Relation between Eigenvalues and Singular Values in the Problem of Stability Maintenance of Ellipsoidal Estimates

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Abstract

Problem of defining of relation between eigenvalues and singular values of matrix and matrix-valued functions of the matrix is considered. The problem arises at the interfaces between methods of systems synthesis with use of generalized modal control and quality evaluation of these systems with the help of ellipsoidal quality estimates. Solution is oriented on maintenance of stability of ellipsoidal quality estimates by means of stabilisation of state matrix eigenvalues and eigenvectors.

1 Introduction. Problem statement.

Nowadays one of design methods of multi-input multi-out (MIMO) control systems, which are invariant to dimensionality of input vector, is the method of modal control. Its generalized version [1,2], putting the problem of maintenance of desirable structure of eigenvalues, as well as of eigenvectors, allows to deliver to eigenvalues the guaranteed stability in conditions of variations or indeterminacy of parameters of matrix components of model description of initial controlled plant. Ellipsoidal quality estimates (or principal components) [3] represent the extreme elements of algebraic spectrum of singular values of researched system criteria matrix for each version of investigated processes (controllability and observability Gramians, cross-Gramian, fundamental and transitional matrices, covariances matrix, etc.). In this connection, if the synthesis problem of MIMO-systems with ellipsoidal quality estimates of the guaranteed stability, which is algorithmically based on the possibility of modal control, is put, there is necessity of defining relation between eigenvalues and singular values of criteria matrices.

2 Basic Result

Consider linear continuous MIMO-system

$$\dot{x}(t) = Fx(t) + Gg(t); x(0), y(t) = Cx(t) \tag{2.1}$$

obtained by controlled plant

$$\dot{x}(t) = Ax(t) + Bu(t); x(0), y(t) = Cx(t); \tag{2.2}$$

and regulator,

$$u(t) = K_g g(t) + Kx(t), \tag{2.3}$$

realized direct connection on exogenous inputs and state feedback.
In (2.1) $\div$ (2.3) x, u, y, g are state vector, input, output and exogenous input accordingly, $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$, $u \in \mathbf{R}^r$, F, G, C, A, B, K_g, K are state matrix of system (1), input, output, state matrix of plant (2), control matrix, matrix of direct connections on exogenous inputs and state feedback matrix accordingly; $F, A \in \mathbf{R}^{n\times n}$, $G, C^T \in \mathbf{R}^{n\times m}$, $B, K^T \in \mathbf{R}^{n\times r}$, $K_g \in \mathbf{R}^{r\times m}$. Basic results are formulated by following propositions.

Proposition 2.1. *Vectors* $\alpha = col\left\{\alpha_i;\ i = \overline{1,n}\right\}$ *and* $\lambda = col\left\{\lambda_i;\ i = \overline{1,n}\right\}$ *composed of singular values* α_i *and eigenvalues* λ_i *of state matrix* F *of the system (1) are connected by vector-matrix ratio*

$$\alpha = \Pi_{\alpha\lambda}\lambda, \tag{2.4}$$

$$\Pi_{\alpha\lambda} = col\left\{(U_i)^T\, Mdiag\left\{\left(M^{-1}V_i\right)^j; j = \overline{1,n}\right\}; i = \overline{1,n}\right\}, \tag{2.5}$$

where U *and* V *are singular bases in singular value decomposition of the matrix* $F = U\Sigma V^T$ *,* $(\circ)_i$ *is* i *-th column,* $(\circ)^i$ *is* i *-th row of appropriate matrix components,* M *and* Λ *are eigenvectors and eigenvalues matrices.*

The degree of proximity of the matrix $\Pi_{\alpha\lambda}$ to diagonal or signature matrix is determined by the degree of proximity of co-ordinated elements of the left singular base and eigenvectors of F. In accordance with growth of difference of the geometric spectrum of the matrix F from the left singular base, the difference of $\Pi_{\alpha\lambda}$ from diagonal form increases.

