Robust Stability and Stabilization of n-D Systems

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Abstract

In this paper, first, the stability margin and stabilizability margin of multidimensional (nD) systems are considered. In particular, the concept of stabilizability margin is introduced for the first time and it is defined to be the largest stability margin that a closed-loop feedback system can reach by menas of any stabilizing compensator. The investigation of stabilizability margin is of special interest in nD system theory, as such a problem does not happen to a conventional 1D system without hidden mode. A general computational algebraic procedure is then proposed, based on quantifier elimination formulation and cylindrical algebraic decomposition method, for both computation of the stability/stabilizability margins and construction of a stabilizing compensator which can reach a stability margin for the resulant closed-loop system as close to the stabilizability margin as desirable.

1 Introduction

In modern digital signal processing and control systems, robustness of stability is an essential problem that requires particular consideration. Indeed, due to the finite word length effect, a nominal stable system may have unstable behaviours. Extensive research work has been conducted on designing robustly stable systems. See, e.g., [1, 16, 10, 12, 6]. One approach is to design the system not only to be merely stable but also possess a sufficiently large stability margin, which can prevent the occurrence of some kind of unfavarable behaviour of the system. For example, the robustness of the stability of a 2-D system against parameter variation could be related to the its stability margin, see [12] for details. A large stability margin also makes the impulse response of a system fast to reach the steady state.

Due to its importance, there has been a large literature on the computation of the stability margin. See, e.g., [6] and the references cited there.

Another robustness issue which is as well important as the stability margin but has attracted less attention is the stabilizability margin of multidimensional linear systems, which is defined for the first time in this article. The stabilizability margin is defiened to be the largest stability margin that a closed-loop system can have by using any stabilizing compensator in a feedback loop. The investigation of the stabilizability margin has a particular interest in multidimensional system theory. In fact, even if a multidimensional system, e.g., a digital filter, may be designed a priori to be stable with a certain stability margin, stabilization techniques can be applied to enlarge this margin without re-designing the filter, to adapt to an operation environment that requires stronger robustness.

The purpose of this work is, first, to formulate the robust stabilization problem of multidimensional systems within the framework of input/output transfer function approach [15]; and second, to present a general computational algebraic method both for computing the stability/stabilizability margins and for constructing the stabilizing compensators.

A quantifier elimination (QE) approach is proposed here as a general method for the computational problems involved. Such an approach has been applied in linear systems theory since 1970's, mainly for solving the stability test problems and for output feedback stabilizer design, see [2, 9]. But to our best knowledge, this approach is used for the robustness issues for the first time in this paper.

In Section 2 the stability and stabilization problems for single input single output (SISO) n-D systems are formulated. Stability and stabilizability margins of a n-D system will be defined in Section 3. Section 4 presents a quantifier elimination (QE) formulation for computing both the stability and stabilizability margins, which can be solved based on the *Cylindrical Algebraic Decompostion* (CAD) [4, 5], a powerful computational algebraic method for solving algebraic problems.

Section 5 provides a QE formulation for the construction of a stabilizing compensator that may approximately achieve the stabilizability margin. In Section 6, the robust stability and stabilization problems of multi-input/multi-output (MIMO) systems are addressed. Some related problems are discussed in Section 7.

2 Stability and Stabilization of n-D Systems

In this section we review the stability and stabilization problems of an n-dimensional linear shift invariant SISO (single-input-single-output) system described by rational transfer function

$$p(z_1, \ldots, z_n) = \frac{f(z_1, \ldots, z_n)}{g(z_1, \ldots, z_n)}, \tag{2.1}$$

where $f(z_1, \ldots, z_n)$ and $g(z_1, \ldots, z_n)$ are relatively prime (free from common factor) polynomials in variables $z_1, \ldots, z_n$ (for simplicity, we also use a single $\boldsymbol{z}$ for expressing the n variables) with real or complex coefficients, i.e., $f(\boldsymbol{z}),\ g(\boldsymbol{z}) \in \boldsymbol{R}[\boldsymbol{z}]$, or $f(\boldsymbol{z}),\ g(\boldsymbol{z}) \in \boldsymbol{C}[\boldsymbol{z}]$. In the following we call $p(\boldsymbol{z})$ a **complex system** when $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ are complex polynomials, in contrast we call it a **real syetem** if $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ are real polynomials. Note that a real system may also be considered as a complex system. Let r be a positive real number, let

$$\Delta^n(0; r) = \{(z_1, \ldots, z_n) \in \boldsymbol{C}^n | \, |z_1| < r, \ldots, |z_n| < r\}$$

