Algebraic Algorithm for 2D Stability Test

Based on a Lyapunov Equation

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Abstract

Some improvements have been proposed for the algorithm of Agathoklis such that 2D stability test can be realized by totally algebraic operations.

1 Introduction

The stability test is the most important and fundamental problem for analysis and design of systems. The stability of the nD (n Dimensional) system determined, as the case for 1D systems, by the locus of the roots of the characteristic polynomial of the system. However, the stability test of nD systems is much more difficult than the 1D case because the multivariable polynomial has infinite number of roots.

The latest results of the stability tests of 2D systems are as follows,

- Bose's method based on resultant theory[1].
- Agathoklis' method for Roesser's model by means of Lyapunov equation[2].
- Hu's method by mean of polynomial array which may be viewed as an extension of Jury's table[3].

Among these method, only Hu's algorithm is a totally algebraic, while the other two require some numerical calculations in a certain step. For example, Agathoklis' algorithm needs to calculate general eigenvalues of some matrix. However, due to the numerical calculation error, it is difficult to check the absolutes of that eigenvalues is really equal to 1.

Authors proposed that Bose's 2D stability test can be fulfilled by totally algebraic algorithm[4]. The purpose of this research is to improve Agathoklis's method such that the tests can be realized by totally algebraic operations.

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2 Agathoklis's stability test

Consider the 2D Roesser's state-space model for a 2D system:

$$\begin{aligned}
\begin{bmatrix} x^h(I+1,j) \\ x^v(I,j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(I,j) \\ x^v(I,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(I,j) \\
y(I,j) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h(I,j) \\ x^v(I,j) \end{bmatrix}
\end{aligned} \tag{2.1}$$

where $x^h \in R^n$, $x^v \in R^m$ are horizontal and vertical states respectively, u is the input and y is the output. A_{11}, A_{12}, A_{21}, A_{22}, B_1, B_2, C_1, C_2 are real matrices with appropriate dimensions. The stability of the 2D system (2.1) depends on the zeros of the characteristic polynomial $C(z_1, z_2)$ given by

$$C(z_1, z_2) = \det \begin{bmatrix} I_n - z_1 A_{11} & -z_1 A_{12} \\ -z_2 A_{21} & I_m - z_2 A_{22} \end{bmatrix}. \tag{2.2}$$

The condition for internal stability is given by

$$C(z_1, z_2) \neq 0 \text{ for } (z_1, z_2) \in \bar{U}^2, \quad \bar{U}^2 = \{(z_1, z_2) || z_1| \leq 1, |z_2| \leq 1\}. \tag{2.3}$$

Agathoklis *et al* proposed the following stability conditions to test (2.3) based on the Lyapunov approach.

Theorem 2.1 *[2] Necessary and sufficient conditions that the system (2.1) is internally stable are*

I)

$$|\lambda_I[A_{22}]| < 1 \tag{2.4}$$

ii) the matrix equation

$$H^T(e^{-j\omega_0}) \bar{P} H(e^{j\omega_0}) - \bar{P} = -\bar{Q} \tag{2.5}$$

has a hermitian positive definite solution $\bar{P}$ for any given constant positive definite matrix $\bar{Q}$ and an arbitrary $\omega_0 \in [0, 2\pi]$ where

$$H(e^{j\omega}) = A_{11} + A_{12}(I_m e^{j\omega} - A_{22})^{-1} A_{21} \tag{2.6}$$

iii)

$$\det(I_{n^2} - H^T(e^{-j\omega}) \otimes H^T(e^{j\omega}) \neq 0 \ \textit{ for all } \omega \in [0, 2\pi]. \tag{2.7}$$

Theorem 2.1 requires test of the stability of two constant matrices A_{22}, $H(e^{j\omega_0})$, and testing of the condition (2.7) for all $\omega \in [0, 2\pi]$. Based on Theorem 2.1, [2] proposed 2D stability conditions testing the only eigenvalues of constant matrices.

3 algebraic approach

The conditions proposed in [2] needs to calculate the eigenvalues of constant matrices and to check whether absolute of eigenvalues equal to 1. It is difficult to check that absolute values of the eigenvalues are really equal to 1 because of the numerical calculation error. Therefore, we propose the algebraic method to test the conditions of Theorem 2.1.

It is easy to check the condition I) algebraically by using , e.g., Jury criterion. The condition ii) is equal to $|\lambda[H(e^{j\omega_0})]| < 1$ for an arbitrary $\omega_0 \in [0, 2\pi]$, then it can be tested algebraically by the same way as the condition I). So, we discuss the condition iii).

Let $z = e^{j\omega}$, the condition iii) is equal to

$$\det(I_{n^2} - H^T(z^{-1}) \otimes H^T(z)) \neq 0 \quad \text{for } |z| = 1. \tag{3.8}$$

From now on we consider the condition (3.8) instead of (2.7)

The following Lemma gives the key idea for algebraic approach by revealing symmetric properties of the rational polynomial (3.8).

