Factorization of M-D Polynomial Matrices for Design of M-D Multirate Systems

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Abstract

The problem of the design of effective 2-D and 3-D multirate systems with prescribed properties is considered using tools from commutative algebra. Results for factoring 2-channel polyphase matrices are presented. After such a factorization, the number of computations may be reduced. For a 3-channel multirate system, an algorithmic version of Suslin's stability theorem may be useful for factoring the polyphase matrices.

1 Introduction

The growing demand for *processing* and *compression* of still two-dimensional (2-D) images and video (3-D) signals in *telecommunications* and *multimedia technology* motivates the fact that increasingly more attention is being paid to multi-dimensional (M-D) systems.

An attempt to tackle the problem of nonseparable ("true") M-D multirate systems' design is presented. It is supposed that nonseparable systems may have better compression and approximation results, but they are more difficult to design.

An important aspect of the implementation of these systems is the number of computations. The theory of Gröbner bases for ideals and modules over a multivariate polynomial ring, $\mathcal{K}[z_1, z_2, \ldots, z_n]$, when $\mathcal{K}$ is an arbitrary but fixed field and $z_1, z_2, \ldots, z_n$ are independent variables, is applied to solve the problem of factorization of the M-D polyphase matrices in order to reduce the complexity of the calculations. An algorithmic proof of Suslin's stability theorem provides a method for finding an explicit factorization of a given polynomial matrix into elementary matrices.

2 Design of M-D PR LP filter banks

The usual requirements that M-D FBs should meet are the following:

- perfect reconstruction (PR) property;

- linear phase (LP) property;
- the filters should be FIR;
- nonseparable lattices and FBs are desirable;
- the frequency responses should be quite smooth at the edges of the stop-bands.

Different approaches exist in M-D filter bank design. One of the most efficient is the polynomial approach, which gives the result directly. Some new results in the theory of FBs design with these properties based on the theory of Gröbner bases were obtained in [2, 5, 6]. In [3] Bernstein polynomials were used to design M-D filter banks. Nevertheless the 3-D case was considered in an inadequate way.

2-D case. It is assumed the 2-channel case and the quincuncial type of downsampling, which is the simplest nonseparable downsampling lattice [12]. The quincunx sublattice is generated by

$$V = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The PR condition can be written then as

$$H_0(z_1, z_2)H_1(-z_1, -z_2) - H_1(z_1, z_2)H_0(-z_1, -z_2) =$$
$$= z_1^{-2k_1-1} z_2^{-2k_2},$$

where $H_0(z_1, z_2), H_1(z_1, z_2)$ are the low-pass and high-pass filters of the analysis filter bank and k_1 and k_2 are arbitrary.

As it was shown in [3, 9, 10] Bernstein polynomials can be applied in order to design the FBs with the properties mentioned above. In this case the following low-pass analysis filter was found

$$H_0(z_1, z_2) = \frac{1}{2^{4N}} \sum_{i=0}^{N} \sum_{j=0}^{N-i} g_{i,j} \begin{pmatrix} N \\ i \end{pmatrix} \begin{pmatrix} N \\ j \end{pmatrix} (-1)^{i+j}$$
$$(1 - z_1^{-1})^{2i}(1 + z_1^{-1})^{2(N-i)} \cdot (1 - z_2^{-1})^{2j}(1 + z_2^{-1})^{2(N-j)},$$

with $g_{i,j}$ chosen according to the given FB's properties.

The values of N (and M for the high-pass filter) allow one to adjust the smoothness of the frequency responses for the low-pass and high-pass filters.

For the case $N = 1, M = 1$ the polyphase matrices are

$$\mathbf{H_p}(a, b) = \begin{bmatrix} 1 & B \\ B & C \end{bmatrix}, \qquad \mathbf{F_p}(a, b) = \begin{bmatrix} C & -B \\ -B & 1 \end{bmatrix},$$

and $B = 1/4 \cdot (1 + b)(1 + a), C = 1/16 \cdot (1 + 2b + 2a + b^2 - 28ab + a^2 + 2ab^2 + 2a^2b + a^2b^2)$, where $a = z_1^{-1}, b = z_2^{-1}$.