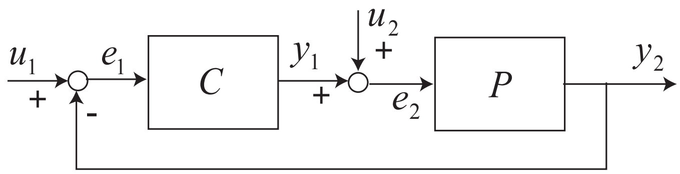

Figure 1: Feedback system.

be open polydisc in $\boldsymbol{C}^n$. Let $\bar{\Delta}^n(0;r)$ denote its closure. Let

$$\bar{U}^n = \bar{\Delta}^n(0;1)$$

be the closed unit polydisc in $\boldsymbol{C}^n$. Let

$$V(f(\boldsymbol{z})) = \{(z_1,\ldots,z_n) \in \boldsymbol{C}^n |\ f(z_1,\ldots,z_n) = 0\},$$

$$V(g(\boldsymbol{z})) = \{(z_1,\ldots,z_n) \in \boldsymbol{C}^n |\ g(z_1,\ldots,z_n) = 0\}$$

be two complex varieties defined by $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$, respectively. The system $p(\boldsymbol{z}) = f(\boldsymbol{z})/g(\boldsymbol{z})$ is by definition **(structurally) stable** if and only if $g(\boldsymbol{z})$ is free from 0 in $\bar{U}^n$, i.e.,

$$V(g(\boldsymbol{z})) \cap \bar{U}^n = \emptyset,$$

and the rational function $f(\boldsymbol{z})/g(\boldsymbol{z})$ and the polynomial $g(\boldsymbol{z})$ are said to be *stable* in this case. If a system is stable, then it is also BIBO (bounded input bounded output) stable, i.e., the output of the system is bounded for bounded input [7].In the case that the system p is not stable, one may try to use a stabilizing compensator with transfer function $c(\boldsymbol{z}) = h(\boldsymbol{z})/k(\boldsymbol{z})$ in a standard feedback configuration shown in Fig.1 [7], to obtain a stable closed-loop feedback system with inputs $u_1,\ u_2$, and outputs $y_1,\ y_2$, where u_2 denotes a disturbance input to the system.

The overall input-output relation can be written as:

$$\begin{aligned} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} \dfrac{c(\boldsymbol{z})}{1+p(\boldsymbol{z})c(\boldsymbol{z})} & \dfrac{-p(\boldsymbol{z})c(\boldsymbol{z})}{1+p(\boldsymbol{z})c(\boldsymbol{z})} \\ \dfrac{p(\boldsymbol{z})c(\boldsymbol{z})}{1+p(\boldsymbol{z})c(\boldsymbol{z})} & \dfrac{p(\boldsymbol{z})}{1+p(\boldsymbol{z})c(\boldsymbol{z})} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{h(\boldsymbol{z})g(\boldsymbol{z})}{k(\boldsymbol{z})g(\boldsymbol{z})+h(\boldsymbol{z})f(\boldsymbol{z})} & \dfrac{-h(\boldsymbol{z})f(\boldsymbol{z})}{k(\boldsymbol{z})g(\boldsymbol{z})+h(\boldsymbol{z})f(\boldsymbol{z})} \\ \dfrac{h(\boldsymbol{z})f(\boldsymbol{z})}{k(\boldsymbol{z})g(\boldsymbol{z})+h(\boldsymbol{z})f(\boldsymbol{z})} & \dfrac{k(\boldsymbol{z})f(\boldsymbol{z})}{k(\boldsymbol{z})g(\boldsymbol{z})+h(\boldsymbol{z})f(\boldsymbol{z})} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \quad (2.2)$$

It can be shown that each entry of the transfer matrix has no pole in $\bar{U}^n$ if and only if $k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z})$ is free from 0 in $\bar{U}^n$. The polynomial $k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z})$ is refered to as the *characteristic polynomial* of the closed-loop system.

Theorem 1. *[13] A necessary and sufficient condition for the existence of a controller $c(\boldsymbol{z}) = h(\boldsymbol{z})/k(\boldsymbol{z})$ such that*

$$k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}) \neq 0 \text{ in } \bar{U}^n \tag{2.3}$$

is that $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ have no commen zero in $\bar{U}^n$, i.e.,

$$V(f(\boldsymbol{z})) \cap V(g(\boldsymbol{z})) \cap \bar{U}^n = \emptyset,$$

This condition can be derived from Cartan's Theorem A, a well known fact in complex analysis in several variables. See [8] for details. Roughly speaking, the condition implies that around any point in $\bar{U}^n$, a unit is contained in the ideal generated by $f(\boldsymbol{z})$ and $g(\boldsymbol{z})$ in the ring of local analytic function germs, Cartan's Theorem ensures that this property can be "extended" globally over $\bar{U}^n$. In this case $p(\boldsymbol{z})$ is said to be **stabilizable**. If a stabilizable system $p(\boldsymbol{z})$ is real, that is, it possesses a transfer function with real coefficients, then the controller $c(\boldsymbol{z})$ can also be chosen to be real [13, Theorem 3.1.21].

