Matrix Functions in Homomorphic Signal Processing

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Abstract

Yamada et al. [1] suggested replacing the traditional cepstrum operator used in homomorphic signal processing by a finite-dimensional alternative called isomorphic operator. This paper sheds another light on the isomorphic operator in terms of two matrix functions: exponential and logarithmic. Closed form formulas for 1–D and 2–D cases are presented.

1 Introduction

In [1] Yamada et al. introduced an alternative to the cepstrum transform called isomorphic operator Ψ. The operator transforms a multidimensional sequence of finite support in the first quadrant into another sequence of finite support in the first quadrant such that

$$\Psi(x \circledast y) = \Psi(x) + \Psi(y) \tag{1.1}$$

for any two sequences x, y of finite support; here $\circledast$ denotes the finite (truncated) convolution. Together with a coordinate transformation (described also in [1]) the operator Ψ is a suitable tool for homomorphic signal processing in any dimension. Its major asset is that not only the domain of Ψ but also the range consists only of finite support sequences. The definition of Ψ, however, is not very transparent and certain properties remained unrevealed.

We first show that Ψ is equivalent to the logarithmic function in the Banach algebra S_n of sequences with finite support in the first quadrant where multiplication is the finite convolution $\circledast$ and its inverse Ψ^{-1} is equivalent to the exponential function in the same Banach algebra. Next we show that in the two practically most important cases, 1–D and 2–D, S_n is isomorphic to the Banach algebra of triangular Toeplitz and block–Toeplitz matrices respectively. This will enable us to express Ψ and Ψ^{-1} in terms of matrix operations; we derive closed-form formulas (3.5) and (3.8) for Ψ and (3.4) and (3.7) for Ψ^{-1}.

2 Finite Support Sequences

Let M be a fixed positive integer, let $\mathbf{Z}^M$ denote the set of all M-tuples of integers and $Q = \{q \in \mathbf{Z}^M; q = (q_1, \ldots, q_M),\ q_i \geq 0,\ i = 1, \ldots, M\}$ be the first quadrant in $\mathbf{Z}^M$. Let S

be the linear space of all M-dimensional sequences $\{x(k)\}_{k\in Q}$. For an $n = (n_1, \ldots, n_M) \in Q$ we define

$$Q_n = \big\{q \in Q;\ q = (q_1, \ldots, q_M),\ 0 \le q_i \le n_i,\ i = 1, \ldots, M\big\}$$

and denote S_n the linear subspace of S comprising all sequences with support in Q_n. We identify the linear space S with the linear space F of all formal power series

$$X(z) = \sum_{k\in Q} x(k)\, z^k = \sum_{k\in Q} x(k_1, \ldots, k_M)\, z_1^{k_1} \cdots z_M^{k_M}$$

and for $n \in Q$ we introduce the projection operator $P_n\colon S \to S_n$ by $P_n(x) = x_n$, where

$$x_n(k) = \begin{cases} x(k) & \text{for} \quad k \in Q_n \\ 0 & \text{otherwise} \end{cases}$$

In S_n we define an operation $\circledast$ of finite convolution

$$x \circledast y = P_n(x * y),$$

where $*$ denotes usual linear convolution in S:

$$(x * y)(k) = \sum_{l\in Q} x(l)\, y(k - l), \quad k \in \mathbf{Z}^M$$

and a norm

$$||x|| = \sum_{k\in Q_n} |x(k)|.$$

S_n is then a commutative Banach algebra with unity

$$\delta(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \ne 0 \end{cases}$$

and hence admits the exponential function Exp and logarithmic function Log [2]:

$$\operatorname{Exp} x = \delta + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.2}$$

where $x^k = x \circledast x \circledast \ldots \circledast x$ (k-times) and

$$y = \operatorname{Log} x \quad \text{iff} \quad x = \operatorname{Exp} y.$$

Note that S_n has only one homomorphism h: $h(x) = x(0)$. This leads to simple conditions for invertibility and existence of logarithm.

Proposition 2.1. *A sequence x has a convolution $\circledast$ inverse in S_n if and only if $x(0) \ne 0$.*

Proof. An element x is invertible in a Banach algebra if and only if $h(x) \neq 0$ for all homomorphisms h of the Banach algebra [2]. $\square$

Proposition 2.2. *A sequence x has a logarithm in S_n if and only if it is invertible in S_n.*

Proof. An element x has a logarithm in a Banach algebra A if and only if $\log h(x)$ is continuous on the space of all homomorphisms of A [3]. $\square$

In [1] an operator $\Psi\colon S_n^o \to S_n$ was defined such that (1.1) holds; the domain of Ψ is $S_n^o = \{x \in S_n;\ x(0) > 0\}$. The definition of Ψ was shown [1] to be equivalent to

$$\Psi\big(X(z)\big) = P_n\big(\log X(z)\big),$$

where $\log X(z)$ is expressed in terms of its Taylor series coefficients in $z_0 = 0$. It was also proved in [1] that Ψ is invertible and

$$\Psi^{-1}\big(X(z)\big) = P_n\big(\exp X(z)\big).$$

The relation of Ψ to the logarithmic and exponential functions in S_n is shown in the next two theorems.

