Unitary Constellation Design with Application to Space-Time Coding

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Abstract

Recently unitary space-time modulation has been presented to give a solution to more reliable multiple antennas communications. Good-performing unitary constellations are required to be designed to fit into this system. Although the basic principle is well understood and certain criteria and their transformations for designing such constellations have been presented, we don't have a general method to design good-performing constellation for any number of transmit and receive antennas and for any transmission rate.

In this paper, we design unitary constellations which have good diversity product in some specific situations and we analyze the asymptotic behavior of some special constellations. We describe a new design criterion for low SNR channel, also we present a mixed code scheme which features good performance unitary constellations. These constellations all come with efficient decoding algorithms.

1 Introduction

One way to acquire high transmission rate on a wireless channel is to use multiple transmit or receive antennas. However, either because of rapid changes in the channel parameters or because of limited system resources, it is reasonable to assume that both the transmitter and the receiver don't know about the channel state information. In [7], unitary space-time modulation is presented to give a solution to a system with such assumption.

Consider a wireless communication system with M transmit antennas and N receive antennas operating in a Rayleigh flat-fading channel. We assume time is discrete and at each time slot, signals are transmitted simultaneously from the M transmitter antennas. We can further assume that the wireless channel is quasi-static over a time block of length T, then the basic equation between the received signals and the transmitted signals is as follows:

$$R = \sqrt{\rho/M} SH + W$$

where $S = (s_{t,m})_{T\times M}$, $H = (h_{m,n})_{M\times N}$, $R = (r_{t,n})_{T\times N}$, $W = (w_{t,n})_{T\times N}$

Here $s_{t,m}$ is the complex signal sent by the m-th transmit antenna at time t, $h_{m,n}$ is a statistically independent multiplicative complex Gaussian fading coefficient between the m-th transmit antenna and the n-th receive antenna. And $w_{t,n}$ is statistically independent additive complex Gaussian noise, $r_{t,n}$ is the complex signal received by the n-th antenna at time t.

Consider the model described in [7], the transmitted codewords are given by $T \times M$ matrices $\sqrt{T}\Phi_k$, $k = 1, 2, \cdots, L$, where L is the number of codewords and $T \geq M$ and $\Phi_k^*\Phi_k = I_M$, the $M \times M$ identity matrix. And we further assume that $\delta_m(\Phi_l^*\Phi_{l'})$ is the m-th singular value of $\Phi_l^*\Phi_{l'}$. It is shown in [7] that pairwise probability of mistaking Φ_l for $\Phi_{l'}$ using the Maximum Likelihood Decoding satisfies

$$P_{\Phi_l,\Phi_{l'}} \leq \frac{1}{2}\prod_{m=1}^{M}[1 + \frac{(\rho T/M)^2}{4(1+\rho T/M)}(1 - \delta_m^2(\Phi_l^*\Phi_{l'}))]^{-N}$$

and the Maximum Likelihood detection is given by

$$\Phi_{ML} = \arg \max_{\Phi_l \in \{\Phi_1,\Phi_2,\cdots,\Phi_L\}} P(R|\Phi_l)$$

where

$$P(R|\Phi_l) = \frac{exp(-tr[I_T - \frac{\rho T}{\rho T+N}\Phi_l\Phi_{l'}^*]RR^*)}{\pi^{MT}(1+\frac{\rho T}{N})^{MN}}$$

To guarantee one can have reliable transmission under the above model, we should design a constellation of Φ_k's such that for every $l \neq l'$ $P_{\Phi_l,\Phi_{l'}}$ is as small as possible .

In [6], a design criterion for high SNR is presented and the problem has been converted to maximize the minimum "distance" between every two elements of a set of unitary matrices. Researchers use different approaches to construct codes according to this criterion. For instance, finite group constellation in [4], orthogonal design in [12]. We also present a series of constellation as a subset of $SU(2)$. It can be proven that this kind of constellation is the best in a asymptotic sense. Some ideas about how to design reducible constellation is presented using the "permutation" function we found. A design criterion for low SNR channel is also presented, we design a series of codes for this design criterion algebraically. In the end of this paper, some ideas about mixed codes will be described. For some special mixed codes with symmetric structure, we have an efficient decoding algorithm which is essential for a real-time data transmission.

2 Design criterion for high SNR channel

In the following discussion, we will assume the same notations in the previous section. One can verify that at high SNR case, it is the design objective to construct $\Phi_1, \Phi_2, \cdots, \Phi_n$ such that

$$\min_{l\neq l'}\prod_{m=1}^{M}(1-\delta_m^2(\Phi_l^*\Phi_{l'}))$$

is as large as possible.

In order to compare different dimensional constellation's performance, the diversity product of a constellation $\mathcal{V}$ is defined in [6]:

$$\zeta\mathcal{V}=\min_{l\neq l'}(\prod_{m=1}^{M}(1-\delta_m(\Phi_l^*\Phi_{l'})^2))^{\frac{1}{2M}}.$$

However it is hard to design general codewords according to the above criterion. In [6], a unitary differential space-time coding scheme is presented. In this case, $T=2M$ and

$$\Phi_k=\begin{pmatrix} I \\ \Psi_k \end{pmatrix}$$

where Ψ_k is a $M\times M$ unitary matrix.