To extend the obtained results to case of matrix-valued function $f(F)$ of the matrix F

$$f(F) = a_0 I + a_1 F + a_2 F^2 + \cdots + a_p F^p + \cdots \tag{2.6}$$

we shall remind, that it is generated by power series [6] on scalar variable ϑ

$$f(\vartheta) = a_0 + a_1\vartheta + a_2\vartheta^2 + \ldots + a_p\vartheta^p + \cdots, \quad a_i, \vartheta \in R.$$

The matrix-valued function $f(F)$ saves the similarity relation in the form $f(F)M = Mf(\Lambda)$ and geometric spectrum of the eigenvectors, and its eigenvalues $f(\lambda_i), i = \overline{1,n}$ are determined by function $f(\vartheta)$ on the spectrum $\lambda_i, i = \overline{1,n}$ of the matrix F. Infortunately, in generally the matrix-valued function $f(F)$ does not save geometric spectrum of singular bases and has not indicated functional connection of algebraic spectrum of singular values.

Proposition 2.2. *Vectors* $\alpha_f = col\left\{\alpha_{fi};\ i = \overline{1,n}\right\}$ *and* $\lambda_f = col\left\{\lambda_{fi};\ i = \overline{1,n}\right\}$ *composed from singular values* α_{fi} *and eigenvalues* λ_{fi} *of matrix-valued function* $f(F) : \mathbf{R}^{n\times n} \Rightarrow \mathbf{R}^{n\times n}$ *are connected by vector-matrix ratio*

$$\alpha_f = \Pi_f \lambda, \tag{2.7}$$

where the relations matrix Π_f *is given by*

$$\Pi_f = col\left\{(U_{fi})^T\, Mdiag\left\{\left(M^{-1}V_f i\right)^j;\, j = \overline{1,n}\right\};\, i = \overline{1,n}\right\}, \tag{2.8}$$

where U_f *and* V_f *are singular bases in singular value decomposition of the matrix-valued function* $f(F) = U_f \Sigma_f V_f^T$.

The established relation between eigenvalues and singular values of the state matrix for dynamic system and its matrix function is used hereinafter for the solution of the delivered problem of maintenance of stability of ellipsoidal quality estimates. It is supposed, that the parametrical variations are such, that the methods of the sensitivity theory are aplicable within the framework of the first order sensitivity functions. There are solutions of the problem of the analysis of parametrical sensitivity of eigenvalues [1,4] and of singular values [3] in separate kind. In the paper the relation between sensitivity functions of eigenvalues and singular values is installed and it is shown, that the stability of ellipsoidal quality estimates can be supplied by control of eigenvalues sensitivity.

Let state matrix F and as corollary the matrix function $f(F)$ depends on vector parameters $q \in \mathbf{R}^p$ with nominal value q_0 as $F(q)$ and $f(F(q))$. Then algebraic spectrum of eigenvalues and singular values and the geometric spectrum of the eigenvectors and singular bases also depend on q such that for $q \neq q_0$ $F(q) = U(q)\Sigma(q)V^T(q), F(q) = M(q)\Lambda(q)M^{-1}(q)$, where $\Sigma(q) = diag\left\{\alpha_i(q), i = \overline{1,n}\right\}$, $\Lambda(q) = diag\left\{\lambda_i(q), i = \overline{1,n}\right\}$, $\alpha_i(q) = \alpha_i + \Delta\alpha_i\left(q_0, \Delta q\right)$, $\lambda_i(q) = \lambda_i + \Delta\lambda_i\left(q_0, \Delta q\right)$.

The finite increments of singular values $\Delta\alpha_i$ and eigenvalues $\Delta\lambda_i$ with use of first order sensitivity functions α_{iq_k} and λ_{iq_k} to variations of k-th element q_k of parameters vector q are defined by $\Delta\alpha_i\left(q_0, \Delta q_k\right) = \alpha_{iq_k}\Delta q_k$, $\Delta\lambda_i\left(q_0, \Delta q_k\right) = \lambda_{iq_k}\Delta q_k$, which permit for variations of singular values $\Delta\alpha_i\left(q_0, \Delta q_k\right)$ and eigenvalues $\Delta\lambda_i\left(q_0, \Delta q_k\right)$ caused by variations of all elements $\Delta q_k, k = \overline{1,p}$ of parameters vector Δq to write

