Difference equations and n-D discrete systems

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Abstract

Discrete systems are often *described* by difference equations. With some attributes of initial state type, a difference equation *defines* a system, i.e. a set of (input,output) pairs. For systems defined by linear difference equations (with constant or variable coefficients) on arbitrary sets $A \subset \mathbf{Z}^n$ sufficient conditions of their stability are formulated. It is also shown that these conditions enable to obtain various growth estimates for outputs of such systems. Examples illustrate the results.

1 Introduction

Discrete systems can be described as operators acting on functions with countable domains. These functions are also called discrete functions or sequences. In this paper the domains of discrete functions will be subsets of $\mathbf{Z}^n$ denoted by capitals $A, B, ..$ For their elements we shall use Greek letters, like $\alpha \in A$. The semigroup structure of $\mathbf{Z^n}$ with component-wise addition is assumed, and e.g. $A + B = \{\gamma : \gamma = \alpha + \beta, \alpha \in A, \beta \in B\}$.

Linear n-D discrete systems are often described by partial difference equations of the form

$$\sum_{\beta \in B} a(\alpha, \beta) y(\alpha + \beta) = x(\alpha). \tag{1.1}$$

Here, $\alpha \in A \subset \mathbf{Z}^n, B \subset \mathbf{Z}^n, 1 < |B| < +\infty$. The domain A of the input, called also the index set, will be an infinite countable set and x is a mapping $x : A \to \mathbf{C}$. The mapping $y : A + B \to \mathbf{C}$, satisfying equation (1.1) will be called its solution or output. For $x \equiv 0$ the equation is called homogeneous and the system is called zero input. If the coefficients $a(\alpha, \beta)$ are independent of α then the corresponding system is commonly called shift-invariant.

All the results discussed below are also valid with non-essential changes for systems of equations (1.1), i.e. when the coefficients in (1.1) are square matrices of order N and x, y are vectors of corresponding dimension. It should be mentioned however that conditions like $a \neq 0$ have to be replaced in such case by $\det A \neq 0$.

Most of the literature is dealing with systems described by equation (1.1) with the assumption that either $A = Z^n$ or $A = Z^n_+$. In this paper we will consider the index set A to be any non-finite proper subset of Z^n so as to widen the scope of possible applications.

The corresponding results of existence and uniqueness of solution of equation (1.1) are given in [3]. It has been proved that for any set A there always exists an ordering of the set

A and a corresponding set $G \subset Z^n$ such that equation (1.1) has a unique solution assuming given values g on the initial set G and that this solution is recursively computable. We cannot describe all the details of this result here. We want to concentrate our attention to some of its corollaries. Three of them will be discussed:

1. The ordering, hence also the initial set G of an equation is not unique. Therefore the equation (1.1) does not define a (single) system, but rather an (infinite) set of systems, which may include systems with mutually essentially different behavior.

2. Recursive computation of the solution y enables to formulate stability conditions. The proofs follow by induction.

3. Simple transformations of the equation (1.1) yield growth estimates of outputs for systems defined by (1.1).

Although the proofs are mostly simple, we prefer to illustrate the results by examples only and discuss the general cases bearing in mind all the restrictions as they follow from proven results (see [2], [3], [7]). To formulate some conclusions we will analyze in detail from various points of view the following simple example:

Consider the difference equation

$$a\,y(i+1,k+1) + b\,y(i,k+1) + c\,y(i+1,k) + d\,y(i,k) = x(i,k), \tag{1.2}$$

with nonzero constant coefficients and with the index set A given by $0 \le i \le N, k \ge 0$ for a fixed positive integer N .

2 Systems

Take item 1. of the above list and use the proven existence and uniqueness results for equation (1.2):

- Lexicographic ordering of the set A demands initial values y to be given on the set $G_1 = \{(i,k) : i\,k = 0\}$. We shall mark this situation as $A(G_1)$. The solution can be computed recursively from any given initial values $g : G_1 \to C$ expressing the term $y(i+1,k+1)$ from (1.2).

- In the strip A a 'reverse' (with respect to the first variable) ordering can be adopted. The initial set becomes $G_2 = \{(i,k) : (i-N)\,k = 0\}$. Again recursive computation gives the unique solution for any given $g : G_2 \to C$ now with the term $y(i,k+1)$ as the 'leading' one. Denote this situation as $A(G_2)$.

