On the Convergence of Nonsystematic Turbo Codes

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Abstract

In this paper, we study the convergence behavior of nonsystematic feedback convolutional encoders used as constituent encoders in a parallel concatenated (turbo) coding scheme. We use mutual information based EXIT charts to visualize the decoding trajectory of different nonsystematic encoders employing an iterative MAP decoding algorithm. It is observed that encoders with low weight feedforward inverses perform better under low signal-to-noise ratio conditions. Catastrophic encoders, which have an infinite weight feedforward inverse, can also be made to converge by doping the code with some systematic bits. We also present some BER performance curves for nonsystematic turbo codes and compare them to systematic turbo codes.

1 Introduction

Turbo codes were introduced by Berrou et al. [1] as a class of parallel concatenated codes using two systematic feedback convolutional encoders. Systematic encoders provide reliable extrinsic information about the information bits when little a-priori information is available. This is critical in the initial stages of the iterative decoding process.

Nonsystematic encoders, on the other hand, have larger values of d_2, the minimum weight codeword produced by a weight two information sequence, than systematic encoders of the same complexity [2]. This implies that nonsystematic turbo codes have larger values of effective free distance d_{eff} than systematic turbo codes [3]. Thus, for large interleaver sizes, nonsystematic turbo codes will exhibit better performance in the errorfloor region than systematic turbo codes of the same complexity. However, not all nonsystematic encoders provide reliable extrinsic information when little a-priori information is available. In other words, many nonsystematic turbo codes exhibit weak convergence behavior with iterative decoding, and thus their performance in the waterfall region is poor. In this paper, we identify a class of nonsystematic encoders with good convergence properties that make attractive choices for use in nonsystematic turbo codes.

There exists a class of nonsystematic convolutional encoders, called quick-look-in (QLI) convolutional encoders [4], which have the property that one can simply extract the information sequence from the parity sequence. These nonsystematic QLI encoders have a feedforward encoder inverse with only two non-zero terms, the minimum for any nonsystematic encoder, and thus they possess an "almost systematic"-like property.

The QLI property for feedforward convolutional encoders was first defined by J. L. Massey and Costello in [4]. Recently, P. C. Massey and Costello have extended this property to feedback convolutional encoders, resulting in a class of rate 1/2 nonsystematic recursive QLI codes [5] that perform well as constituent codes in a parallel concatenated (turbo) coding scheme.

In this paper, we explore the relationship between the feedforward inverses of nonsystematic convolutional encoders and the convergence behavior of these encoders in a turbo configuration. There are different measures that have been proposed in the literature for tracking the exchange of extrinsic information between the constituent decoders in an iterative decoding scheme [6, 7, 8]. In this paper, we use the mutual information based EXtrinsic Information Transfer charts, since they give the most accurate estimates of the convergence threshold [9]. As an example we consider a series of 8-state rate 1/2 nonsystematic feedback encoders, each having a different weight in the encoder inverse realization. The results indicate that encoders with low-weight feedforward inverses provide reliable initial extrinsic information estimates and that it is important to select two constituent encoders that complement each other with iterative decoding to minimize the convergence threshold.

The paper is organized as follows. In the next section, we classify nonsystematic feedback encoders based on the weight of their feedforward inverse. In Section 3, we present results on the convergence behavior and BER performance of several parallel concatenated coding schemes with nonsystematic feedback constituent encoders. Finally, Section 4 summarizes the main results and anticipates some future work.

2 Inverses of Nonsystematic Feedback Convolutional Encoders

Let $\mathbf{G}(D) = \left[\frac{\mathbf{g}^{(1)}(D)}{\mathbf{g}^{(0)}(D)}\ \frac{\mathbf{g}^{(2)}(D)}{\mathbf{g}^{(0)}(D)}\right]$ represent a rate 1/2 nonsystematic feedback convolutional encoder. To recover the information sequence from the output of the encoder, one needs an inversion matrix, i.e., a 2×1 matrix $\mathbf{G}^{-1}(D)$ that satisfies

$$\mathbf{G}(D)\mathbf{G}^{-1}(D) = D^l \tag{2.1}$$

for some $l \geq 0$.

