Optimal Control for a Class of Differential Linear Repetitive Processes

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Abstract

Differential repetitive processes are a class of continuous-discrete 2D linear systems of both systems theoretic and applications interest. The feature which makes them distinct from other classes of such systems is the fact that information propagation in one of the two independent directions only occurs over a finite interval. Applications areas include iterative learning control and iterative solution algorithms for classes of dynamic nonlinear optimal control problems based on the maximum principle. In this paper, we first develop new results on optimal and sub-optimal control for an important sub-class of differential linear repetitive processes and then proceed to extend the well known maximum and ϵ-maximum principles to this sub-class. The end goal of the research programme for which this paper forms part of the output is the development of numerically reliable algorithms for the synthesis of optimization based control schemes for these processes.

1 Introduction

The essential unique characteristic of a repetitive (termed multipass in the early literature) process can be illustrated by considering machining operations where the material or workpiece involved is processed by a sequence of sweeps, termed passes, of the processing tool. Assume that the pass length α (i.e. the duration of a pass of the processing tool), which is finite by definition, has a constant value for each pass. Then in a repetitive process the output vector, or pass profile, $y_k(t)$, $0 \le t \le \alpha$, (t being the independent spatial or temporal variable) produced on pass k acts as a forcing function on the next pass and hence contributes to the dynamics of the new pass profile $y_{k+1}(t)$, $0 \le t \le \alpha$, $k \ge 0$.

Industrial examples (see, for example, [9]) include long-wall coal cutting and metal rolling operations. Also problem areas exist where adopting a repetitive process setting for analysis has clear advantages over alternatives. This is especially true for classes of iterative learning control schemes (see, for example, [1]) and of iterative solution algorithms for classes of dynamic nonlinear optimal control problems based on the maximum principle (see, for example, [7]).

The basic unique control problem for repetitive processes is that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction (i.e. in the k - direction in the notation for variables used here). Early approaches to stability analysis and controller design for (linear single-input single-output) repetitive processes and, in particular, long-wall coal cutting were based on first converting the system into an equivalent infinite-length single-pass process [2]. This, for example, resulted in a scalar differential/algebraic system to which standard scalar inverse-Nyquist stability criteria were then applied. In general, however, it was soon established that this approach to analysis (and controller design) would, except in a few very restrictive special cases, lead to incorrect conclusions [5]. The basic reason for this is that such an approach effectively neglects their finite pass length repeatable nature together with the effects of resetting the initial conditions before the start of each new pass. To remove this difficulty, a rigorous stability theory for linear repetitive processes has been developed [5, 8] using an abstract model in a Banach space setting which includes all examples with linear dynamics and a constant pass length as special cases.

Given a suitable stability theory, it is a natural progression to consider the structure of control schemes for these processes and the development of suitable controller design/synthesis tools. In this latter respect, one obvious way to proceed is to minimize a suitably defined cost function. This is the subject area of this paper where we first develop new results on optimal and sub-optimal control on an important sub-class of differential linear repetitive processes and then proceed to extend the well known maximum and ϵ-maximum principles to this case.

2 Background

The state space model of the differential linear repetitive processes considered here has the form over $0 \leq t \leq \alpha,\ k \geq 0$

$$\dot{y}_k(t) = A_1 y_k(t) + A_2 y_{k-1}(t) + b u_k(t) \tag{2.1}$$

Here on pass k, $y_k(t)$ is the $n \times 1$ pass profile vector and $u_k(t)$ is the scalar control input.

To complete the process description, it is necessary to specify the initial, or so-called boundary, conditions, i.e. the pass state initial vector sequence and the initial pass profile. The structure of these is of critical importance since it is known that if they are an explicit function of the previous pass profile then this alone can cause instability [6]. In this work, however, we consider the simplest possible form, i.e.

$$\begin{aligned} x_k(0) &= d_k,\ k \geq 1 \\ y_0(t) &= f(t),\ 0 \leq t \leq \alpha \end{aligned} \tag{2.2}$$

Here d_k is an $n \times 1$ vector with known constant entries and $f(t)$ is an $n \times 1$ vector whose elements are known piecewise continuous functions of t.

