THE BANG-BANG PRINCIPLE FOR THE GOURSAT-DARBOUX PROBLEM*

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ABSTRACT. In the paper, the bang-bang principle for a control system connected with a system of linear nonautonomous partial differential equations of hyperbolic type (the so-called Goursat-Darboux problem or continuous Fornasini-Marchesini problem) is proved. Some density result is also obtained.

1. INTRODUCTION

Let us consider the following control system with the concentrated parameters

$$\dot{x} = A(t)x + B(t)u$$

for $t \in [0,1]$ *a.e.*,

$$\begin{aligned} x(0) &= 0, \\ u &\in M, \end{aligned}$$

where $x \in \mathbb{R}^n,\ M \subset \mathbb{R}^m,\ A(t) \in \mathbb{R}^{n\times n},\ B(t) \in \mathbb{R}^{n\times m}$. One of the fundamental results of the control theory for the systems of the above type states that the set $\mathcal{A}_M(1)$ of the points that can be attained from 0 at the time $t = 1$ with the aid of the measurable controls taking their values in a fixed compact set $M \subset \mathbb{R}^m$ is compact and convex in $\mathbb{R}^n$ and, if $M^* \subset \mathbb{R}^m$ is a compact set such that convex hull of it coincides with the convex hull of M ($coM = coM^*$) then

$$\mathcal{A}_M(1) = \mathcal{A}_{M^*}(1)$$

(time $t = 1$ can be replaced by any $T \in [0,1]$). This result is known in the literature as the bang-bang principle (cf. [8], [10]).

An important question, from the practical point of view, is the possibility of choosing the control function $u : [0,1] \to M$ which steers the system to a given point and is piecewise constant function([1]). This problem is investigated in [9] and [1]. It is proved there that the set $\mathcal{A}_M^{PC}(1)$ of the points that can be attained from 0 at the time $t = 1$ with the aid of the piecewise constant controls taking their values in a fixed compact set $M \subset \mathbb{R}^m$ is, in general, dense in $\mathcal{A}_M(1)$ (a density theorem). The sets $\mathcal{A}_M^{PC}(1)$ and $\mathcal{A}_M(1)$ coincide if M is the closed convex hull of a finite number of points.

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[1]We recall that a function $u : [0,1] \to \mathbb{R}^m$ is called a piecewise constant function if there exists a finite partition $0 = x_0 < x_1 < ... < x_k = 1$ of the interval $[0,1]$ such that u is constant on (x_{i-1}, x_i) for every $i = 1, ..., k$.

The aim of the paper is to prove the analogous theorems for the following control system with distributed parameters

$$z_{xy} = A_0(x,y)z + A_1(x,y)z_x + A_2(x,y)z_y + B(x,y)u \tag{1.1}$$

for $(x,y) \in P = [0,1] \times [0,1]$ *a.e.*,

$$z(x,0) = 0,\ z(0,y) = 0 \tag{1.2}$$

for $x, y \in [0,1]$,

$$u \in M,$$

where $z \in \mathbb{R}^n,\ M \subset \mathbb{R}^m,\ A_0,\ A_1,\ A_2 : P \to \mathbb{R}^{n\times n},\ B : P \to \mathbb{R}^{n\times m}$. In the automatic control theory such system is called a continuous Fornasini-Marchesini system. It is in fact a continuous version of 2-D discrete model of Fornasini-Marchesini type (cf. [5], [2]).

In the proof of the bang-bang principle, we shall use some facts concerning the integration of the set-valued functions (Aumann integral). To prove a density theorem we shall prove some extension (to the case of functions of two variables) of theorem on the density of piecewise constant functions of one variable taking their values in some fixed set, in the space of integrable functions which take their values in the same set (cf. [1]).