The code generated by $\mathbf{G}(D)$ is called a *recursive quick-look in* (*RQLI*) convolutional code if

$$\frac{\mathbf{g}^{(1)}(D)}{\mathbf{g}^{(0)}(D)} + D^{\beta}\frac{\mathbf{g}^{(2)}(D)}{\mathbf{g}^{(0)}(D)} = D^{\alpha}, \tag{2.2}$$

for some non-negative integers α, β.

In this case, it can be seen that $\mathbf{G}(D)$ has a simple weight two feedforward inverse $\mathbf{G}^{-1}(D) = \left[1\ D^{\beta}\right]^T$ of delay α, i.e., $\mathbf{G}(D)\mathbf{G}^{-1}(D) = D^{\alpha}$. (We refer to the total number of non-zero terms in $\mathbf{G}^{-1}(D)$ as its weight.)

Moreover, if $\mathbf{V}(D) = \left[\mathbf{v}^{(1)}(D)\ \mathbf{v}^{(2)}(D)\right] = \left[\mathbf{u}(D)\frac{\mathbf{g}^{(1)}(D)}{\mathbf{g}^{(0)}(D)}\ \mathbf{u}(D)\frac{\mathbf{g}^{(2)}(D)}{\mathbf{g}^{(0)}(D)}\right]$ represents a codeword, we obtain:

$$\mathbf{V}(D)\mathbf{G}^{-1}(D) = \mathbf{v}^{(1)}(D) + D^{\beta}\mathbf{v}^{(2)}(D) = D^{\alpha}\mathbf{u}(D), \tag{2.3}$$

i.e., the information sequence $\mathbf{u}(D)$ can be recovered (without decoding) from the codeword $\mathbf{V}(D)$ using an encoder inverse with only weight two. (There may sometimes be practical reasons to try recovering the information sequence $\mathbf{u}(D)$ from the received codeword $\mathbf{V}(D)$ without decoding.)

From (2.3), we see that an error in recovering a particular bit in $\mathbf{u}(D)$ can be caused either by an error in $\mathbf{v}^{(1)}(D)$ or by an error in $\mathbf{v}^{(2)}(D)$. Thus we say that RQLI codes have an *error amplification factor* of $A = 2$. In general, for a noncatastrophic encoder with feedforward inverse $\mathbf{G}^{-1}(D) = \left[\{\mathbf{g}^{(1)}(D)\}^{-1}\ \ \{\mathbf{g}^{(2)}(D)\}^{-1}\right]^T$, the error amplification factor is given by:

$$A = w_H\left[\{\mathbf{g}^{(1)}(D)\}^{-1}\right] + w_H\left[\{\mathbf{g}^{(2)}(D)\}^{-1}\right], \tag{2.4}$$

i.e., RQLI encoders have the minimum possible value of A of any nonsystematic feedback encoder. (Catastrophic encoders do not possess a feedforward inverse, and thus their error amplification factor is infinite.) Systematic feedback encoders, on the other hand, have a trivial feedforward inverse with $A = 1$, but their inferior d_2 make them less desirable than nonsystematic encoders in some turbo coding applications.

Encoders with inverses $\mathbf{G}^{-1}(D)$ of weight three are called Easy-Look-In (ELI) [10], and encoders whose inverses have small weight, but weight greater than three, are called Nearly-Quick-Look-In (NQLI) [5]. Constructing the inverses of nonsystematic convolutional encoders and conditions on their existence are summarized in section 2.3 of [11].

3 Using Nonsystematic Feedback Encoders in a Turbo Configuration

In this section we present some nonsystematic turbo codes using QLI, ELI, NQLI, and catastrophic encoders. We also present some partially systematic codes obtained by doping (replacing) some of the parity bits of a catastrophic nonsystematic encoder with systematic bits. We use mutual information based EXIT charts to study the convergence behavior of these turbo coding schemes.