In practice, a repetitive process will only complete a finite number of passes. Here we denote this number by N. Also the most obvious approach to optimal control of these processes is, by analogy with the standard linear systems case, to define a quadratic cost function for each pass and the overall cost function as the sum of these N pass cost functions. Here, however, we interested in the case when there is a terminal constraint on the value at the end of each pass (a feature which is of particular interest in the iterative learning control application [1]) which is assumed to be of the form

$$H_k y_k(\alpha) = g_k,\ k = 1, 2, \cdots, N \tag{2.3}$$

where H_k is an $m \times n$ matrix with known entries and $g_k(\alpha)$ is an $m \times 1$ vector with known entries.

The class of admissible control signals is defined as follows.

Definition 2.1. *Consider differential linear repetitive processes of the form (2.1) and (2.2). Then for $k = 1, 2, \cdots, N$, the piecewise continuous function $u_k : [0, \alpha] \to \mathbb{R}$ is termed an admissible control on pass k if it satisfies the condition $|u_k(t)| \le 1,\ 0 \le t \le \alpha$.*

The optimization problem we wish to solve can now be stated as finding the admissible controls $u_1(t), u_2(t), \cdots, u_N(t)$ such that the corresponding pass profiles $y_1(t), y_2(t), \cdots, y_N(t)$ of (2.1) and (2.2) maximize the cost function

$$J(u) = \sum_{k=1}^{N} p_k^T y_k(\alpha) \tag{2.4}$$

subject to the constraint (2.3), where p_k is a given $n \times 1$ vector.

In this paper, we solve this problem by extending the so-called constructive methods [3, 4] to differential linear repetitive processes. One major motivation for this approach, as opposed to alternatives, is the belief that it may lead to efficient numerical methods and hence optimization controller design algorithms for these processes. Further study of such a possibility is, however, left here as a topic for future research.

3 Optimality and Sub-optimality Conditions

The problem (2.1)-(2.4) can be represented in the following integral form :

$$\max_{u_1,\ldots,u_N} J(u) = \max_{u_1,\ldots,u_N} \sum_{j=1}^{N} \int_0^{\alpha} c_j(\tau) u_j(\tau) d\tau \tag{3.1}$$

subject to the constraints

$$\begin{cases} \int_0^\alpha g_{11}(\tau)u_1(\tau)d\tau & = h_1, \\ \int_0^\alpha [g_{21}(\tau)u_1(\tau) + g_{22}(\tau)u_2(\tau)]d\tau & = h_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \int_0^\alpha [g_{N1}(\tau)u_1(\tau) + ... + g_{NN}(\tau)u_N(\tau)]d\tau & = h_N, \end{cases} \quad |u_k(\tau)| \leq 1,\ k = 1, 2, \cdots, N \tag{3.2}$$

where

$$c_j(\tau) \ = \ \sum_{k=j}^{N} p_k^T K_{k+1-j}(\alpha - \tau)b,$$

$$g_{kj}(\tau) = H_k K_{k+1-j}(\alpha - \tau)b, \quad j \leq k, \quad k = 1, 2, \cdots, N,$$

$$h_k = g_k - \sum_{j=1}^{k} H_k K_j(\alpha) d_{k+1-j} - \int_0^\alpha H_k K_k(\alpha - \tau) A_2 f(\tau) d\tau$$

and the $K_i(t)$ satisfy

$$\dot{K}_1(t) = A_1 K_1(t), \quad K_1(0) = I_n, \quad \dot{K}_i(t) = A_1 K_i(t) + A_2 K_{i-1}(t), \quad K_i(0) = 0, \quad i = 2, ..., N.$$

Now we require the following.

Definition 3.1. *For each fixed k, $1 \leq k \leq N$, the points $0 < \tau_{k1} < \tau_{k2} < ... < \tau_{km} < \alpha$ are termed supporting points and their collection $\tau^k_{supp} = \{\tau_{k1}, ..., \tau_{km}\}$ is termed the support on pass k for the problem (2.1)-(2.4) if the matrix $G^k_{supp} = \{g_{kk}(\tau_{k1}), ..., g_{kk}(\tau_{km})\}$ is non-degenerate. A pair $\{\tau^k_{supp},\ u_k(t), k = 1, 2, \cdots, N\}$ consisting of a support τ^k_{supp} and admissible control functions $u_k(t)$, $0 \leq t \leq \alpha$, is termed a supporting control function for the problem (2.1)-(2.4).*