3 Stability Margin and Stabilizability Margin

In practice, due to the uncertainties in the system parameters and the finite word length effect, a nominal stable system may have unstable behaviours. Extensive research work have been conducted on designing robustly stable systems. See, e.g., [1, 6, 10, 12, 16]. One approach is to design the system not only to be merely stable but also possess a sufficiently large stability margin, which can prevent the occurrence of some kind of unfavarable behaviour of the system. A large stability margin also makes the impulse response of a system fast to reach the steady state. There are various definitions of stability margin for a linear n-D system, the definition we take here is a simplest one, but the ideas and results that follow this definition may be trivially modified for other definitions.

Definition 1. *Let $p(\boldsymbol{z}) = f(\boldsymbol{z})/g(\boldsymbol{z})$ be a system. Its* **stability margin** *σ_0 is defined as*

$$\sigma_0 = \sup_{\sigma}\{\sigma \in \Sigma\}, \quad \textit{with}$$

$$\Sigma = \{\sigma \in \boldsymbol{R} : \ V(g(\boldsymbol{z})) \cap \bar{\Delta}^n(0; 1+\sigma) = \emptyset\}. \tag{3.4}$$

Remark: Since $V(g(\boldsymbol{z}))$ is a closed set and $\bar{\Delta}^n(0; 1+\sigma)$ is compact, the condition $V(g(\boldsymbol{z})) \cap \bar{\Delta}^n(0; 1+\sigma) = \emptyset$ means that the distance between $V(g(\boldsymbol{z}))$ and $\bar{\Delta}^n(0; 1+\sigma)$ is greater than 0, thus there exists some $\sigma_1 > \sigma$ such that $V(g(\boldsymbol{z})) \cap \bar{\Delta}^n(0; 1+\sigma_1) = \emptyset$ still holds. Therefore

Σ is an open set with upper bound σ_0. This means that $V(g(\boldsymbol{z}))$ intersects with $\bar{\Delta}^n(0; 1+\sigma)$ just on their boundaries. It is readily seen that a system is stable if and only if its stability margin $\sigma_0 > 0$.

Along this line of formulation, the stabilization of a system can be viewed as the construction of a compenator that yields a positive stability margin of the closed-loop feedback system. A system is stabilizable if such a compensator exists. Here naturally arises the question that what is the upper bound of the stability margin of the closed-loop system that can be achieved with any compensator? This upper bound is defined here as the "stabilizability margin" of the original system.

Definition 2. *Let $p(\boldsymbol{z}) = f(\boldsymbol{z})/g(\boldsymbol{z})$ be an n-D system. Its* **stabilizability margin** σ_s *is defined as*

$$\sigma_s = \sup_{\sigma}\{\sigma \in \Sigma_s\}, \ \textit{with}$$

$$\Sigma_s = \{\sigma \in \boldsymbol{R} : \exists c(\boldsymbol{z}) = h(\boldsymbol{z})/k(\boldsymbol{z}),$$

$$\forall z \in \Delta^n(0; 1+\sigma) \ k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}) \neq 0)\}. \tag{3.5}$$

Proposition 1.

$$\Sigma_s = \{\sigma \in \boldsymbol{R} : V(f(\boldsymbol{z})) \cap V(g(\boldsymbol{z})) \cap \bar{\Delta}^n(0; 1+\sigma) = \emptyset\}. \tag{3.6}$$

This fact does by no means follow trivially from the definition, however, the proof for Theorem 1 applies to this proposition by substituting $\bar{U}^n$ with $\bar{\Delta}^n(0; 1+\sigma)$, which has the same geomtric property necessary for the proof [13]. Also as an analogue to the stabilzation case, it could be shown that if $p(\boldsymbol{z})$ is a real system, then its stabilizability margin can be achieved by a real compensator.