We shall consider system (1.1) in the space $AC_0^1(P,\mathbb{R}^n)$ (cf. [13]) of solutions, which consists of all functions z having the representation

$$z(x,y) = \int_0^x \int_0^y l(s,t)dsdt,\ (x,y) \in P,$$

with $l \in L^1(P,\mathbb{R}^n)$. A function $z \in AC_0^1(P,\mathbb{R}^n)$ possesses *a.e.* on P the partial derivatives

$$z_{xy}(x,y) = l(x,y),$$

$$z_x(x,y) = \int_0^y l(x,t)dt,\ z_y(x,y) = \int_0^x l(s,y)ds$$

and, of course, satisfies the initial conditions (1.2). In an elementary way one can check that $AC_0^1(P,\mathbb{R}^n)$ with the norm

$$\|z\|_{AC_0^1(P,\mathbb{R}^n)} = \int_0^1 \int_0^1 |z_{xy}(x,y)|\, dxdy$$

is the Banach space.

On the control functions u we assume that they belong to $L^1(P,\mathbb{R}^n)$. The functions $A_0,\ A_1,\ A_2 : P \to \mathbb{R}^{n\times n},\ B : P \to \mathbb{R}^{n\times m}$ are assumed to be elements of $L^\infty(P,\mathbb{R}^{n\times n})$, $L^\infty(P,\mathbb{R}^{n\times m})$, respectively. In paper [3] it is proved that for any control $u \in L^1(P,\mathbb{R}^m)$ there exists exactly one solution $z^u \in AC_0^1(P,\mathbb{R}^n)$ of (1.1) and $z^{u_n} \underset{n\to\infty}{\longrightarrow} z^{u_0}$ in $AC_0^1(P,\mathbb{R}^n)$ as $u_n \underset{n\to\infty}{\longrightarrow} u_0$ in $L^1(P,\mathbb{R}^m)$.

2. Bang-bang principle

Let us consider an operator

$$G : L^1(P, \mathbb{R}^m) \ni u \longmapsto z^u(1,1) \in \mathbb{R}^n.$$

Obviously, it is linear and bounded. Consequently, there exists a function $g \in L^\infty(P, \mathbb{R}^{n\times m})$ such that

$$G(u) = \int_0^1 \int_0^1 g(x,y)u(x,y)dxdy$$

for $u \in L^1(P, \mathbb{R}^m)$.

Now, let us fix a set $M \subset \mathbb{R}^m$ and consider the sets

$$\begin{aligned} \mathcal{U}_M &= \{u \in L^1(P, \mathbb{R}^m);\ u(x,y) \in M \text{ for } (x,y) \in P \ a.e.\}, \\ \mathcal{U}_{coM} &= \{u \in L^1(P, \mathbb{R}^m);\ u(x,y) \in coM \text{ for } (x,y) \in P \ a.e.\}. \end{aligned}$$

We shall show that if M is compact, then the sets

$$\begin{aligned} \mathcal{A}_M(1,1) &= G(\mathcal{U}_M) = \{\int_0^1 \int_0^1 g(x,y)u(x,y)dxdy;\ u \in \mathcal{U}_M\}, \\ \mathcal{A}_{coM}(1,1) &= G(\mathcal{U}_{coM}) = \{\int_0^1 \int_0^1 g(x,y)u(x,y)dxdy;\ u \in \mathcal{U}_{coM}\} \end{aligned}$$

are convex compact and coincide (the point $(1,1)$ can be replaced by any point $(X,Y) \in P$).

Let us define a single-valued function

$$f : P \times \mathbb{R}^m \ni (x,y,u) \longmapsto g(x,y)u \in \mathbb{R}^n,$$

the set-valued functions

$$\begin{aligned} F &: P \ni (x,y) \longmapsto f(x,y,M) = g(x,y)M \in 2^{\mathbb{R}^n} \\ coF &: P \ni (x,y) \longmapsto cof(x,y,M) = g(x,y)coM \in 2^{\mathbb{R}^n} \end{aligned}$$

and denote by $\int_0^1 \int_0^1 F(x,y)dxdy$ the set

$$\{\int_0^1 \int_0^1 v(x,y)dxdy;\ v \in L^1(P, \mathbb{R}^n),\ v(x,y) \in F(x,y) \text{ for } (x,y) \in P \ a.e.\}$$

and by $\int_0^1 \int_0^1 coF(x,y)dxdy$ the set

$$\{\int_0^1 \int_0^1 v(x,y)dxdy;\ v \in L^1(P, \mathbb{R}^n),\ v(x,y) \in coF(x,y) \text{ for } (x,y) \in P \ a.e.\}.$$

In fact, they are the Aumann integrals of the set-valued functions of two variables $F(x,y)$, $coF(x,y)$, respectively (cf. [4]).