Let $\{\tau^k_{supp}, u_k(t), k = 1, 2, \cdots, N\}$ be a given support control function, and for each $k = N, ..., 1$ find the $m \times 1$ vector of multipliers $\nu^{(k)}$ as the solution of the following linear algebraic system

$$\begin{cases} (\nu^{(N)})^T G^N_{supp} - c^{(N)}_{supp} & = 0, \\ (\nu^{(N-1)})^T G^{N-1}_{supp} + (\nu^{(N)})^T F^N_{(N-1)supp} - c^{(N-1)}_{supp} & = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (\nu^{(1)})^T G^1_{supp} + (\nu^{(2)})^T F^2_{1supp} + ... + (\nu^{(N)})^T F^N_{1supp} - c^{(1)}_{supp} & = 0 \end{cases} \tag{3.3}$$

where for $j = 1, 2, \cdots, N-1$

$$c^{(k)}_{supp} = (c_k(\tau_{k1}), ..., c_k(\tau_{km})),\ k = 1, 2, \cdots, N,\ F^k_{jsupp} = (g_{kj}(\tau_{j1}), ..., g_{kj}(\tau_{jm})),\ k > j.$$

Using these multipliers, define the function $\Delta(t) = (\Delta_1(t), ..., \Delta_N(t))$ as $\Delta(t) = (\hat{\nu})^T \hat{G}(t) - c(t)$, where $\hat{\nu} = (\nu^{(1)}, ..., \nu^{(N)})$, $c(t) = (c_1(t), ..., c_N(t))$, and let $\hat{G}(t)$ denote the $(mN \times mN)$ triangular matrix whose rows are formed from the $m \times 1$ vector functions $g_{ij}(t)$ of (3.2). The support control function $\{\tau^k_{supp}, u_k(t), k = 1, 2, \cdots, N\}$ is termed non-degenerate for the problem (2.1)-(2.4) if $d\Delta_k(\tau_j)/dt \neq 0$, $\forall\, \tau_j \in \tau^k_{supp}$, $\quad k = 1, 2, \cdots, N$ and we have the following result.

Theorem 3.1. *A sufficient condition for a supporting control function* $\{\tau^k_{supp},\ u^0_k(t),\ k = 1, 2, \cdots, N\}$ *to be optimal for the problem (2.1)-(2.4) is that*

$$u^0_k(t) = \operatorname{sign}(\Delta_k(t)),\ 0 \leq t \leq \alpha,\ k = 1, 2, \cdots, N. \tag{3.4}$$

If the control function is non-degenerate then this last condition is both necessary and sufficient.

The proof of sufficiency in the above result follows immediately from the fact that $\Delta J(u) \geq 0$ for any admissible control u under the given conditions. The necessity part is proved by establishing a contradiction through construction of a particular control function. These are routine exercises and hence the details are omitted here.

The optimality conditions for the supporting control functions here can be expressed in the classical maximum principle form. In particular, let $\psi_k(t),\ k = 1, 2, \cdots, N,\ 0 \leq t \leq \alpha$, be the solutions of the following set of differential equations

$$\frac{d\psi_k(t)}{dt} = -A_1^T \psi_k(t) - A_2^T \psi_{k+1}(t), \quad \frac{d\psi_N(t)}{dt} = -A_1^T \psi_N(t),\ \psi_k(\alpha) = p_k - H_k^T \nu^k. \tag{3.5}$$

and for each $k = 1, 2, \cdots, N$, define the Hamiltonian function as

$$\mathcal{H}_k(y_{k-1}(t), y_k(t), \psi_k(t), u_k(t)) = \psi_k^T(t)(A_1 y_k(t) + A_2 y_{k-1}(t) + b u_k(t)),\ 0 \leq t \leq \alpha. \tag{3.6}$$

Then the following Corollary of Theorem 3.1 shows that the optimality conditions (3.4) are of the maximum principle form.