	Encoder	**Inverse**
Systematic:	$G(D) = \left[\begin{array}{cc} 1 & \frac{1+D+D^3}{1+D^2+D^3} \end{array}\right]$	$G^{-1}(D) = \left[\begin{array}{c} 1 \\ 0 \end{array}\right]$
Quick-Look-In:	$G(D) = \left[\begin{array}{cc} \frac{D+D^2}{1+D^2+D^3} & \frac{1+D+D^3}{1+D^2+D^3} \end{array}\right]$	$G^{-1}(D) = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]$
Easy-Look-In:	$G(D) = \left[\begin{array}{cc} \frac{1+D^3}{1+D^2+D^3} & \frac{1+D+D^3}{1+D^2+D^3} \end{array}\right]$	$G^{-1}(D) = \left[\begin{array}{c} 1+D \\ D \end{array}\right]$
Nearly-Quick-Look-In:	$G(D) = \left[\begin{array}{cc} \frac{1+D+D^2+D^3}{1+D^2+D^3} & \frac{1+D+D^3}{1+D^2+D^3} \end{array}\right]$	$G^{-1}(D) = \left[\begin{array}{c} D^2 \\ 1+D+D^2 \end{array}\right]$
Catastrophic:	$G(D) = \left[\begin{array}{cc} \frac{1+D^3}{1+D^2+D^3} & \frac{1+D+D^2+D^3}{1+D^2+D^3} \end{array}\right]$	$G^{-1}(D) = \left[\begin{array}{c} \frac{1}{1+D} \\ \frac{D}{1+D} \end{array}\right]$

Table 1: Inverses of several 8-state, Rate 1/2 Constituent Encoders

We consider turbo codes obtained by the parallel concatenation of two constituent encoders. The iterative decoders for these codes are based on SISO modules that employ the MAP algorithm [12]. Assuming binary encoders and BPSK modulation, when the interleaver is very large and random, the successive extrinsic information estimates entering the SISO modules are independent and identically distributed. Moreover, the extrinsic information passed from one SISO module to the other can be assumed to be Gaussian distributed, with a *symmetric* density function [13].

Through independent simulations of the two SISO modules, we can compute the input mutual information between an information bit and the corresponding a-priori input ($\mathrm{I_a}$) and the output mutual information between the same information bit and the corresponding a-posteriori extrinsic estimate ($\mathrm{I_e}$) for each decoder, using the symmetric Gaussian approximation. We can then plot, for a given value of E_b/N_0, transfer curves of the output mutual information ($\mathrm{I_e}$) as a function of the input mutual information ($\mathrm{I_a}$) for each decoder. To show the evolution of extrinsic information as iterations progress, we plot the output mutual information of decoder 1 ($\mathrm{I_{e1}}$) versus its input mutual information ($\mathrm{I_{a1}}$), and the input mutual information of decoder 2 ($\mathrm{I_{a2}}$) versus its output mutual information ($\mathrm{I_{e2}}$) on the reverse axes. The behavior of the iterative decoding algorithm is then described by a step-wise curve [8]. A wide separation between the two curves implies fast convergence of the iterative decoding algorithm, whereas any contact or crossing of the two curves implies that the algorithm does not converge.

Table 1 lists the different rate R=1/2, 8-state constituent encoders considered, along with their respective encoder inverses. In Figure 1 the transfer curves of each of the nonsystematic encoders is shown for $E_b/N_0 = 0$ dB. For zero a-priori information, we notice that lower weight encoder inverses lead to larger values of output mutual information, i.e., the QLI encoder gives the best initial estimate, then the ELI encoder, which has a weight three inverse, followed by the NQLI encoder with a weight four encoder inverse. To construct rate R=1/3

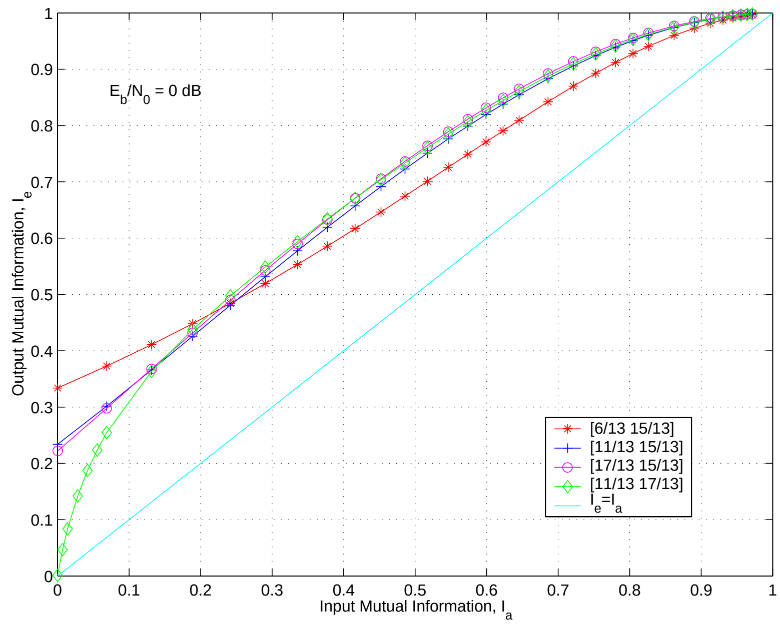

Figure 1: *Transfer curves for different nonsystematic R=1/2 constituent encoders used in an R=1/3 turbo configuration.*