$$\Delta\alpha_i\left(q_0, \Delta q\right) = row\left\{\alpha_{iq_k};\ k = \overline{1,p}\right\}\Delta q,$$

$$\Delta\lambda_i\left(q_0, \Delta q\right) = row\left\{\lambda_{iq_k};\ \ k = \overline{1,p}\right\}\Delta q$$

where $\left(\circ\left(q\right)\right)_{|q=q_0} = \left(\circ\right), \quad \left(\circ\left(q\right)\right)_{q_k} = \frac{\partial}{\partial q_k}\left(\circ\left(q\right)\right)_{|q=q_0}$.

Computation of sensitivity function α_{iq_k} and λ_{iq_k} is carried out with the help of the following propositions.

Proposition 2.3. *Sensitivity functions of eigenvalues λ_{iq_k} and singular values α_{iq_k} are related by ratio*

$$\alpha_{iq_k} = col\left\{\Pi^i_{\alpha\lambda q_k}\lambda + \Pi^i_{\alpha\lambda}\lambda_{q_k};\ i=\overline{1,n}\right\} \tag{2.9}$$

where the row-matrix $\Pi^i_{\alpha\lambda q_k}$ is given by

$$\Pi^i_{\alpha\lambda q_k} = (U_i)^T_{q_k} M diag\left\{(M^{-1}V_i)^j; j=\overline{1,n}\right\} + (U_i)^T M_{q_k} diag\{ (M^{-1}V_i)^j; \\ j=\overline{1,n}\} - (U_i)^T M diag\left\{(M^{-1}M_{q_k}M^{-1}V_i)^j; j=\overline{1,n}\right\} + \\ + (U_i)^T M diag\left\{(M^{-1}V_{iq_k})^j; j=\overline{1,n}\right\}$$

and the vector-functions of eigenvalues sensitivity λ_{q_k} is defined [1] *by*

$$\lambda_{q_k} = col\left\{(M^{-1}F_{q_k}M)_{ii}; i=\overline{1,n}\right\}. \tag{2.10}$$

Proposition 2.4. *Sensitivity functions of eigenvalues $f(\lambda_{q_k})$ and singular values α_{fq_k} of matrix-valued function $f(F)$ are related:*

$$\alpha_{fq_k} = col\left\{\left(\Pi^i_{fq_k}\lambda_f + \Pi^i_f \frac{\partial\lambda_f}{\partial\lambda_i}\lambda_{iq_k}\right)_{|q=q_0};\ i=\overline{1,n}\right\},$$

where the row-matrix $\Pi^i_{fq_k}$ is defined by

$$\Pi^i_{fq_k} = (U_{fi})^T_{q_k} M diag\left\{(M^{-1}V_{fi})^j; j=\overline{1,n}\right\} + (U_{fi})^T M_{q_k} diag\left\{(M^{-1}V_{fi})^j;\right. \\ \left. j=\overline{1,n}\right\} - (U_{fi})^T M diag\left\{(M^{-1}M_{q_k}M^{-1}V_{fi})^j; j=\overline{1,n}\right\} + (U_{fi})^T M\times \\ \times diag\left\{(M^{-1}V_{fiq_k})^j; j=\overline{1,n}\right\}$$

and the sensitivity vector-functions of eigenvalues $f(\lambda_{q_k})$ of matrix-valued function $f(F)$ is given [1] by $\lambda_{fq_k} = col\left\{[M^{-1}f_{q_k}(F)M]_{ii}, i=\overline{1,n}\right\}$, where $f_{q_k}(F) = M_{q_k}M^{-1}f(F) - f(F)M_{q_k}M^{-1} + M diag\left\{\frac{\partial f(\lambda_i)}{\partial\lambda_i}_{|q=q_0}\lambda_{iq_k}, i=\overline{1,n}\right\}M^{-1}$.

Computation of sensitivity functions of the eigenvectors and singular bases of appropriate matrices are made as in [1,3,4].