Example 1. *Let $f(z_1, z_2) = 1 - 4z_1z_2$, $g(z_1, z_2) = z_1$. As $V(f(z_1, z_2)) \cap V(g(z_1, z_2)) = \emptyset$, we have $\Sigma_s = \boldsymbol{R}$, therefore $\sigma_s = \infty$. Actually, with the compensator $c(z_1, z_2) = 1/4z_2$, the characteristic polynomial of the closed-loop system is 1, thus the system has an infinite stability margin.*

Example 2. $f(z_1, z_2) = 1 - 4z_1z_2$, $g(z_1, z_2) = 1 + z_1 - z_2$.

$$V(f(z_1, z_2)) \cap V(g(z_1, z_2)) =$$

$$\{(\frac{-1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2}), \ (\frac{-1+\sqrt{2}}{2}, \frac{1+\sqrt{2}}{2})\}.$$

It is easy to check that

$$V(f(z_1,z_2)) \cap V(g(z_1,z_2)) \cap \bar{\Delta}^n(0;1+\sigma) = \emptyset \textit{ iff } \sigma < \frac{\sqrt{2}-1}{2},$$

thus

$$\Sigma_s = \{\sigma \in \boldsymbol{R}: \ \sigma < \frac{\sqrt{2}-1}{2}\}.$$

Therefore, the stabilizability margin is $\sigma_s = \frac{\sqrt{2}-1}{2}$. *With the compensator* $c(z_1,z_2) = \frac{1}{2(1+\sqrt{2})}$, *we can obtain a closed-loop system whose characteristic polynomial is*

$$4(\frac{1+\sqrt{2}}{2} + z_1)(\frac{1+\sqrt{2}}{2} - z_2).$$

Its stability margin is $\sigma_0 = \frac{\sqrt{2}-1}{2}$, *which is the best one achievable by any compensator.*

4 A General Method for Computing Stability and Stabilizability Margins

4.1 Stability margin computation

Write the complex variables $\boldsymbol{z}$ in real coordinates $x_1, y_1, \cdots, x_n, y_n$ as

$$\boldsymbol{z} = (z_1, \cdots, z_n) = (x_1 + iy_1, \cdots, x_n + iy_n).$$

Write the polynomial $g(z)$ as

$$g(\boldsymbol{z}) = u(x_1, y_1, \cdots, x_n, y_n) +$$

$$iv(x_1, y_1, \cdots, x_n, y_n),$$

where u and v are real polynomials.

Since Σ is defined as

$$\Sigma = \{\sigma \in \boldsymbol{R}: \ V(g) \cap \bar{\Delta}^n(0;1+\sigma) = \emptyset\},$$

its complementary subset is

$$\Sigma^c = \{\sigma \in \boldsymbol{R}: \ \exists z \in \bar{\Delta}^n(0;1+\sigma), \ g(z) = 0\}. \tag{4.7}$$

$$\Sigma^c = \{\sigma \in \boldsymbol{R}: (\exists x_1 \cdots \exists y_n)(x_1^2 + y_1^2 \leq (1+\sigma)^2$$

$$\wedge \cdots \quad \wedge x_n^2 + y_n^2 \leq (1+\sigma)^2 \wedge \quad u = 0 \ \wedge v = 0)\}. \tag{4.8}$$

Σ^c can be descibed by algebraic conditions imposed on its argument σ which can be obtained by *eliminating the quantifiers* $\exists x_1 \cdots \exists y_n$, such an operation is called **Quantifier Elimination (QE)**.

A general computational method based on the **Cylindrical Algebraic Decomposition (CAD)** [4] of algebraic sets has been developed for solving the QE problems.

Example 3. *Let $g(z_1, z_2) = z_1 z_2 - 4$. Though it is easy to inspect that the stability margin is 1, in the following we show a method in the essentials of CAD for solving the QE problem.*

Writting $g(z_1, z_2)$ as $g(z_1, z_2) = u(x_1, y_1, x_2, y_2) + iv(x_1, y_1, x_2, y_2) = (x_1 x_2 - y_1 y_2 - 4) + i(x_1 y_2 + x_2 y_1)$, the problem is to eliminate $(\exists x_1 \exists y_1 \exists x_2 \exists y_2)$ from $(\exists x_1 \exists y_1 \exists x_2 \exists y_2)(x_1^2 + y_1^2 \leq (1+\sigma)^2 \wedge x_2^2 + y_2^2 \leq (1+\sigma)^2 \wedge \ u = 0 \ \wedge v = 0)$. The semi-algebraic set $V(g(z_1, z_2)) \cap \bar{\Delta}^n(0; 1+\sigma)$ is defined by

$$\begin{aligned}
&\text{(i)} \quad x_1 x_2 - y_1 y_2 - 4 = 0, \\
&\text{(ii)} \quad x_1 y_2 + x_2 y_1 = 0, \\
&\text{(iii)} \quad x_1^2 + y_1^2 - (1+\sigma)^2 \leq 0, \\
&\text{(iv)} \quad x_2^2 + y_2^2 - (1+\sigma)^2 \leq 0.
\end{aligned}$$