We have

Lemma 2.1. *If $M \subset \mathbb{R}^m$ is a compact set, then*

$$\mathcal{A}_M(1,1) = \int_0^1 \int_0^1 F(x,y)dxdy.$$

Proof. The inclusion

$$\mathcal{A}_M(1,1) \subset \int_0^1 \int_0^1 F(x,y)dxdy$$

is obvious.

So, let $z \in \int_0^1 \int_0^1 F(x,y)dxdy$, i.e.

$$z = \int_0^1 \int_0^1 v(x,y)dxdy$$

where $v \in L^1(P, \mathbb{R}^n)$ and $v(x,y) \in F(x,y) = g(x,y)M$ for $(x,y) \in P$ *a.e.* The implicit function theorem for the set-valued function (cf. [6]) implies that([2]) there exists a measurable function $u : P \to \mathbb{R}^m$ such that $u(x,y) \in M$ for $(x,y) \in P$ *a.e.* (so $u \in \mathcal{U}_M$) and

$$v(x,y) = g(x,y)u(x,y), \ (x,y) \in P \ a.e.$$

Thus,

$$z = \int_0^1 \int_0^1 g(x,y)u(x,y)dxdy \in \mathcal{A}_M(1,1)$$

and the proof is completed. ∎

Since the convex hull of a compact subset of $\mathbb{R}^n$ is compact, therefore from the above theorem it follows that

$$\mathcal{A}_{coM}(1,1) = \int_0^1 \int_0^1 coF(x,y)dxdy$$

provided $M \subset \mathbb{R}^m$ is compact.

Now, we shall prove the main result of the paper

Theorem 2.1. *If $M \subset \mathbb{R}^m$ is a compact set, then the set $\mathcal{A}_M(1,1) \subset \mathbb{R}^n$ is convex compact. Moreover, if $M^* \subset \mathbb{R}^m$ is a compact set such that $coM = coM^*$, then*

$$\mathcal{A}_M(1,1) = \mathcal{A}_{M^*}(1,1).$$

[2]We apply the implicit function theorem to the single-valued function $f(x,y,u) = g(x,y)u$ and the constant set-valued function $\Gamma(x,y) = M$.

Proof. First we shall prove the second part of the theorem. From [4] we have([3])

$$\int_0^1 \int_0^1 F(x,y)dxdy = \int_0^1 \int_0^1 coF(x,y)dxdy.$$

So,

$$\mathcal{A}_M(1,1) = \mathcal{A}_{coM}(1,1).$$

Analogously,

$$\mathcal{A}_{M^*}(1,1) = \mathcal{A}_{coM^*}(1,1).$$

Thus, using the fact that $coM = coM^*$, we obtain

$$\mathcal{A}_M(1,1) = \mathcal{A}_{M^*}(1,1).$$

To prove the first part of the theorem let us recall some classical result of the functional analysis which states that the set $\mathcal{U}_{coM}$ is convex and weakly compact in $L^2(P,\mathbb{R}^m)$ provided $M \subset \mathbb{R}^m$ is compact. Of course, the functional $G\mid_{L^2(P,\mathbb{R}^m)}$ (the restriction of G to $L^2(P,\mathbb{R}^m)$) is linear and continuous on $L^2(P,\mathbb{R}^m)$ (in $L^2(P,\mathbb{R}^m)$ we consider the classical norm). Consequently, the set

$$\mathcal{A}_{coM}(1,1) = G(\mathcal{U}_{coM}) = (G\mid_{L^2(P,\mathbb{R}^m)})(\mathcal{U}_{coM})$$

is convex and compact in $\mathbb{R}^n$. This fact and the second part of the theorem imply the convexity and compactness of $\mathcal{A}_M(1,1)$. ∎