turbo codes using these nonsystematic convolutional encoders, we must parallel concatenate the R=1/2 nonsystematic encoders with an R=1 nonsystematic encoder. In Figure 2, the transfer curves of several R=1, 8-state and 4-state nonsystematic encoders are shown for $E_b/N_0 = 0$ dB. We separately consider three different cases using the R=1/2 QLI, ELI, and catastrophic encoders and applying the EXIT chart analysis to find the R=1 nonsystematic encoders best matched to each of the R=1/2 nonsystematic encoders.

For the 8-state, R=1/2 QLI encoder in Table 1, the following R=1 nonsystematic encoders give the best match: the 8-state big numerator accumulator (BNA) encoder $\frac{1+D^2+D^3}{1+D}$ ([13/3]) [5], the 8-state encoders $\frac{1+D+D^3}{1+D^2+D^3}$ ([15/13]) and $\frac{1+D+D^2+D^3}{1+D^2+D^3}$ ([17/13]), and the 4-state encoder $\frac{1+D^2}{1+D+D^2}$ ([5/7]). The EXIT charts for each of these combinations is shown in Figure 3 for $E_b/N_0 = 0$ dB. For the R=1/2 ELI encoder in Table 1, the following R=1 nonsystematic encoders give the best match: the 8-state BNA [13/3], the 8-state encoder $\frac{D+D^2}{1+D^2+D^3}$ ([6/13]), the 4-state encoder [5/7] and the 4-state BNA $\frac{1+D+D^2}{1+D}$ ([7/3]) (see Figure 4).

The Bit Error Rate (BER) performance of the corresponding nonsystematic rate R=1/3 turbo codes for an interleaver size of N=4096 bits and 30 decoding iterations on an AWGN channel is shown in Figure 5. Since, the “tunnel” between the transfer curves is largest for the R=1/2, 8-state ELI encoder combined with the R=1, 8-state BNA encoder, we expect this turbo code to have the best performance in the waterfall region. This is observed in the BER performance curve shown in Figure 5. In fact, we see from Figure 5 that the performance of this code in the waterfall region is even better than the parallel concatenation of two systematic 8-state encoders $[1 \frac{1+D+D^3}{1+D^2+D^3}]$ ([1 15/13]).

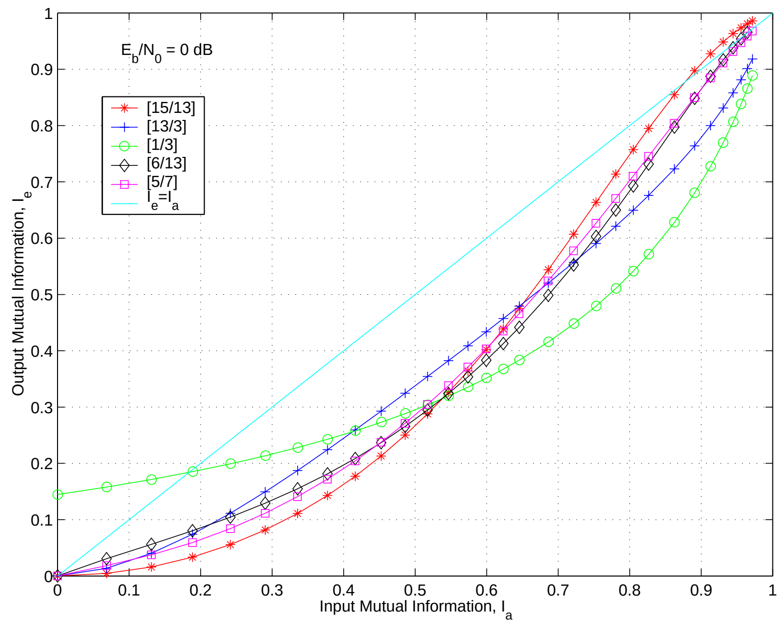

Figure 2: *Transfer curves for several different nonsystematic R=1 constituent encoders used in an R=1/3 turbo code configuration.*