Eliminating y_2 in (i) *and* (ii), *we have*

$$x_1(x_1 x_2 - 4) + x_2 y_1^2 = 0,$$

$$x_2 = \frac{4x_1}{x_1^2 + y_1^2},$$

for $x_1^2 + y_1^2 \neq 0$, as is ensured by equation (i). *This says that the projection in $\boldsymbol{R}^3$ is a surface parametrized by x_1 and y_1. From equations* (i) *and* (ii) *it can be deduced that*

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = 16,$$

therefore the inequalities (iii) *and* (iv) *can be rewritten as*

$$\frac{16}{(1+\sigma)^2} \leq x_1^2 + y_1^2 \leq (1+\sigma)^2,$$

which yields

$$16 \leq (1+\sigma)^4 \ \textit{or equivalently}$$

$$\sigma \geq 1 \quad \vee \quad \sigma \leq -3.$$

Therefore we have

$$\Sigma = \{\sigma \in \boldsymbol{R} : \sigma < 1 \quad \wedge \quad \sigma > -3\} = (-3, \ 1).$$

Thus the stability margin is $\sigma_0 = 1$.

A software package "QEPCAD" for solving QE problems based on CAD has been developed by H. Hong [5] and has been applied in some systems theoretic problems [9]. The package may potentially be used for computing the n-D system stability margin.

4.2 Stabilizability Margin Computation

The complementary subset of Σ_s is

$$\Sigma_s^c = \{\sigma \in \boldsymbol{R}:\ \exists \boldsymbol{z} \in \bar{\Delta}^n(0; 1+\sigma)\ (f(\boldsymbol{z}) = 0\ \wedge\ g(\boldsymbol{z}) = 0)\}. \tag{4.9}$$

Writing $f(\boldsymbol{z})$ as

$$f(\boldsymbol{z}) = s(x_1, \cdots, y_n) + it(x_1, \cdots, y_n),$$

where $s(x_1, \cdots, y_n)$ and $t(x_1, \cdots, y_n)$ are real polynomials.

$$\Sigma_s^c = \{\sigma \in \boldsymbol{R} : (\exists x_1 \cdots \exists y_n)(x_1^2 + y_1^2 \leq (1+\sigma)^2 \wedge \cdots\ \wedge x_n^2 + y_n^2 \leq (1+\sigma)^2$$

$$\wedge s(x_1, \cdots, y_n) = 0\ \wedge t(x_1, \cdots, y_n) = 0\ \wedge u(x_1, \cdots, y_n) = 0\ \wedge v(x_1, \cdots, y_n) = 0)\}. \tag{4.10}$$

In the same way as in the case for stability margin computation, this set can be expilicitly computed by eliminating the existential quantifiers $\exists x_1 \cdots \exists y_n$, by applying the CAD method. And the stabilizability margin

$$\sigma_s = \sup_{\sigma}\{\sigma \in \Sigma_s\}$$

can be computed.

5 Construction of Stabilizer by QE

Given a stabilizable n-D plant $p(\boldsymbol{z}) = f(\boldsymbol{z})/g(\boldsymbol{z})$ with stabilizability margin σ_s, the problem is to construct two polynomials $h(\boldsymbol{z}),\ k(\boldsymbol{z}) \in \boldsymbol{R}[\boldsymbol{z}]$ such that

$$k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}) \neq 0,\ \ \boldsymbol{z} \in \bar{\Delta}^n(0; 1+\sigma),$$

for a desirable $\sigma,\ 0 < \sigma < \sigma_s$.

Remark: The resulting closed-loop system has a stability margin which is greater than σ but is possibly smaller than σ_s.

If an upper bound is given for the total degree of the polynomials to be constructed, then a fixed finite number of coefficients suffice to characterize all the candidate polynomials. The stabilizer construction problem is reduced to one of determining the fixed number of coeffients such that the above inequlity holds. In practice, a controller of very high degree is of insignificant use, therefore, a reasonable way for constructing a stabilizer is to search the space of polynomials of increasing degrees, step by step, and stop whenever either a stabilizer is found or the total degree grows too high.

Provided that the upper bound of the total degree is given, the construction of a stabilzer can be formulated as a QE problem as follows.