Proposition 2.5. *Sensitivity functions M_{iq_k} of eigenvectors of the state matrix F of the system (1) and singular bases U_{iq_k} and V_{iq_k} are given by [3]*

$$M_{iq_k} = \sum_{l=1}^{n}\delta^k_{il}M_l \quad \delta^k_{il} = \tfrac{(M^{-1})^l F_{q_k}M_i}{\lambda_i-\lambda_l};\ i\neq l;\ i,l=\overline{1,n};\ k=\overline{1,p}$$

$$U_{iq_l} = \sum_{l=1}^{\nu}\gamma^k_{il}U_l; V_{iq_k} = \sum_{l=1}^{\nu}\beta^k_{il}V_l;\ k=\overline{1,p};\ i=\overline{1,\nu};\ where$$

$$\gamma^j_{il} = \tfrac{\alpha_i (U^T)^l \Pi_{q_k}V_i + \alpha_{lk}(U^T)^i |Pi_{q_k}V_l}{\alpha_i^2-\alpha_l^2};\ i\neq l;\ \gamma_{il}=0;\ i,l=\overline{1,n};\ k=\overline{1,p},$$

$$\beta^k_{il} = \tfrac{\alpha_i (U^T)^i \Pi_{q_k}V_l + \alpha_k (U^T)^l \Pi_{q_k}V_i}{\alpha_i^2-\alpha_l^2};\ i\neq l;\ \beta_{ii}=0;\ i,l=\overline{1,n};\ k=\overline{1,p}$$

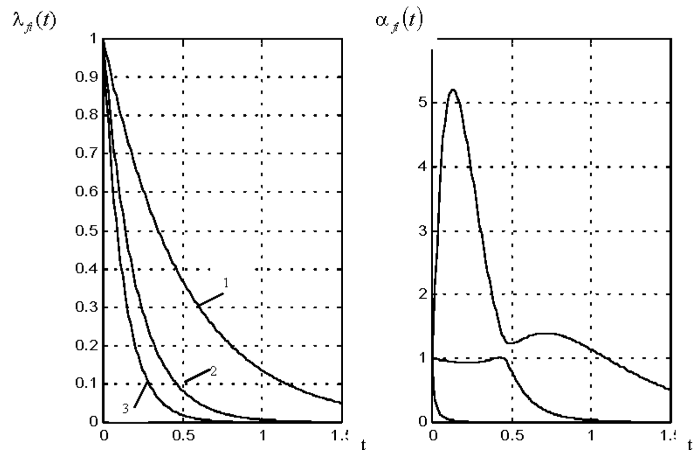

Figure 1:

Example It is required to define relation between of eigenvalues and singular values of matrix function $f(F,t)$, represented by fundamental matrix $f(F) = exp(Ft)$ for control system (1) with the state matrix in Frobenius basis and spectrum of eigenvalues $\{\lambda_1 = -2;\ \lambda_2 = -5;\ \lambda_3 = -8\}$. Computation of Π_f (2) at moments t=0; 0.13; 1.49 sec. gives respectively

$$\Pi_f = \begin{bmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{bmatrix}, \Pi_f = \begin{bmatrix} \text{-8.26 51.04 -42.64} \\ \text{2.32 - 2.21 0.89} \\ \text{0.01 - 0.16 0.29} \end{bmatrix}, \Pi_f = \begin{bmatrix} \text{10.71 -48.84 38.24} \\ \text{-0.003 4.56 -5.42} \\ \text{0.00 0.00 0.023} \end{bmatrix}.$$

On Fig.1, a) the eigenvalues $f(\lambda_1,t) = exp(-2t)$, $f(\lambda_2,t) = exp(-5t)$ and $f(\lambda_3,t) = exp(-8t)$ (curves 1, 2 and 3) and on Fig.1,b) the singular values $\alpha_{fi}(t), i = \overline{1,3}$ of the fundamental matrix $f(F) = exp(Ft)$ are shown.

Conclusion. From the established relation between eigenvalues and singular values, between sensitivity functions of eigenvalues and singular values of state matrix of system and matrix-valued functions of the matrix follow, that it is possible to formulate requirements imposed on eigenvalues sensitivity for stability maintenance of ellipsoidal quality estimates.

References

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