Lemma 3.1 $\det(I_{n^2} - H^T(z^{-1}) \otimes H^T(z))$ *is the self-inversive rational polynomial given by*

$$\det(I_{n^2} - H^T(z^{-1}) \otimes H^T(z)) = \det(I_{n^2} - H^T(z) \otimes H^T(z^{-1})) \tag{3.9}$$

Proof. Note that $\det(I - A \otimes B) = \det(I - B \otimes A)$ (see Appendix A) for the same size of square matrices A and B, we see that (3.9) is true.

The self-inversive rational polynomial $f(z)(= f(z^{-1}))$ can be represented as $f(z) = n(z)/d(z)$ where $n(z)$ and $d(z)$ are self-inversive polynomials, namely $n(z) = n(z^{-1})$, $d(z) = d(z^{-1})$. Thus, from Lemma 3.1, there exist self-inversive polynomials $h_n(z)$, $h_d(z)$ such that

$$\frac{h_n(z)}{h_d(z)} \triangleq \det(I_{n^2} - H^T(z^{-1}) \otimes H^T(z)). \tag{3.10}$$

Because the degree of the numerator of $H(z)$ is $m - 1$ at most, the degree of $h_n(z)$ is only $2(m-1)n^2$. Let d_n be the degree of $h_n(z)$, the self-inversive polynomial $h_n(z)$ can be described as

$$h_n(z) = f_{n0} + \sum_{k=1}^{d_n} f_{nk}(z^k + z^{-k}). \tag{3.11}$$

$h_n(z)$ can be transformed to real polynomial of x by using $x = (z + z^{-1})/2$ as [5]

$$h_n(z) = f_n Z^{(d_n)} = f_n D_{d_n} x_{d_n} \triangleq \hat{h}_n(x), \quad I = 1, 2 \tag{3.12}$$

where $f_n = [f_{n0}\ f_{n1} \cdots\ f_{nd_n}]$C $x_k = [1\ x\ x^2 \cdots x^k]^T$

$$Z^{(d_n)} = \begin{bmatrix} 1 \\ z^{-1} + z^1 \\ z^{-2} + z^2 \\ \vdots \\ z^{-d_n} + z^{d_n} \end{bmatrix} = \begin{bmatrix} 1 \\ d_1 x_1 \\ d_2 x_2 \\ \vdots \\ d_{d_n} x_{d_n} \end{bmatrix} = D_{d_n} x_{d_n},$$

$$d_k = [d_{k,0}\ d_{k,1} \cdots d_{k,k}].$$

d_k is defined as

$$z^k + z^{-k} = \sum_{j=0}^{k} d_{k,j} x^j = d_k x_k \tag{3.13}$$

and calculates recursively as

$$\begin{aligned} d_{I,0} = d_{i-2,0}, \qquad & d_{i,j} = 2d_{i-1,j-1} - d_{i-2,j}, \\ & i \geq 3,\ j = 1, 2, \ldots, i \end{aligned} \tag{3.14}$$

with the initial condition $[d_{1,0}\ d_{1,1}] = [0\ 2]$, $[d_{2,0}\ d_{2,1}\ d_{2,2}] = [-2\ 0\ 4]$ and

$$D_{d_n} = \begin{bmatrix} 1 \quad 0 \\ d_1 \\ d_2 \\ \vdots \\ d_{d_n} \end{bmatrix} = \begin{bmatrix} 1 & & 0 & & \\ d_{1,0} & d_{1,1} & & & \\ d_{2,0} & d_{2,1} & d_{2,2} & & \\ \vdots & \vdots & \vdots & \ddots & \\ d_{d_n,0} & d_{d_n,1} & d_{d_n,2} & \cdots & d_{d_n,d_n} \end{bmatrix}$$

As $h_d(z)$ are also self-inversive polynomials, $h_d(z)$ can be converted into polynomials $\hat{h}_d(x)$ in the variable $x = (z + z^{-1})/2$ by the method mentioned above. Also, note that $\bar{z} = z^{-1}$ on the unit circle $|z| = 1$, all the zeros of $h_n(z)$, $h_d(z)$ on the unit circle are located within the interval $-1 < x < 1$. The condition (3.10) can be rewritten as

$$\frac{\hat{h}_n(x)}{\hat{h}_d(x)} \neq 0 \quad \text{for } -1 \leq x \leq 1. \tag{3.15}$$

Note that the polynomial $\hat{h}_d(x)$ is bounded within the interval of $-1 \leq x \leq 1$, the condition (3.15) is equal to

$$\hat{h}_n(x) \neq 0 \quad \text{for } -1 \leq x \leq 1. \tag{3.16}$$

The condition (3.16) can be easily tested algebraically by using the Sturm's theorem[6]. Let $g(s)$ be polynomials given arbitrarily. It is assumed that $g(x)$ and its derived function $g'(x)$ have no common zeros. Define the sequence begun with $g(x)$, $g_1(x)(= g'(x))$;

$$g(x),\ g_1(x),\ g_2(x),\ \cdots, g_l(x) \tag{3.17}$$

where

$$g_{i-1}(x) = h_i(x) g_i(x) - g_{i+1}(x), \quad i = 1, 2, \ldots, l-1 \tag{3.18}$$