Suppose that the total degree of both $h(z)$ and $k(z)$ is upper bounded by M.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers which denotes a composite power index such that

$$\boldsymbol{z}^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

$|\alpha| = \alpha_1 + \cdots + \alpha_n \leq M$. Write $h(\boldsymbol{z}),\ k(\boldsymbol{z})$ as

$$h(\boldsymbol{z}) = \sum_{|\alpha| \leq M} c_\alpha z^\alpha, \quad k(\boldsymbol{z}) = \sum_{|\alpha| \leq M} d_\alpha z^\alpha.$$

Rewrite $\boldsymbol{z}$ in real coordinates

$$\boldsymbol{z} = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n).$$

The polynomial $k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z})$ can be rewriten as a complex polynomial in real variables

$$c_\alpha,\ d_\alpha,\ x_j,\ y_j,\quad 0 \leq |\alpha| \leq M,\ 1 \leq j \leq n$$

as

$$k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}) = s(c_\alpha,\ d_\alpha,\ x_j,\ y_j;\quad 0 \leq |\alpha| \leq M,\ 1 \leq j \leq n)$$

$$+it(c_\alpha,\ d_\alpha,\ x_j,\ y_j;\quad 0 \leq |\alpha| \leq M,\ 1 \leq j \leq n),$$

where s and t are two real polynomials. h/k is a desired stabilizer if and only if

$$\forall x_1 \forall y_1 \cdots \forall x_n \forall y_n (x_1^2 + y_1^2 \leq 1 + \sigma \wedge \cdots \wedge x_n^2 + y_n^2 \leq 1 + \sigma \rightarrow s \neq 0 \vee t \neq 0),$$

or equivalently

$$\forall x_1 \forall y_1 \cdots \forall x_n \forall y_n (x_1^2 + y_1^2 > 1 + \sigma \vee \cdots \vee x_n^2 + y_n^2 > 1 + \sigma \vee s \neq 0 \vee t \neq 0).$$

Eliminating the bound variables $x_j,\ y_j,\ \ 1 \leq j \leq n$, a set of algebraic equality and inequality constraints on the free variables $c_\alpha,\ d_\alpha,\ \ 0 \leq |\alpha| \leq M$ are obtained, by solving which one can find a desirable stabilizer if any exisits.

Remark: A QE formulation for constructing a stabilizing compensator that achieves exactly the stabilizablity margin is also possible but requires a longer tedious description. Indeed, such a formulation can be obtained by transforming the intuitive geometric condition that the two sets $V(k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}))$ and $\bar{\Delta}^n(0; 1 + \sigma_s)$ just touch each other on their boundaries into a rigorous predictive logic formula composed of a set of algebraic equalities and inequalities.

6 Robust Stability and Stabilization of MIMO n-D Systems

Here a MIMO system is described by a *Matrix Fraction Description* (MFD). Some facts concerning the stability and stabilizability of such a system are summarized as follows.

A *left* MFD of $P(\boldsymbol{z})$ is defined as

$$P(\boldsymbol{z}) = D^{-1}(\boldsymbol{z})N(\boldsymbol{z}), \tag{6.11}$$

where $D(\boldsymbol{z})$ is an $m \times m$ and N an $m \times l$ polynomial matrix.

$$[D(\boldsymbol{z})\ N(\boldsymbol{z})] = \begin{bmatrix} d_{11} & \cdots & d_{1m} & n_{11} & \cdots & n_{1l} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ d_{m1} & \cdots & d_{mm} & n_{m1} & \cdots & n_{ml} \end{bmatrix}. \tag{6.12}$$

Let $\alpha_1(\boldsymbol{z}),\ \alpha_2(\boldsymbol{z}), \cdots, \alpha_M(\boldsymbol{z})$, $M = \begin{pmatrix} m+l \\ l \end{pmatrix}$, be the maximal order minors of $[D(\boldsymbol{z})\ N(\boldsymbol{z})]$, with $\alpha_1(\boldsymbol{z}) = \det D(\boldsymbol{z})$.

Let $d(\boldsymbol{z})$ be the greatest common divisor of $\alpha_i(\boldsymbol{z})$, $b_i = \alpha_i(\boldsymbol{z})/d(\boldsymbol{z}),\ i = 1, \ldots, M$ are called the "generating polynomials" or "reduced minors" of the system, which are independent of the MDF (see [11]). If $d(\boldsymbol{z})$ is a nonzero constant, then the MFD is said to be *minor coprime*, this property is irrelevant to our formulation here.

