma \\ \gamma & \beta \end{smallmatrix}\right)$, for example, we have $\eta(P) = \sqrt{\alpha\beta/(\alpha\beta - \gamma^2)}$. Roughly speaking, the measure $\eta(\cdot)$ administers a proximity evaluation of a positive definite matrix to the set of positive diagonal matrices. It is easy to see that $\eta(P) \ge 1$ for any positive definite matrix P, and we can also readily check that the set $\{P \,;\, P > O \text{ and } \eta(P) = 1\}$ consists of positive diagonal matrices. The following result specialized in the case of $\delta = 1$ is what we have already shown in [8].

Proposition 3.4. *Let $0 \le \delta \le 1$. If there exists $P > O$ satisfying the constraint (1.10) and*

$$P - \left(\delta^2 \eta(P)^2 - \delta^2 + 1\right) A^{\mathsf{T}} P A > O\,, \tag{3.23}$$

then (3.19) is stable irrespective of the saturation arithmetics belonging in Ω_δ.

The result above is predicated on the satisfaction of the condition in Theorem 1.2, so it actually is not much useful. We can give a wider scope to this approach by the method used in the last section.

Theorem 3.5. *Let $\boldsymbol{A}$ be of (2.15), and let $0 \le \delta \le 1$. If there exists $\boldsymbol{P} > O$ satisfying the constraint (2.17) and*

$$\boldsymbol{P} - \left(\delta^2 \eta(\boldsymbol{P})^2 - \delta^2 + 1\right) \boldsymbol{A}^{\mathsf{T}} \boldsymbol{P} \boldsymbol{A} > O\,, \tag{3.24}$$

then (3.19) is stable irrespective of the saturation arithmetics belonging in Ω_δ.

References

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