Theorem 2.1. $\Psi^{-1}(x) = \operatorname{Exp} x$ *for any* $x \in S_n$.

Proof. Since $x \circledast y = P_n(x * y)$ for any $x, y \in S_n$,it follows from (2.2) (due to linearity and continuity of P_n) $\operatorname{Exp} x = \exp\left(P_n(X(z))\right) = \Psi^{-1}(x)$. $\square$

Theorem 2.2. *If $x \in S_n$ and $x(0) > 0$, then* $\Psi(x) = \operatorname{Log} x$.

Proof. Let $y = \operatorname{Log} x$. Then $x = \operatorname{Exp} y = \Psi^{-1}(y)$, hence $y = \Psi(x)$. $\square$

3 Matrix Computations

For $M = 1, 2$ we now describe Ψ in terms of matrix operations. First, let $M = 1$ and n be a positive integer. Let $T^{(n)}$ denote the linear space of all lower-triangular Toeplitz matrices of order $n + 1$, i.e. matrices

$$\boldsymbol{X} = \begin{pmatrix} x_0 & 0 & 0 & \dots & 0 \\ x_1 & x_0 & 0 & \dots & 0 \\ x_2 & x_1 & x_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_n & x_{n-1} & x_{n-2} & \cdots & x_0 \end{pmatrix}. \tag{3.3}$$

Each such matrix is uniquely determined by its first column $x = (x_0, x_1, \dots, x_n)^T$. Moreover, if $x, y \in S_n$ determine matrices $\boldsymbol{X}, \boldsymbol{Y} \in T^{(n)}$ respectively, then $x \circledast y$ determines the matrix $\boldsymbol{XY}$. Hence the following theorem holds.

Theorem 3.1. *The linear space $T^{(n)}$ with usual matrix multiplication and the norm*

$$||\boldsymbol{X}|| = \sum_{i=0}^{n} |x_i|$$

is a commutative Banach algebra which is isometrically isomorphic to S_n.

By combining Theorems 2.1, 2.2 and 3.1 we can express $\operatorname{Log} x$ and $\operatorname{Exp} x$, hence also $\Psi(x)$ and $\Psi(x)^{-1}$ in terms of matrix logarithmic and exponential functions.

Theorem 3.2. *Let $x = \{x_0, x_1, \ldots, x_n\}$ and $\boldsymbol{X}$ be the matrix (3.3). Then*

$$\exp \boldsymbol{X} = \mathrm{e}^{x_0} \left(\boldsymbol{I} + \sum_{k=1}^{n-1} \frac{1}{k!} \Big(\boldsymbol{X} - x_0 \boldsymbol{I} \Big)^k \right) \tag{3.4}$$

and $\Psi^{-1}(x)$ equals to the first column of the matrix $\exp \boldsymbol{X}$. If $x_0 \neq 0$, then

$$\log \boldsymbol{X} = \big(\log x_0 \big) \boldsymbol{I} + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k\, x_0^k} \Big(\boldsymbol{X} - x_0 \boldsymbol{I} \Big)^k \tag{3.5}$$

and $\Psi(x)$ equals to the first column of the matrix $\log \boldsymbol{X}$.

Proof. Any function f of a matrix $\boldsymbol{A}$ is determined by a polynomial p whose values on the spectrum of $\boldsymbol{A}$ coincide with those of f: $p^{(j)}(\lambda_i) = f^{(j)}(\lambda_i)$ for each eigenvalue λ_i and $j = 0, 1, \ldots, m_i - 1$, where m_i is the geometric multiplicity (index) if λ_i [4]. Since $\boldsymbol{X}$ has a single eigenvalue x_0 of multiplicity n (both algebraic and geometric), this amounts to $p^{(j)}(x_0) = f^{(j)}(x_0), \quad j = 0, 1, \ldots n - 1$. Then

$$p(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{n!} \, (x - x_0)^i$$

Formulas (3.4) and (3.5) follow by setting $f(x) = \exp x$ and $f(x) = \log x$ respectively. ☐

Moreover, if $x_1 = x_2 = \cdots = x_{l-1} = 0$, then geometric multiplicity of x_0 decreases and

$$(\boldsymbol{X} - x_0 \boldsymbol{I})^k = \boldsymbol{O} \quad \text{for} \quad k > \left[\frac{n}{l} \right] + 1$$

($[\cdot]$ denotes the integer part) and the number of nonzero terms in (3.5) and (3.4) reduces considerably.