$P(\boldsymbol{z}) = D^{-1}(\boldsymbol{z})N(\boldsymbol{z})$ is stable iff

$$V(b_1(\boldsymbol{z})) \cap \bar{U}^n = \emptyset. \tag{6.13}$$

Theorem 2. *[14, 11] Let $P = D^{-1}N$ be an n-D MIMO system.*
Let $b_j,\ j = 1, \cdots, M,\ M = \begin{pmatrix} m+l \\ m \end{pmatrix}$, be the reduced minors of $[D(\boldsymbol{z})\ N(\boldsymbol{z})]$. Let $I = (b_1(\boldsymbol{z}), ..., b_M(\boldsymbol{z}))$ be the ideal generated by the reduced minors. Then the system $P(\boldsymbol{z})$ is stabilizable iff

$$V(I) \cap \bar{U}^n = \emptyset. \tag{6.14}$$

If $P(\boldsymbol{z}) = D^{-1}(\boldsymbol{z})N(\boldsymbol{z})$ is real, then a real stabilizing compensator exists.

Definition 1′. the Stability margin σ_0 *of a system $P(\boldsymbol{z}) = D^{-1}(\boldsymbol{z})N(\boldsymbol{z})$ is defined as*

$$\sigma_0 = \sup_{\sigma}\{\sigma \in \Sigma\}, \quad \text{with}$$

$$\Sigma = \{\sigma \in \boldsymbol{R}:\ V(b_1(\boldsymbol{z})) \cap \bar{\Delta}^n(0; 1+\sigma) = \emptyset\}. \tag{6.15}$$

Definition 2′. the Stabilizability margin σ_s *of* $P(\boldsymbol{z})$ *is the largest stability margin that can be achieved by any compensator. Let*

$$\Sigma_s = \{\sigma \in \boldsymbol{R} : \exists h_1(\boldsymbol{z}), \cdots, h_M(\boldsymbol{z}) \in \boldsymbol{C}[\boldsymbol{z}],$$

$$\forall \boldsymbol{z} \in \bar{\Delta}^n(0; 1+\sigma)\ (h_1(\boldsymbol{z})b_1(\boldsymbol{z}) + \cdots + h_M(\boldsymbol{z})b_M(\boldsymbol{z}) \neq 0)\}. \tag{6.16}$$

$$\sigma_s = \sup_{\sigma}\{\sigma \in \Sigma_s\}.$$

Based on these facts, the following results can be given.

Proposition 2. *Let* $P(\boldsymbol{z}) = D^{-1}(\boldsymbol{z})N(\boldsymbol{z})$ *be an n-D MIMO system. Let* $b_j(\boldsymbol{z})$, $j = 1, \cdots, M$, $M = \begin{pmatrix} m+l \\ m \end{pmatrix}$, *be the reduced minors of* $[D(\boldsymbol{z})\ \ N(\boldsymbol{z})]$. *Let* $I = (b_1(\boldsymbol{z}), ..., b_M(\boldsymbol{z}))$ *be the ideal generated by the reduced minors. Then*

$$\Sigma_s = \{\sigma \in \boldsymbol{R} : V(I) \cap \bar{\Delta}^n(0; 1+\sigma) = \emptyset. \tag{6.17}$$

If $P = D^{-1}(\boldsymbol{z})N(\boldsymbol{z})$ *is real, then a real compensator can be chosen to get a closed-loop system with stability margin no less than a given* σ, *if* $\sigma < \sigma_s$.

A proof of this proposition can be established modifying a proof for stabilizability theorem of MIMO n-D systems (e.g., [11, Theorem 1]) by replacing $\bar{U}^n$ with $\bar{\Delta}^n(0; 1+\sigma)$.

It is straightforward to modify the method described in Section 4 for the computation of the stability margin of the MIMO systems. Let us proceed to consider the construction of a stabilizing compensator. Let σ be a number such that $0 < \sigma < \sigma_s$. If there has been found M polynomials $h_1(\boldsymbol{z}), \cdots, h_M(\boldsymbol{z})$ such that

$$h_1(\boldsymbol{z})b_1(\boldsymbol{z}) + \cdots + h_M(\boldsymbol{z})b_M(\boldsymbol{z}) \neq 0 \ \ \forall z \in \bar{\Delta}^n(0; 1+\sigma),$$

then by a slight generalization of the constructive proof of [11, Theorem 1], one can obtain a stabilizing compensator that yields a closed-loop stability margin σ. Thus the problem is reduced to the construction of polynomials $h_1, \cdots, h_M$ such that

$$h_1(\boldsymbol{z})b_1(\boldsymbol{z}) + \cdots + h_M(\boldsymbol{z})b_M(\boldsymbol{z}) \neq 0 \ \ \forall z \in \bar{\Delta}^n(0; 1+\sigma).$$

By setting an upper bound on the total degree of each polynomial, the above problem has a QE formulation, and can be solved by the cylindrical algebraic decomposition method.