Corollary 3.1. *A sufficinet condition for optimality of the admissible supporting control* $\{\tau^k_{supp},\ u^0_k(t),\ k = 1, 2, \cdots, N, \}$ *is that along the corresponding trajectories* $y^0_k(t),\ \psi_k(t)$ *of (2.1) and (3.5) the Hamiltonian function (3.6) attains its maximum value, i.e.*

$$\mathcal{H}_k(y^0_{k-1}(t), y^0_k(t), \psi_k(t), u^0_k(t)) = \max_{|v| \leq 1} \mathcal{H}_k(y^0_{k-1}(t), y^0_k(t), \psi_k(t), v),\ k = 1, 2, \cdots, N,\ 0 \leq t \leq \alpha. \tag{3.7}$$

This condition is also necessary in the case when the supporting control action is non-degenerate.

Next we establish the classical maximum principle for the admissible controls of a process of the form considered in this work. This result will be stated as a consequence of the so-called ϵ-optimality conditions which are expected to play a major role in numerical methods for computing the controls. Let $\{u_k^0(t),\ k=1,2,\cdots,N,\}$ be an optimal control signal for the problem defined by (2.1)-(2.4) and also let $J(u^0)$ denote the corresponding optimal cost functional value. Then we can introduce the following definition of an ϵ-optimal control function.

Definition 3.2. *We say that the admissible control function* $\{u_k^\epsilon(t),\ k=1,2,\cdots,N,\}$ *is* ϵ*-optimal for the problem defined by (2.1)-(2.3) if the corresponding solution* $\{y_k^\epsilon(t),\ k=1,2,\cdots,N,\ 0\le t\le\alpha,\}$ *of (2.1) satisfies the inequality* $J(u^0)-J(u^\epsilon)\le\epsilon$.

Now we calculate the estimate (or measure of non-optimality) of arbitrary supporting control function $\{u_k,\ \tau_{supp}^k,\ k=1,2,\cdots,N,\ 0\le t\le\alpha\}$. In particular, we define the estimate of sub-optimality as the number $\beta=\beta(\tau_{supp},u)$ obtained as the solution of the following relaxed optimization problem for (2.1)-(2.4):

$$\Delta J(u)\to\max_{\Delta u_k},\quad |u_k(t)+\Delta_k u(t)|\le 1,\ k=1,2,\cdots,N,\ 0\le t\le\alpha. \tag{3.8}$$

It is also easy to see that

$$\beta=\beta(\tau_{supp},u)=\sum_{k=1}^{N}\int_{PL_k^+}\Delta_k(t)(u_k(t)+1)dt+\sum_{k=1}^{N}\int_{PL_k^-}\Delta_k(t)(u_k(t)-1)dt, \tag{3.9}$$

where

$$PL_k^+=\{t\in[0,\alpha]:\ \Delta_k(t)>0\},\ PL_k^-=\{t\in[0,\alpha]:\ \Delta_k(t)<0\}$$

and we have the following result.

Theorem 3.2. *(ϵ-maximum principle) For any* $\epsilon\ge 0$, *admissible control signal* $\{u_k(t),\ k=1,2,\cdots,N,\ 0\le t\le\alpha\}$ *has the* ϵ*-optimality property if, and only if,* $\exists$ *support* $\{\tau_{supp}^k,\ k=1,2,\cdots,N,\}$ *such that the associated Hamiltonian attains the* ϵ*- maximum value*

$$\mathcal{H}_k(y_{k-1}(t),y_k(t),\psi_k(t),u_k(t))=\max_{|v|\le 1}\mathcal{H}_k(y_{k-1}(t),y_k(t),\psi_k(t),v)-\epsilon_k(t),\ 0\le t\le\alpha,$$

$$\sum_{k=1}^{N}\int_0^\alpha\epsilon_k(t)dt\le\epsilon \tag{3.10}$$

along the solutions $y_k(t),\psi_k(t),\ k=1,2,\cdots,N,\ 0\le t\le\alpha,$ *of (2.1)-(2.3) and (3.5).*