3-D case. It is also assumed the 2-channel case and one of the simplest nonseparable downsampling lattice that is generated by $V = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

A similar technique gives for the case $N = 1, M = 1$ the polyphase matrix $\mathbf{H_p}(a,b) = \begin{bmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{bmatrix}$ where $H_{00} = 1/2, H_{01} = H_{10} = -1/32(ab^2d + a^2 - 4abd + b^2d^2 + ad - 4a^2bd - 4ab^2d^2 - 4a^2d - 4abd^2 + a^2b^2d^2 + a^3d - 4a^2bd^2 + ab^2d^3 + a^2d^2)/(abd)$ and $H_{11} = 1/512(a^4d^4 + 20a^2b^2d^2 - 8a^2bd^2 + 2ab^2d^3 + 2a^5d - 8d^3a^2b + a^2d^2 + 2a^3d - 8a^5d^3b + a^4 - 8b^3d^5a^2 + 16b^3d^3a^2 + a^4b^4d^4 - 8a^4b^3d^4 + 20d^4a^2b^2 + 16a^2b^2d^3 + 16b^3d^4a^2 + 2a^3b^4d^5 + b^4d^6a^2 + 16a^3b^3d^3 + b^4a^2d^2 + 2a^3d^5b^2 + 20b^4d^4a^2 - 8a^3b^3d^2 - 8a^4b^3d^3 - 8a^3b^4d^4 + 20a^4d^4b^2 + 16a^4d^3b^2 - 8b^4d^3a^2 - 8b^4d^5a^2 + 2a^5d^3b^2 + 16a^3b^3d^4 + 16a^3d^4b^2 - 8a^3b^3d^5 + 16a^3b^2d^2 - 8a^2b^3d^2 - 408a^3d^3b^2 + 2a^3b^4d^3 + 20a^4b^2d^2 - 8a^3d^2 - 8b^3d^4a - 8a^3d^4b - 8b^3d^3a + a^6d^2 + 2a^3b^2d + 16a^3bd^2 + 2a^5d^3 - 8a^5bd^2 + 16a^4bd^2 + 2b^4d^3a - 8a^5d^2 - 8a^4d + 16a^4d^3b - 8a^4d^4b - 8a^4bd + 20a^4d^2 + 2b^4d^5a + b^4d^4 + 16a^3d^3b - 8a^4d^3 + 2a^3d^3 - 8a^3bd - 8b^4d^4a)/(a^2b^2d^2)$.

3 Factorization of two-channel 2-D and 3-D polyphase matrices

Any $M-$channel filter bank is represented by $M \times M$ polyphase polynomial matrices. A polyphase matrix may be factored into a product of **elementary** and **diagonal** matrices by application of a Gaussian elimination procedure (an elementary matrix $[i,j,f]$ is a matrix which coincides with the identity except for possibly a single off-diagonal entry f in the ij-position).

The main reasons behind factorization of polyphase matrices are:

- to reduce the number of required computations (additions, multiplications),
- to obtain 'good' coefficients (integers, powers of two and so on) for the filters.

As a result, the following factorization was obtained in the 2-D case (for $N = 1$ and $M = 1$ - see [10, 11]): $\mathbf{H_p} = \mathbf{H}_1 \cdot \mathbf{H}_2 \cdot \mathbf{H}_3$, where

$$\mathbf{H_1} = \begin{bmatrix} 1 & 0 \\ B & 1 \end{bmatrix}, \qquad \mathbf{H_2} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \cdot ab \end{bmatrix}, \qquad \mathbf{H_3} = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$$

and $B = 1/4 \cdot (1+b)(1+a)$.

Similar results were obtained for $N = M = 2$ and $N = M = 3$. It should be mentioned that this procedure may be applied for any values of N and M.

In the 3-D case the result was: $\mathbf{H_p} = \mathbf{H}_1 \cdot \mathbf{H}_2 \cdot \mathbf{H}_3$, where

$$\mathbf{H_1} = \begin{bmatrix} 1 & 0 \\ B & 1 \end{bmatrix}, \qquad \mathbf{H_2} = \begin{bmatrix} 1/2 & 0 \\ 0 & -ad \end{bmatrix}, \qquad \mathbf{H_3} = \begin{bmatrix} 1 & B \\ 0 & 1 \end{bmatrix}$$

and $B = -1/16((d + a)(ad + 1)(a + b^2d) - 4ad(b^2d + bd + abd + b + ab + a))/(abd)$.