7 Concluding Remarks

- If the stabilizability margin of a system is $\sigma_0 = \infty$, this is the case that

  $$V(f(\boldsymbol{z})) \cap V(g(\boldsymbol{z})) = \emptyset,$$

  there exist two polynomials $h(\boldsymbol{z})$ and $k(\boldsymbol{z})$ such that

  $$k(\boldsymbol{z})g(\boldsymbol{z}) + h(\boldsymbol{z})f(\boldsymbol{z}) = 1,$$

  which can be calculated by the Grobner base method [3].

- In the 2-D case, the method presented in [7, 17] for constructing a stabilizing compensator actually yields a closed-loop system with the largest stability margin posibly obtained by any compensator, which is equal to the stabilizability margin of the original plant. However, for an n-D system with $n > 2$, to the knowledge of the authors, there even exists no alternative mathematicaly rigorous method for constructing a stabilizing compensator, not mentioning for constructing one that achieves the stabilizability margin.

- The QE formulation can also be applied to solve other robust stabilization problems, including the strong stabilization problem which requires the stabilizing compensator itself to be stable [18], and the problem of simultaneously stabilizing a finite number of plants by a single compensator.

References

[1] P. Agathoklis, E. I. Jury and M .Mansour, *The Margin of Stability of 2-D Linear Discrete Systems*, IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-30, No. 6 (1982), pp. 869-873.

[2] N. K. Bose ed., *Multidimensional Systems: Theory & Applications*, IEEE Press, (1982)

[3] B. Buchberger, *Gröbner Bases: An Algorithmic Method in Polynomial Ideal Theory*, in N. K. Bose ed., Multidimensional Systems Theory, D. Reidel Publishing Company, Dordrecht, Holland (1985).

[4] G. E. Collins, *Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition*, Lecture Notes in Computer Science, Vol.33 (1975), pp. 134-183.

[5] G. E. Collins and H. Hong, *Partial Cylindrical Algebraic Decomposition and Quatifier Elimination*, Journal of Symbolic Computation, Vol. 12, (1991), pp. 299-328.

[6] E. Curtin and S. Saba, *Stability and Stability Margin Tests for Multidimensional Filters*, IEEE Trans. Circuits and Systems-I, Vol. 46, No. 7 (1999), pp. 806-809.

[7] J. P. Guiver and N. K. Bose, *Causal and Weakly Causal 2-D Filtes with Applications in Stabilization*, in N. K. Bose ed., Multidimensional Systems Theory, D. Reidel Publishing Company, Dordrecht, Holland (1985).

[8] R. C. Gunning, H. Rossi, *Analytic functions of several complex variables*, Prentice Hall, Englewood Cliffs, N. J. (1966).

[9] H. Hong, R. Liska and S. Steinberg, *Testing Stability by Quatifier Elimination*, Journal of Symbolic Computation, Vol. 24 (1997), pp. 161-187.

[10] M. Kawamata and T. Higuchi, *A Unified Approach to the Optimal Synthesis of Fixed-Point State-Space Digital Filters*, IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-33, No. 4 (1985), pp. 911-920.

[11] Z. Lin, *Feedback Stabilizability of MIMO n-D Linear Systems*, Multidimensional Systems and Signal Processing, Vol. 9 (1998), pp. 149-172.

[12] W-U. Lu, A. Antoniou and P. Agathoklis, *Stability of 2-D Digital Filters under Parameter Variations*, IEEE Trans. Circuits and Systems, Vol. CAS-33, No. 5 (1986), pp. 476-482.

[13] S. Shankar and V. R. Sule, *Algebraic Geometric Aspects of Feedback Stabilization*, SIAM J. Control and Optimization, Vol. 30 (1992), No. 1, pp. 11-30.

[14] V. R. Sule, *Feedback Stabilization over Commutative Rings: the Matrix Case*, SIAM J. Control and Optimization, Vol. 32, No.6 (1994), pp. 1675-1695.

[15] M. Vidyasagar, *Control System Synthesis*, The MIT Press, (1985)

[16] E. Walach and E. Zeheb, *N-Dimensional Stability Margins Computation and a Variable Transformation*, IEEE Trans. Acoust., Speech, Signal Processing, Vol. ASSP-30, No. 6 (1986), pp. 887-893.

[17] L. Xu, O. Saito and K. Abe, *Output Feedback Stabilizability and Stabilization Algorithms for 2-D Systems*, Multidimensional Systems and Signal Processing, Vol. 5 (1994), pp. 41-60.

[18] J. Q. Ying, *On the Strong Stabilizability of MIMO N-Dimensional Linear Systems*, SIAM Journal on Control and Optimization, Vol. 38, No. 1, (2000), pp. 313-325.