Lemma 3.2 *[6] Let $V(x_0)$ be the number of sign changes of sequence (3.17) corresponded to $g(x_0)$ and N_0 assume the number of roots of equation $g(x) = 0$ for the interval of $a < x \leq b$. The following relation is hold.*

$$V(a) - V(b) = N_0 \tag{3.19}$$

Based on this lemma, if $V(-1) - V(1) = 0$, $\hat{h}_n(x)$ dose not have roots on real region $-1 < x \leq 1$. One may reduce testing the condition (3.16) to checking $V(-1) - V(1) = 0$ and $\hat{h}_n(-1) \neq 0$ algebraically.

Now we have following stability test algorithm;

step 1 Test $|\lambda_i[A_{22}]| < 1$ by using Jury criterion etc.

step 2 Test $|\lambda_i[H(1)]| < 1$ for $\omega_0 = 0$ by using Jury criterion etc.

step 3 Calculate $\hat{h}_n(x)$ and check the condition (3.16) by the Sturm's theorem.

4 examples

Consider the stability of the following 2D system[2].

$$\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|cc} -0.5 & 0.75 & 0.3895 & 0.03895 \\ 0 & 0 & 0 & 0 \\ \hline 0.1423 & 0 & -0.4 & 0.02 \\ -0.0342 & 0 & -0.6 & 0.03 \end{array}\right] \tag{4.20}$$

where $n = 2$, $m = 2$.

step 1 $\det(zI - A_{22}) = z^2 + 37/100z = z(z + 37/100)$. This polynomial is obviously 1D stable.

step 2 $\det(zI - H(1)) = z(z + \frac{1273387641}{2740000000})$. This polynomial is also 1D stable.

step 3

$$\begin{aligned} \frac{h_n(z)}{h_d(z)} &= \det(I - H^T(-z) \otimes H^T(z)) && (4.21) \\ h_n(z) &= 3478920240964073679 + 1215156998392190720(z + z^{-1}) && \\ &\quad + 11575161000000000(z^2 + z^{-2}) && (4.22) \\ h_d(z) &= 45476 \times 10^{14} + 148 \times 10^{16}(z + z^{-1}) && (4.23) \end{aligned}$$

Let $x = (z + z^{-1})/2$, $h_n(z)$ and using the transformation (3.12), we have

$$\hat{h}_n(x) = -46300644 \times 10^9 x^2 + 2430313996784381440x + 3502070562964073679. \tag{4.24}$$

Constructing the Sturm's functions $f_0(x)$, $f_1(x)$, $f_2(x)$ for $\hat{h}_n(x)$ and computing $V(-1)$, $V(1)$, we have $V(-1) = 1$, $V(1) = 1$. And the value of $\hat{h}_n(-1)$ is not 0.

Therefore, we conclude that the system (4.20) is 2D stable.

Noting that the above $\hat{h}(x)$ is only a second degree polynomial, while [2] needs to calculate 18×18 matrixes eigenvalues.

5 conclusions

Some implements are proposed for algorithm of Agathoklis such that the 2D stability test can be realized by total algebraic operations. The stability test using proposed algorithm usually involve very complicated symbolic manipulations. Therefore, the Stability Testing Package, which consists of 2D stability testing functions implemented Bose's, Hu's and Agathoklis's method, has been developed for use with the MATLAB and (Extended) Symbolic Math Toolbox[7]. Among these method, algorithm of Agathoklis is modified so that the stability test can be realized by totally algebraic manipulations by using algorithm mentioned in this paper.

References

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A the proof of $\det(I - A \otimes B) = \det(I - B \otimes A)$

We first show $\det(A \otimes B) = \det(B \otimes A)$ in case A and B are the same size. Assuming that $A = [a_{ij}] \in R^{n\times n}$, $B = [b_{ij}] \in R^{n\times n}$. From the definition of Kronecker product

$$\det(A \otimes B) = \begin{vmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{vmatrix}. \tag{A.25}$$

The column of $(i-1)n+j$ is exchanged for $(j-1)n+i$ $(i, j = 1, 2, \ldots, n,\ i < j)$ to move b_{ij} into the (i, j) block matrix. It's just the same with rows. The number of exchange is even number $n(n-1)$. Then (A.25) can be rewritten as

$$\begin{vmatrix} b_{11}A & b_{12}A & \cdots & b_{1n}A \\ b_{21}A & b_{22}A & \cdots & b_{2n}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}A & b_{n2}A & \cdots & b_{nn}A \end{vmatrix} \tag{A.26}$$

So, $\det(A \otimes B) = \det(B \otimes A)$.

Next, in the same manner as before, we exchange columns and rows. Diagonal elements $1 - a_{ii}b_{jj}$ are always moved to diagonal locations. Then we conclude that $\det(I - A \otimes B) = \det(I - B \otimes A)$.