Next, let $M = 2$ and $n = (n_1, n_2)$, where n_1 and n_2 are positive integers. Let $B^{(n)}$ denote the linear space of all triangular block Toeplitz matrices $\boldsymbol{X}$ whose blocks $\boldsymbol{X}_k$ are also triangular and Toeplitz ($k = 0, 1, \ldots, n_2$):

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_0 & \boldsymbol{O} & \ldots & \boldsymbol{O} \\ \boldsymbol{X}_1 & \boldsymbol{X}_0 & \ldots & \boldsymbol{O} \\ \boldsymbol{X}_2 & \boldsymbol{X}_1 & \ldots & \boldsymbol{O} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{X}_{n_2} & \boldsymbol{X}_{n_2-1} & \cdots & \boldsymbol{X}_0 \end{pmatrix}, \quad \boldsymbol{X_k} = \begin{pmatrix} x_{0,k} & 0 & \ldots & 0 \\ x_{1,k} & x_{0,k} & \ldots & 0 \\ x_{2,k} & x_{1,k} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n_1,k} & x_{n_1-1,k} & \cdots & x_{0,k} \end{pmatrix}. \tag{3.6}$$

Theorem 3.3. *The linear space $B^{(n)}$ with usual matrix multiplication and the norm*

$$||\boldsymbol{X}|| = \sum_{i=0}^{n_1} \sum_{k=0}^{n_2} |x_{i,k}|$$

is a commutative Banach algebra which is isometrically isomorphic to S_n.

Proof. Obviously, it is sufficient to show that (3.6) is the matrix of the linear mapping L: $Lu = x \circledast u$ in a suitably chosen basis of S_n. For the sake of notation clarity we write $x(i,k)$ instead of $x_{i,k}$. Let

$$\delta_k(i) = \begin{cases} 1 & \text{for } \; i = k, \\ 0 & \text{otherwise,} \end{cases} \qquad k \in Q_M;$$

then $\mathscr{B} = \{\delta_{0,0}, \delta_{1,0}, \ldots, \delta_{m_1,0}, \delta_{0,1}, \delta_{1,1}, \cdots \delta_{m_1,m_2}\}$ is a basis of S_n (note that $\delta_{r,s}$ is the $((r+1) + s(m_1+1))$-th element of the basis). If $y_{r,s} = x \circledast \delta_{r,s}$, then

$$y_{r,s}(i,j) = \sum_{k=0}^{i} \sum_{l=0}^{j} x(k,l)\, \delta_{r,s}(i-k, j-l) = \begin{cases} x(i-r, j-s) & \text{if } \; i \geq r \text{ and } j \geq s, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the $((r+1) + s(m_1+1))$-th column of the matrix is

$$\Big(\underbrace{0, \ldots, 0}_{r+s(m_1+1)}, x(0,0), x(1,0), \ldots, x(m_1-r, 0), \underbrace{0, \ldots, 0}_{r}, x(0,1), x(1,1), \ldots, x(m_1-r, 1),$$
$$\ldots, \underbrace{0, \ldots, 0}_{r}, x(0, m_2-s), \ldots, x(m_1-r, m_2-s)\Big),$$

hence the same as the $((r+1) + s(m_1+1))$-th column of the matrix $\boldsymbol{X}$ in (3.6). □

As in 1–D case, the isomorphism between $B^{(n)}$ and S_n enables to identify the exponential functions in both algebras as well as logarithmic functions.

Theorem 3.4. *Let $x = \{x_{i,k}\}$, $i = 0, 1, \ldots, n_1$, $k = 0, 1, \ldots, n_2$, let $\boldsymbol{X}$ be the block matrix (3.6) and let $N = n_1 n_2$. Then*

$$\exp \boldsymbol{X} = \mathrm{e}^{x_{0,0}} \left(\boldsymbol{I} + \sum_{k=1}^{N-1} \frac{1}{k!} \Big(\boldsymbol{X} - x_{0,0} \boldsymbol{I} \Big)^k \right) \tag{3.7}$$

and $\Psi^{-1}(x)$ is determined by the first column of the matrix $\exp \boldsymbol{X}$. If $x_{0,0} \neq 0$, then

$$\log \boldsymbol{X} = \big(\log x_{0,0} \big)\, \boldsymbol{I} + \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k\, x_{0,0}^k} \Big(\boldsymbol{X} - x_{0,0} \boldsymbol{I} \Big)^k \tag{3.8}$$

and $\Psi(x)$ is determined by the first column of the matrix $\log \boldsymbol{X}$.

Proof. The matrix $\boldsymbol{X}$ has a single eigenvalue $x_{0,0}$ of algebraic multiplicity N. Though geometric multiplicity of $x_{0,0}$ is less than N, the proof is the same as in Theorem 3.2. □

4 Conclusion

We have presented an alternative view on the multidimensional isomorphic operator Ψ introduced by Yamada et al. in [1]. The isomorphic operator Ψ finds applications in homomorphic signal processing where it can replace the cepstrum transform or approximate the cepstrum of minimum phase signals. In this paper Ψ has been shown to be equivalent to the logarithmic function in the Banach algebra of triangular Toeplitz matrices and hence expressible in terms of matrix operations. From the theoretical viewpoint we thus get an interesting connection between two rather distant branches of applied mathematics. It is fair to admit, however, that in general the matrix operations will be more time consuming than the original recursive definition.

Acknowledgement: This work has been supported by the Research Program No. J04/ 98:212300013 "Decision Making and Control for Manufacturing" of the Czech Technical University in Prague (sponsored by the Ministry of Education, Youth and Sports of the Czech Republic).

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