It can be noticed that in both the 2-D and 3-D cases most of the coefficients are powers of two.

4 An algorithmic version of Suslin's stability theorem

Suslin's stability theorem [8] appeared in 1977, when A.A. Suslin proved that $E_m(R)$, the subgroup of the group of invertible matrices which is generated by the elementary matrices, is equal to the special linear group $SL_m(R)$, whenever R is a multivariate polynomial ring over a field and $m \geq 3$. Consequently, a multivariate polynomial matrix of order at least three with determinant one can be factored into a product of elementary matrices. Over a univariate polynomial ring over a field, a factorization into elementary matrices can be obtained by applying the Euclidean algorithm to any row or column of the matrix. The Smith normal form algorithm may also be used. These techniques fail, however, if the polynomial ring has at least two variables, since such a polynomial ring is not a Euclidean domain and therefore, the Euclidean algorithm is no longer available to use. An algorithmic version of Suslin's stability theorem is presented in [7]. The algorithm has input a polynomial matrix $\mathbf{A}$ (3×3 or larger)with determinant one and outputs matrices α and β which are explicit products of elementary matrices, such that $\alpha\mathbf{A}\beta$ is the identity matrix. Consequently, $\alpha^{\mathbf{-1}}\beta^{\mathbf{-1}}$ is a factorization of $\mathbf{A}$ into a finite product of elementary matrices. The algorithm consists of three key steps:

- reduction to the special case of $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where a, b, c and d are multivariate polynomials with $ad - bc = 1$,
- generation of solutions over finitely many suitable local rings (which allows division by certain polynomials),
- patching together the local solutions (which involve ratios of polynomials) to obtain a global solution (which involves strictly polynomials).

Unfortunately, the aforementioned algorithm for Suslin's stability theorem is not practical. One reason is the patching step which patches together solutions over local rings in such a way as to obtain a global solution. The Hilbert basis theorem, which states that any ideal of a multivariate polynomial ring over a field is finitely generated, guarantees that only finitely many local solutions are needed to obtain a global solution. However, there is no a priori bound on exactly how many local solutions are necessary.

Can modifications be made to produce practical implementable algorithms? For instance, when the algorithm for Suslin's stability theorem is applied to some matrices, the patching step is not needed; the local case subalgorithm actually yields a global (polynomial) solution.

Thus, a solution can be found much more easily than by using the entire algorithm. One question needing further investigation is whether or not the local case subalgorithm will work in enough cases to be of practical use. For example, consider the matrix [4]

$$\mathbf{H}(x,y) = \begin{pmatrix} 1+g_0+g_1 & (px+q)^4 & 0 \\ (ry+s)^4 & 1-g_0+g_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where p, q, r, and s are real numbers and $g_0 = \sqrt{2}\,(px+q)\,(ry+s)$, $g_1 = (px+q)^2(ry+s)^2$. The following factorization of $\mathbf{H}(x,y)$ was found by using only the local case subalgorithm:

$$\begin{aligned}
&[2,1,(ry+s)^4(1-g_0+g_1)]\cdot[2,3,-(px+q)^3(ry+s)^4]\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\\
&\cdot[2,1,(px+q)(ry+s)^4(1-g_0+g_1)]\cdot[2,3,-(px+q)^3(ry+s)^4]\cdot[2,3,-1]\cdot[3,2,1]\\
&\cdot[2,3,-1]\cdot[2,1,(px+q)^2(ry+s)^4(1-g_0+g_1]\cdot[2,3,-(px+q)^3(ry+s)^4]\cdot[2,3,-1]\\
&\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,1,(px+q)^3(ry+s)^4(1-g_0+g_1)]\cdot[2,3,-(px+q)^3(ry+s)^4]\\
&\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,1,1-g_0+g_1]\cdot[1,2,-1-g_0-g_1]\cdot[3,2,-1]\cdot[2,3,1]\\
&\cdot[3,2,-1]\cdot[2,1,-(\sqrt{2}(ry+s)+(px+q)(ry+s)^2]\cdot[1,2,px+q]\cdot[2,1,-1]\cdot[1,2,1]\\
&\cdot[2,1,-1]\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\\
&\cdot[1,2,-\sqrt{2}(ry+s)-(px+q)(ry+s)^2]\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,3,1]\cdot[3,2,g_1^2]\\
&\cdot[3,1,-(px+q)(1-g_0+g_1)]\cdot[3,2,-1]\cdot[2,3,1]\cdot[3,2,-1]\\
&\cdot[2,1,-\sqrt{2}(ry+s)+(px+q)(ry+s)^2]\cdot[1,2,px+q]\cdot[2,1,-1]\cdot[1,2,1]\cdot[2,1,-1]\\
&\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\\
&\cdot[1,2,-\sqrt{2}(ry+s)-(px+q)(ry+s)^2]\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,3,px+q]\\
&\cdot[3,2,(px+q)^3(ry+s)^4]\cdot[3,1,-(px+q)(1-g_0+g_1)]\cdot[3,2,-1]\cdot[2,3,1]\cdot[3,2,-1]\\
&\cdot[2,1,-\sqrt{2}(ry+s)+(px+q)(ry+s)^2]\cdot[1,2,px+q]\cdot[2,1,-1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\\
&\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,-\sqrt{2}(ry+s)-(px+q)(ry+s)^2]\\
&\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,3,(px+q)^2]\cdot[3,2,(px+q)^2(ry+s)^4]\\
&\cdot[3,1,-(px+q)(1-g_0+g_1)]\cdot[3,2,-1]\cdot[2,3,1]\cdot[3,2,-1]\\
&\cdot[2,1,-\sqrt{2}(ry+s)+(px+q)(ry+s)^2]\cdot[1,2,px+q]\cdot[2,1,-1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\\
&\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,1]\cdot[1,2,1]\cdot[2,1,-1]\cdot[1,2,-\sqrt{2}(ry+s)-(px+q)(ry+s)^2]\\
&\cdot[2,3,-1]\cdot[3,2,1]\cdot[2,3,-1]\cdot[2,3,(px+q)^3]\cdot[3,2,(px+q)(ry+s)^4]\\
&\cdot[3,1,-(px+q)(1-g_0+g_1)]\cdot[2,1,-1]\cdot[1,2,1]\cdot[2,1,-1].
\end{aligned}$$

5 Heuristic methods

Multiplying by elementary matrices corresponds to row and column operations from Gaussian elimination. So, it is reasonable to try and obtain a factorization of a polynomial matrix by mimicking Gaussian elimination. If after fixing a monomial order, there are entries f and

g in some column or row of the matrix to be factored such that the leading term of f (with respect to the monomial order) divides the leading term of g, then the matrix can be reduced to a "simpler" matrix, in the sense that, by multiplying by an appropriate elementary matrix, g can be replaced by $g - LT(g)$ (where $LT(g)$ denotes the leading term of g). In some cases (but not all), a complete factorization may be obtained by repeating this process. For example, using this method one obtains the following factorization of $\mathbf{H}(x, y, z, w)$:

$$\begin{pmatrix} x^2y + x^2z + 2x - 2xy - xz + y - 2 & x^2 - x & xy - y + 1 \\ xyz + xz^2 - yz + xw + 2z - w & xz + 1 & yz + w \\ -xy - xz + y - 1 & -x & -y \end{pmatrix} = [1, 3, -x] \cdot [2, 3, -z]$$
$$\cdot [1, 2, -1] \cdot [2, 1, 1] \cdot [1, 2, -1] \cdot [2, 1, x] \cdot [3, 1, -x] \cdot [2, 3, -1] \cdot [3, 2, -xw + y] \cdot [2, 3, x - 1]$$
$$\cdot [2, 3, -1] \cdot [3, 2, 1] \cdot [2, 3, -1] \cdot [1, 3, w] \cdot [1, 2, -xw - z + w] \cdot [2, 1, -1] \cdot [1, 2, 1] \cdot [2, 1, -1].$$

6 Summary

M-D polynomial matrices appear in many applications including the design of 2-D and 3-D FBs with certain desired properties. The factorization of these matrices may speed up the computation rate. Examples were presented of using the local case subalgorithm of an algorithm for Suslin's stability theorem and of using heuristic methods to factor M-D polynomial matrices. These factorizations allow one to obtain effective realizations of multirate systems which are suitable for a wider range of industrial problems.

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