To prove this theorem, we use the fact that the estimate of sub-optimality can be expressed in the form $\beta=\beta(\tau_{supp},u)=\beta_{supp}+\beta_u$. Here $\beta_u=\sum_{k=1}^{N}\int_0^\alpha c_k(t)(u_k(t)-u_k^0(t))dt$ denotes the non-optimality measure of the given control function $\{u_k(t),\ k=1,2,\cdots,N,\ 0\le t\le\alpha\}$, and

$$\beta_{supp}=\left[\sum_{k=0}^{N}h_k^T(\nu_k-z_k^0)+\sum_{k=1}^{N}\int_0^\alpha(v_k(t)-v_k^0(t))-(w_k(t)-w_k^0(t))dt\right] \tag{3.11}$$

denotes the sub-optimality estimate of the chosen support $\{\tau^k_{supp},\ k = 1, 2, \cdots, N,\}$ which is determined from the solution of the following so-called dual problem for (2.1)

$$\min_{z,v,w} I(z, v, w) = \min_{z,v,w} \sum_{k=1}^{N} \left[h_k^T z_k + \int_0^{\alpha} v_k(t)dt + \int_0^{\alpha} w_k(t)dt \right] \tag{3.12}$$

subject to the constraints

$$\sum_{s=k}^{N} z_s^T g_{sk}(t) - v_k(t) + w_k(t) = c_k(t),\ v_k(t) \geq 0,\ w_k(t) \geq 0,\ k = 1, 2, \cdots, \alpha,\ 0 \leq t \leq \alpha. \tag{3.13}$$

The classical maximum principle follows immediately from the above theorem by setting $\epsilon = 0$. This is stated formally as the following corollary to Theorem 3.2.

Corollary 3.2. *An admissible control* $\{u_k^0(t),\ k = 1, 2, \cdots, N,\ 0 \leq t \leq \alpha\}$ *is optimal if, and only if,* $\exists$ *a support* $\{\tau^{0k}_{supp},\ k = 1, 2, \cdots, N,\}$ *such that the supporting control* $\{u_k^0(t), \tau^{0k}_{supp},\ k = 1, 2, \cdots, N,\ 0 \leq t \leq \alpha\}$ *satisfies the maximum conditions*

$$\max_{|v|\leq 1} \mathcal{H}_k(y^0_{k-1}(t), y^0_k(t), \psi_k(t), v) = \mathcal{H}_k(y^0_{k-1}(t), y^0_k(t), \psi_k(t), u^0_k(t)),\ k = 1, 2, \cdots, N,\ 0 \leq t \leq \alpha.$$

4 Conclusions

This paper has used a supporting control functions setting to study optimization problems for an important sub-class of differential linear repetitive processes. In common with other methods, the approach proposed here enables us to characterize solutions in terms of constructive necessary and sufficient optimality conditions which can then be used for the design of numerical algorithms. Moreover, the design task here can be based on the principle of decreasing the sub-optimality estimate on each iteration, i.e. the iteration $\{\tau^k_{supp},\ u_k(t),\ k = 1, 2, \cdots, N\} \rightarrow \{\hat{\tau}^k_{supp},\ \hat{u}_k(t),\ k = 1, 2, \cdots, N\}$ is completed in such a way that $\beta(\hat{\tau}_{supp}, \hat{u}) < \beta(\tau_{supp}, u)$. Also in accordance with the obtained representation of this estimate, the iterative task here can be separated into the following two stages:
1) apply a transformation of the admissible control functions $\{u_k(t),\ k = 1, 2, \cdots, N\} \rightarrow \{\hat{u}_k(t),\ k = 1, 2, \cdots, N\}$ which decreases the degree of non-optimality of the admissible controls $\beta(\hat{u}) < \beta(u)$, and;
2) variation of the support $\{\tau^k_{supp},\quad k = 1, 2, \cdots, N\} \rightarrow \{\hat{\tau}^k_{supp},\ k = 1, 2, \cdots, N\}$ such that the degree of non-optimality of the support decreases,i.e. $\beta(\hat{\tau}_{supp}) < \beta(\tau_{supp})$.
These transformations essentially invoke the duality theory for the problems defined by (2.1)-(2.4) and (3.12)-(3.13) and also exploit the ϵ-optimality conditions developed in this paper.

The results given here are the first in this very important general area and work is currently proceeding in a number of follow up areas. Results from this research will be reported in due course. It can also be conjectured that the results given here can also be used to construct the differential equations for the switching functions of the optimal control laws that will arise in the design of optimal control feedback control laws for these processes. This area is also currently under investigation.

Acknowledgements

This research is supported in part by Grant F 00-53 from The Research Foundation of Belarus and The UK Engineering and Physical Sciences Research Council.

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