A catastrophic encoder gives very poor estimates for zero a-priori information. So, to make it work at all in a turbo configuration, it must be matched with an encoder that gives good estimates for zero a-priori information. For rate R=1 nonsystematic encoders, the accumulator $\frac{1}{1+D}$ ([1/3]) is such an encoder [14]. It is a self-QLI encoder [5], since the information sequence can be obtained from the parity sequence with a one time unit delay using the weight two feedforward inverse $1 + D$. However, the accumulator provides poor estimates for large a-priori information (see Figure 6).

Figure 6 shows how four different R=1 nonsystematic encoders perform when used with the 8-state, R=1/2 catastrophic encoder from Table 1 for $E_b/N_0 = 0$ dB . We notice that the 8-state BNA provides the best match with the catastrophic encoder, but that the transfer curves cross at very low values of a-priori information. Thus iterative decoding will not converge for this turbo code at 0 dB. Since it is known that systematic bits help in providing a good initial extrinsic information estimate when little a-priori information is available, if some of the nonsystematic bits are replaced by systematic bits, the convergence behaviour will be enhanced. This is demonstrated in Figure 7. This is the idea behind code doping which ten Brink [15] originally introduced for serially concatenated codes.

In Figure 8, the BER performance of the 8-state catastrophic encoder concatenated with the 8-state BNA encoder is shown. Results are shown for two different doping ratios, (1:64) and (1:16). As predicted by the EXIT Charts, the catastrophic encoder by itself won't converge when concatenated with the 8-state BNA encoder, but doping helps in convergence, since the initial estimates of the partially systematic code aid in the early iterations of

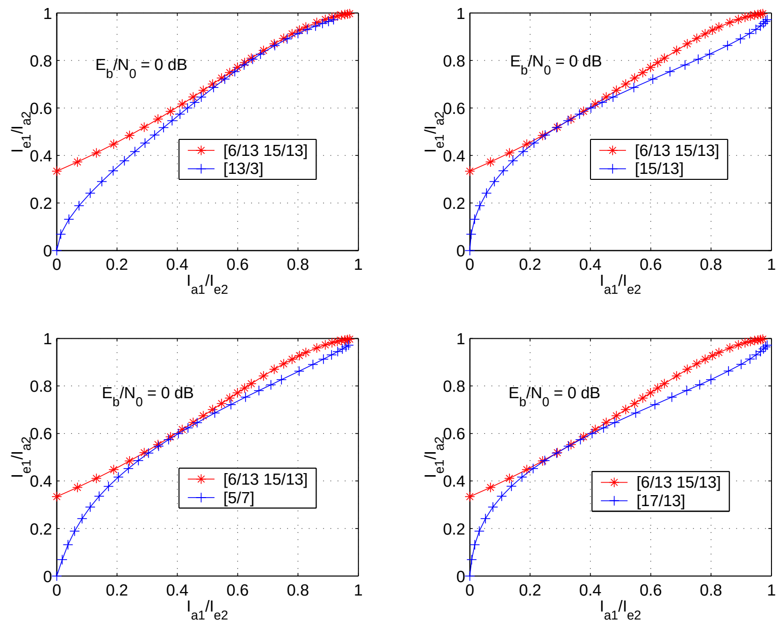

Figure 3: *EXIT charts for R=1/3 turbo codes using an 8-state QLI encoder.*

decoding. In fact, the partially systematic "catastrophic" turbo code with doping ratio (1:16) gives very good BER performance compared to other 8-state turbo codes.

4 Conclusions and Future Work

In this paper, we have looked at the design of nonsystematic turbo codes. It is observed that encoders with low weight inverses have good convergence properties. We have also shown that catastrophic encoders can be made to converge at low SNRs by using code doping, i.e., by replacing some nonsystematic bits with systematic bits. EXIT Charts provide design guidelines for choosing nonsystematic turbo codes with good waterfall performance. In future work, we will examine the performance of nonsystematic turbo codes in the errorfloor region by determining the distance properties of QLI and ELI nonsystematic encoders with good convergence properties and comparing them to systematic encoders.