- Choose two integers p, q of opposite signs. The 'straight line' $p\,i + q\,k = 0$ subdivides the semi-infinite strip A into two parts, the upper part A_u and the lower (finite) part A_l.

The points of the straight line could be adjoint to either of these. It is easy to arrange for a separate ordering of A_u, A_l and when defining the points of this straight line and two or three parallel lines as the initial set G_3, we obtain for any mapping $g : G_3 \to C$ a recursively computable solution of equation (1.2). Although this approach may seem rather esoteric, it yields a unique solution of equation (1.2) for any positive integer values of p, q. Therefore G_3 implies in fact an infinite set of different orderings and different initial sets.

The concept of a system has been dealt with in many basic monographs (see e.g. [8],[4],[6],[5] and many more). All these concepts, be it an input-output model, an input-state-output model, the behavioral model, would consider the $A(G_i)$ cases described above as mutually different systems, although they are described by a single equation. Equation (1.2) is a very simple example. Nevertheless, for general linear discrete systems we may conclude that any linear difference equation (1.1) defines an infinite set of systems. Each of these systems has its own laws of development, its own states and characteristics, which cannot be derived from the equation (1.1) only.

3 Stability

Turn now to item 2 of our list. In what follows, $\|.\|$ will denote any fixed norm in the linear space of sequences. For illustration we will use again equation (1.2).

- Consider $A(G_1)$ as it is described above. The set A can be well ordered (i.e. there exists a sequence $\gamma_j \in A, j = 1, 2, \ldots$ such that $\bigcup_j \gamma_j = A$). Writing the equation in the form $y(i+1, k+1) = 1/a(x(i, k) - \ldots)$ and using the triangular inequality, we may conclude: if $|a| > |b| + |c| + |d|$ and if x and/or g are bounded sequences then there exists positive constants M, K such that $\|y\| \leq M\|x\| + K\|g\|$, i.e. the system $A(G_1)$ is both initial state and input stable (see below).

- Similarly for $A(G_2)$ we obtain: If $|b| > |a| + |c| + |d|$ then the system is initial state and input stable.

- Any of the systems $A(G_3)$ can be treated similarly. Since the 'lower' part of A is finite, we obtain as a sufficient condition of stability the previous one.

Due to ordering of the set A we may the recursive computation pinpoint one element of the mask B for each part of the set A and obtain the desired values by recursive computation. Let this element be called the leading element of the mask and its coefficient the leading coefficient. We can formulate the following definition and sufficient conditions of stability of systems defined by equation (1.1).

Definition 3.1. *The system defined by equation (1.1) together with its initial set G will be called* initial state stable *iff for $x \equiv 0$ and for any initial state $g : G \to \mathbf{C}$ there exists a positive constant K such that $\|y\| \leq K\|g\|$.*

The system defined by equation (1.1) together with its initial set G will be called input stable *iff for the initial state $g \equiv 0$ and for any input $x : A \to \mathbf{C}$ there exists a positive constant K such that $\|y\| \leq K\|x\|$.*

These two concepts of stability are not equivalent. There exist initial state stable systems, which are not input stable. A simple class of such examples in n-D systems are those which exhibit the well-known phenomenon of resonance.

Theorem 3.1. *Let in equation (1.1) the initial set G and the ordering is fixed so that the coefficient $a(\alpha, \beta_0) \neq 0$ for the leading element β_0 and for all $\alpha \in A$.*

If

$$|a(\alpha, \beta_0)| > \sum_{\beta \in B_0} |a(\alpha, \beta)| \quad \text{for all } \alpha \in A, \quad B_0 = B \setminus \{\beta_0\} \tag{3.3}$$

then the system described by (1.1) is both input and initial state stable.

Note that this theorem, which might be called the principle of *leading coefficient dominance* (LCD), is formulated for linear difference equations with variable coefficients. In the last section an example of its use in this rather general setting will be presented. Various non-linear generalizations of Theorem 3.1 as well as its formulation for systems of equations is also possible.