Remark. The equality

$$\mathcal{A}_M(1,1) = \mathcal{A}_{bd(M)}(1,1)$$

where $bd(M)$ is the boundary of M has been obtained (on the basis of a decomposition theorem and the so-called multipliers of Cesari) in [12]. The possibility of the extension of this result to the case of the set of extreme points of M is also noticed.

3. Piecewise constant controls

In [1] (cf. also [11]) the following interesting theorem is proved

Theorem 3.1. *If $M \subset \mathbb{R}^m$ is nonempty, then for every integrable function $l : [0,1] \to M$ and every $\varepsilon > 0$ there exists a picewise constant function $u : [0,1] \to M$ such that*

$$\int_0^1 |l(t) - u(t)|\, dt \le \varepsilon.$$

Now, we shall prove an extension of this theorem to the case of functions of two variables.

We say that a function $u : P \to \mathbb{R}^m$ is piecewise constant if there exists a partition $0 = x_0 < x_1 < ... < x_k = 1$ of the interval $[0,1]$ such that u is constant on $(x_{i-1}, x) \times (x_{j-1}, x_j)$ for every $i, j = 1, ..., k$.

We have the following

[3] This theorem can be applied in our case because the set-valued function $F(x,y)$ is measurable (in the sense of definition given in [IT]), bounded by an integrable function and has the closed values (in fact compact).

Theorem 3.2. *If $M \subset \mathbb{R}^m$ is nonempty, then for every integrable function $l : P \to M$ and every $\varepsilon > 0$ there exists a piecewise constant function $u : P \to M$ such that*

$$\int_0^1 \int_0^1 |l(x,y) - u(x,y)|\, dxdy \leq \varepsilon.$$

Proof. Let us fix an integrable function $l : P \to M$, $\varepsilon > 0$, $u_0 \in M$ and consider a function

$$b_0 : P \ni (x,y) \longmapsto l(x,y) - u_0 \in \mathbb{R}^m.$$

The Tchebychev's inequality (cf. [7]) implies that for any $\nu \in \mathbb{N}$

$$\mu(\{(x,y) \in P;\ |b_0(x,y)| \geq \nu\}) \leq \frac{1}{\nu} \int_0^1 \int_0^1 |b_0(x,y)|\, dxdy$$

(μ denotes the Lebesgue measure in P). From the absolute continuity of the integral it follows that for any $\eta > 0$ there exists $\delta > 0$ such that

$$\int\int_R |b_0(x,y)|\, dxdy < \eta$$

provided $\mu(R) < \delta$. Consequently, there exists $\nu_0 \in \mathbb{N}$ such that

$$\int\int_{R_0} |b_0(x,y)|\, dxdy < \frac{\varepsilon}{2}$$

where $R_0 = \{(x,y) \in P;\ |b_0(x,y)| \geq \nu_0\}$.

If we put

$$b(x,y) = \begin{cases} u_0 & ;\ (x,y) \in R_0 \\ l(x,y) & ;\ (x,y) \in P \setminus R_0 \end{cases}$$

for $(x,y) \in P$, then we have

$$\int_0^1 \int_0^1 |l(x,y) - b(x,y)|\, dxdy$$

$$= \int\int_{R_0} |l(x,y) - b(x,y)|\, dxdy + \int\int_{P \setminus R_0} |l(x,y) - b(x,y)|\, dxdy$$

$$= \int\int_{R_0} |b_0(x,y)|\, dxdy < \frac{\varepsilon}{2}.$$

Let us denote by γ a constant (a finite number) which bounds the function b on P, i.e. $|b(x,y)| \leq \gamma$ for $(x,y) \in P$. The Lusin's Theorem implies that there exists a compact set $H \subset P$ such that

$$\mu(H) > 1 - \frac{\varepsilon}{8\gamma}$$

and the function $H \ni (x,y) \longmapsto b(x,y) \in \mathbb{R}$ is uniformly continuous. In particular, there exists $\sigma > 0$ such that

$$|b(x,y) - b(\overline{x}, \overline{y})| < \frac{\varepsilon}{4}$$

for $(x,y), (\overline{x}, \overline{y}) \in H, |(x,y) - (\overline{x}, \overline{y})| < \sigma$.