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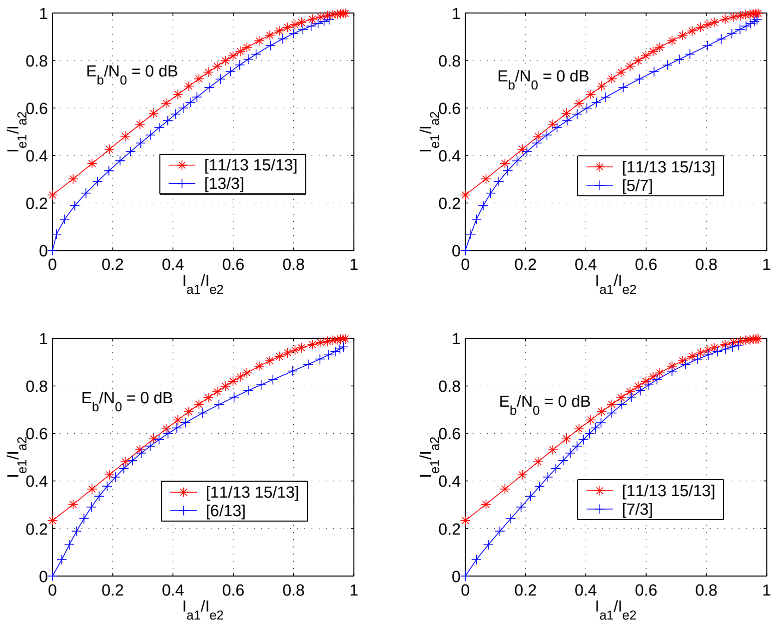

Figure 4: *EXIT charts for R=1/3 turbo codes using an 8-state ELI encoder.*

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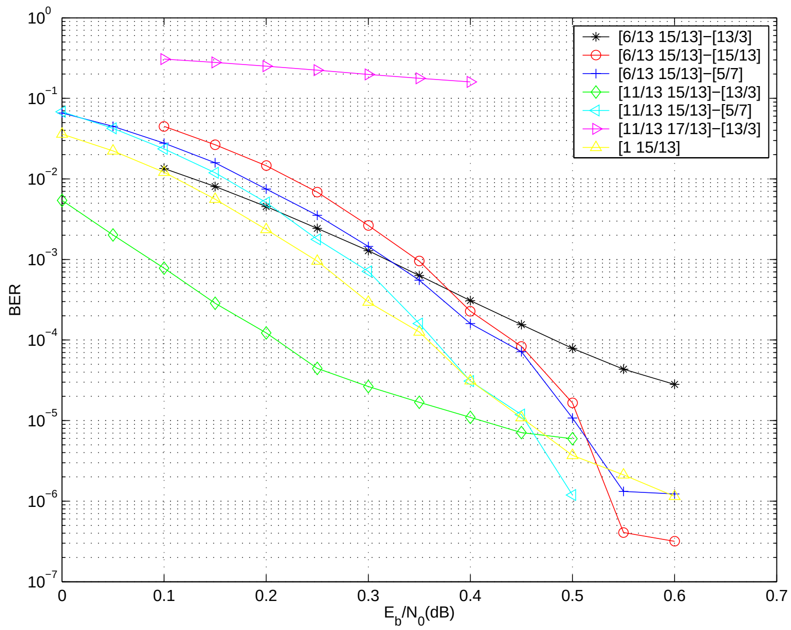

Figure 5: *BER curves for R=1/3 turbo codes, N=4096 bits, 30 iterations, AWGN channel.*

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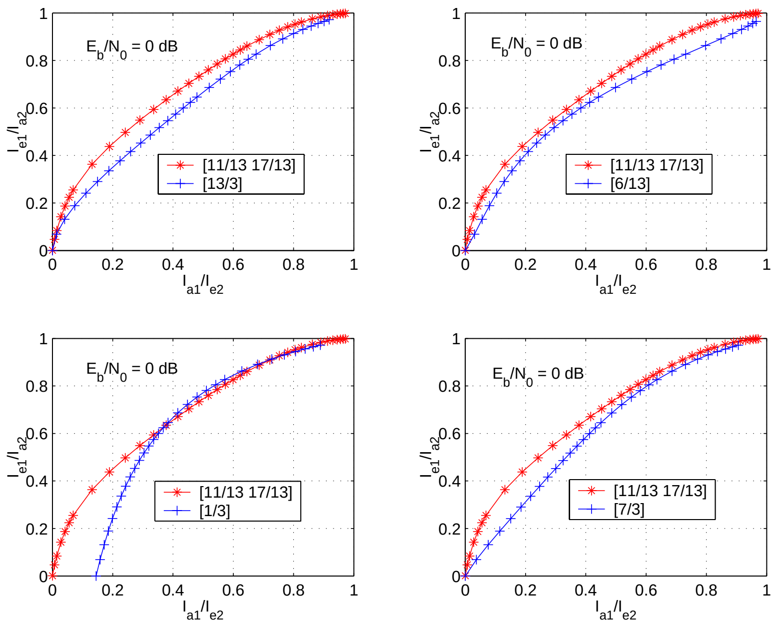