The special case with $A = Z_+^n,\ B \subset -A,\ 0 \in B$ has an almost self-evident corollary: For a polynomial $P(z) = \sum_{\alpha \in B} a_\alpha\, z^\alpha$ with $|a_0| > \sum_{\alpha \in B \setminus \{0\}} |a_\alpha|$ there is $P(z) \neq 0$ for all $|z| = \max_{1 \leq i \leq n} \leq 1$. This result compares the LCD principle to the common BIBO stability criterion in a highly specialized case of an n-D LSI system.

Recall now the equation (1.2) with $a = 1,\ b = 0.5,\ c = 0.2,\ d = 0.1$. The system $A(G_1)$ is stable, The system $A(G_2)$ does not satisfy the above sufficient conditions and straightforward calculations show that this system is unstable.

It is interesting to consider a kind of 'interaction' of the two systems $A(G_i)$. Since both $G_i \subset A, i = 1, 2$, the output of $A(G_1)$ is bounded on the set G_2. Taking these values as initial values of the system $A(G_2)$, the output of $A(G_2)$ must be bounded due to uniqueness of the solution. Hence, for some (in fact for an infinite set) of bounded initial conditions the unstable system $A(G_2)$ has bounded outputs. For ordinary (1-D) difference equations this fact has a simple explanation: The linear space of initial conditions for an unstable homogeneous equation with constant coefficients contains a linear subspace (which might be empty if all the characteristic values are greater than 1 in modulus), which gives rise to bounded solutions. For a homogeneous linear n-D difference equation let $\mathcal{G}$ denote the linear space of all 'initial conditions' $g : G \to C$. The linear space $\mathcal{G}$ may contain a linear subspace $\mathcal{G}_1$ such that all solutions y satisfying initial conditions $g \in \mathcal{G}_1$ are bounded. The construction of $\mathcal{G}_1$ is far from being evident. This phenomenon demands further investigations. Its 'nonhomogeneous counterpart' might be closely connected to the well known problem of second kind singularities originally investigated by Goodman [1].

4 Growth

Introduce a new function η in equation (1.1) as follows: $y(\alpha) = \lambda^\alpha \eta(\alpha)$ with λ to be determined later. We obtain a new equation which may be forced to satisfy the sufficient initial state stability condition by appropriate choice of the vector λ. Evidently, this condition reads

$$\sum_{\beta \in B_0} |a_\beta| \lambda^\beta < |a_{\beta_0}| \lambda^{\beta_0}$$

and we have the following

Corollary 4.1. *Let $\tilde{A}$ be such subset of the index set A of the equation (1.1) that for all $\alpha \in \tilde{A}$ the homogeneous equation (1.1) can be written in explicit form*

$$a(\alpha, \beta_0)\, y(\alpha, \beta_0) = -\sum_{\beta \in B_0} a(\alpha, \beta)\, y(\alpha + \beta), \quad B_0 = B \setminus \{\beta_0\}. \tag{4.4}$$

Then for any $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n), \lambda_i > 0$ satisfying for all $\alpha \in \tilde{A}$ the inequality

$$\sum_{\beta \in B_0} |a(\alpha, \beta)| \lambda^\beta < |a(\alpha, \beta_0)| \lambda^{\beta_0} \tag{4.5}$$

there exists a positive constant M such that

$$\|y\| \leq M\, \lambda^{-\alpha} \|\tilde{g}\|.$$

Here, $\tilde{g}$ represents the initial state on the set $\tilde{G} \subset G$ ensuring the existence and unicity of the solution on $\tilde{A}$.

The inequality (4.5) shows the significance of various values of the 'characteristic polynomial' for equation (1.1) and yields interesting growth estimates.

As an example let a growth estimate for the Stirling numbers of the first kind be derived. These numbers are defined by the following difference equation

$$y(n, m) = (n-1) y(n-1, m) + y(n-1, m-1) \tag{4.6}$$

on the index set $A = \{(n, m) : n \geq 1, 1 \leq m \leq n\}$ where $y(n, n) = 1$ and $y(n, 0) = 0$ both for $n \geq 0$. Using this corollary we obtain the estimate $y(n, m) \leq n! n^\varepsilon$, valid for all $\varepsilon > 0$. Since $y(n, 1) = n!$, this estimate cannot be improved. A similar result can be derived also for the Stirling numbers of the second kind.

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