Let us fix a number $r \in \mathbb{N}$ such that $\frac{\sqrt{2}}{r} < \sigma$ and consider a partition

$$P_{ij} = [\frac{i}{r}, \frac{i+1}{r}] \times [\frac{j}{r}, \frac{j+1}{r}];\ i,j = 0,...,r-1,$$

of the interval P. Let us also define a function $u : P \to \mathbb{R}^m$,

$$u(x,y) = \begin{cases} b(\widetilde{x}_i, \widetilde{y}_j) & ; \ (x,y) \text{ belongs to some } IntP_{ij} \text{ and } (IntP_{ij}) \cap H \neq \emptyset \\ u_0 & ; \ \text{otherwise} \end{cases},$$

where $(\widetilde{x}_i, \widetilde{y}_j)$ is an arbitrary fixed point of $(IntP_{ij}) \cap H$ for any i, j such that $(IntP_{ij}) \cap H \neq \emptyset$. Of course, u is a constant function and

$$u(x,y) \in M$$

for $(x,y) \in P$ $a.e.$ Moreover, we have

$$\begin{aligned} &\int\int_P |b(x,y) - u(x,y)|\, dxdy \\ &= \int\int_H |b(x,y) - u(x,y)|\, dxdy + \int\int_{P \setminus H} |b(x,y) - u(x,y)|\, dxdy \\ &\leq \sum_{i,j=0,(IntP_{ij})\cap H \neq \emptyset}^{r-1} \int\int_{(IntP_{ij})\cap H} |b(x,y) - u(x,y)|\, dxdy + \mu(P \setminus H) 2\gamma \\ &\leq r^2 \frac{1}{r^2} \frac{\varepsilon}{4} + \frac{\varepsilon}{8\gamma} 2\gamma = \frac{\varepsilon}{2}. \end{aligned}$$

Finaly, $u : P \to M$ is the piecewise constant function and

$$\begin{aligned} &\int\int_P |l(x,y) - u(x,y)|\, dxdy \\ &\leq \int\int_P |l(x,y) - b(x,y)|\, dxdy + \int\int_P |b(x,y) - u(x,y)|\, dxdy \leq \varepsilon \end{aligned}$$

which completes the proof. ∎

Let us denote by U_M^{PC} the set of all piecewise constant functions $u : P \to M$. From the above theorem and the continuity of the mapping

$$G : L^1(P, \mathbb{R}^m) \to \mathbb{R}^n,$$
$$G(u) = z^u(1,1),$$

we obtain

Theorem 3.3. *If $M \subset \mathbb{R}^m$ is nonempty, then the set*

$$\mathcal{A}_M^{PC}(1,1) = \{z^u(1,1);\ u \in U_M^{PC}\}$$

is dense in $\mathcal{A}_M(1,1)$.

Finally, from theorems 2.2 and 3.3 we obtain

Theorem 3.4. *If $M, M^* \subset \mathbb{R}^m$ are nonempty compact sets such that $coM = coM^*$, then each point $z_1 \in \mathbb{R}^n$ that can be attained with the aid of a control $u \in U_M$ can be attained with the aid of a control $\overline{u} \in U_{M^*}$. Moreover, for every $\varepsilon > 0$ there exists a control $\widetilde{u} \in U_{M^*}^{PC}$ such that*

$$\left| z_1 - z^{\widetilde{u}}(1,1) \right| < \varepsilon.$$

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