Figure 6: *EXIT charts for R=1/3 turbo codes using an 8-state catastrophic encoder.*

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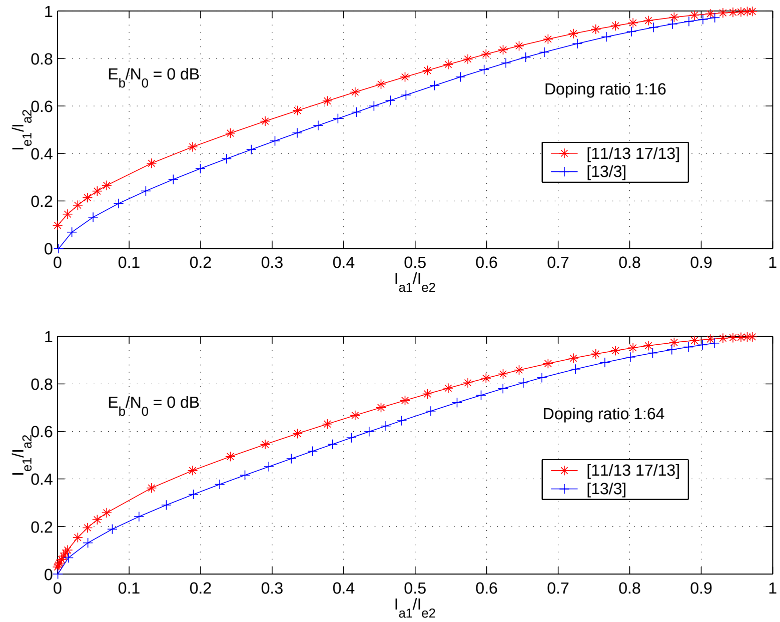

Figure 7: *EXIT charts for R=1/3 doped turbo codes using an 8-state catastrophic encoder.*

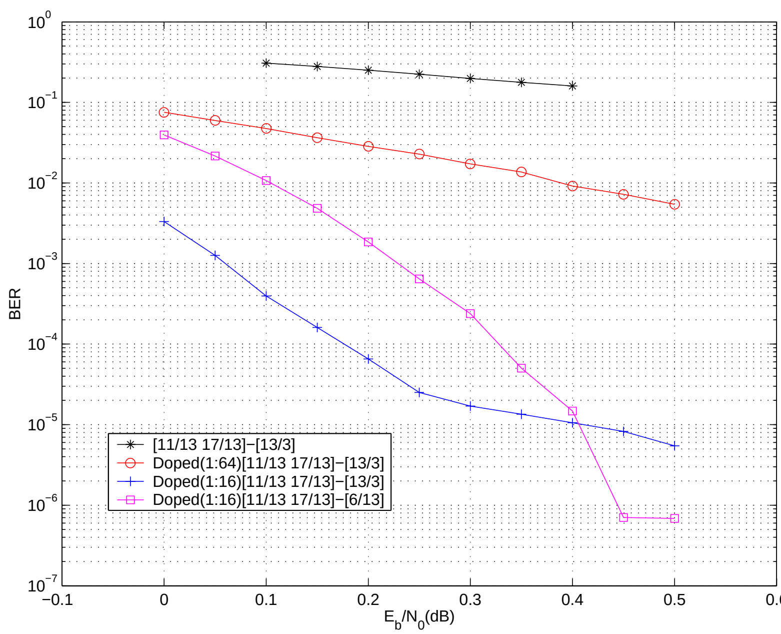

Figure 8: *BER curves for R=1/3 turbo codes using a catastrophic/doped encoder, N=4096 bits, 30 iterations, AWGN channel.*