Since

$$1-\delta_m^2(\Phi_{l'}^*\Phi_l)=\frac{1}{4}\lambda_m(2I_M-\Phi_l^*\Phi_{l'}-\Phi_{l'}^*\Phi_l)=\frac{1}{4}\delta_m^2(I_M-\Psi_{l'}^*\Psi_l)=\frac{1}{4}\delta_m^2(\Psi_{l'}-\Psi_l)$$

so we have

$$\prod_{m=1}^{M}(1-\delta_m^2(\Phi_{l'}^*\Phi_l))^{\frac{1}{2M}}=\frac{1}{2}\prod_{m=1}^{M}\delta_m(\Phi_{l'}-\Phi_l)^{\frac{1}{M}}=\frac{1}{2}|\det(\Psi_{l'}-\Psi_l)|^{\frac{1}{M}}.$$

Basically the design criterion has been transformed to the problem as follows:

Find a constellation of square matrices $\mathcal{V}=\{v_1,v_2,\cdots,v_L\}$, such that the diversity product

$$\zeta\mathcal{V}=\frac{1}{2}\min_{0\leq l<l'\leq L}|\det(v_l-v_{l'})|^{\frac{1}{M}} \tag{2.1}$$

is as large as possible.

We call $\mathcal{V}$ a fully diverse constellation if $\zeta\mathcal{V}>0$.

A lot of efforts have been taken to construct constellations with large diversity product. In [6], cyclic groups are introduced for differential modulation. In [12], a two-antenna differential scheme is introduced using orthogonal designs. In [4], all the finite fully diverse group constellations are classified and many good-performing group constellations are found. Using local optimization technique, Cayley codes are presented in [3]. (Cayley codes use modified design criterion based on this one though). Based on the orthogonal codes, two series of codes as subsets of $SU(2)$ are discovered in [2].

2.1 Cayley transformation

Definition 2.1. For a complex $M \times M$ matrix Y which has no eigenvalues at -1, the Cayley tranform of Y is defined to be

$$Y^c = (I+Y)^{-1}(I-Y)$$

where I is the $M \times M$ identity matrix

Note that $(I+Y)$ is nonsingular when Y has no eigenvalue at -1.

Lemma 2.1. *A matrix with no eigenvalues at -1 is unitary if and only if its Cayley transform is skew-Hermitian.*

Lemma 2.2. *A set of unitary matrices $V_0, V_1, \cdots, V_L$ is fully diverse, i.e., $|\det(V_l - V_{l'})|$ is positive for all $l \neq l'$, if and only if the set of its skew-Hermitian Cayley transforms $V_0^c, V_1^c, \cdots, V_L^c$ is fully-diverse. Moreover, we have*

$$V_l - V_{l'} = 2(I+V_l^c)^{-1}[V_{l'}^c - V_l^c](I+V_{l'}^c)^{-1}.$$

From now on, we let A^c denote the Cayley transform of A.

2.2 2 dimensional constellation design and analysis

In [2], two series of 2×2 constellations are discovered as subsets of $SU(2)$, it is proven that they have better performance than the orthogonal designs, however asymptotically they still have n^2 elements with $O(\frac{1}{n})$ diversity product.

Consider the following matrices as a subset of $SU(2)$, for given integers $n > 0$ and $0 \leq k \leq n$, we define

$$N_0 = 1, N_k = \frac{\pi}{\left\lfloor \arcsin \frac{\sin\frac{\pi}{4n}}{\cos\frac{(n-k)\pi}{2n}} \right\rfloor}, k = 1, 2, \cdots, n$$

$$M_k = \frac{\pi}{\left\lfloor \arcsin \frac{\sin\frac{\pi}{4n}}{\sin\frac{(n-k)\pi}{2n}} \right\rfloor}, k = 0, 1, \cdots, n-1, M_n = 1$$

$$\mathcal{V}_k = \left\{ \begin{pmatrix} a_{k,j} & b_{k,l} \\ -\bar{b}_{k,l} & \bar{a}_{k,j} \end{pmatrix} | a_{k,j} = \cos\frac{(n-k)\pi}{2n} e^{i\frac{2j\pi}{N_k}}, b_{k,l} = \sin\frac{(n-k)\pi}{2n} e^{i\frac{2l\pi}{M_k}} \right\}$$

Take

$$\mathcal{V} = \bigcup_{k=0}^{n} \mathcal{V}_k$$

we have the following theorem

Theorem 2.1. *$\mathcal{V}$ is a fully diverse constellation with $\sum_{k=0}^{n} M_k N_k$ elements having diversity product:*

$$\zeta\mathcal{V} = \sin\frac{\pi}{4n}.$$

Proof. Pick two distinct elements in $\mathcal{V}$,

$$F = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad G = \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix}.$$

One can verify that

$$|\det(F-G)| = \det(F-G) = |a-c|^2 + |b-d|^2.$$

So if $|a| \neq |c|$, then

$$|\det(F-G)| = |a-c|^2 + |b-d|^2 \geq (|a|-|c|)^2 + (|b|-|d|)^2 \geq 2 - 2\cos\frac{\pi}{2n},$$

and one can verify that equality holds if there is $k \in \{0, 1, \cdots, n\}$ such that $F \in \mathcal{V}_k$ and $G \in \mathcal{V}_{k+1}$ or alternatively $G \in \mathcal{V}_k$ and $G \in \mathcal{V}_{k+1}$.

If $|a| = |c|$, then either we have

$$|\det(F-G)| = |a-c|^2 + |b-d|^2 \geq |a-c|^2 \geq 2 - 2\cos\frac{\pi}{2n},$$

or we can have

$$|\det(F-G)| = |a-c|^2 + |b-d|^2 \geq |b-d|^2 \geq 2 - 2\cos\frac{\pi}{2n}.$$

in all the cases, we have

$$|\det(F-G)| \geq 2 - 2\cos\frac{\pi}{2n}$$

and we can see in the above discussion, the minimum can be reached, so for this constellation, it follows from (2.1) that

$$\zeta\mathcal{V} = \frac{1}{2}(2 - 2\cos\frac{\pi}{2n})^{\frac{1}{2}} = \sin\frac{\pi}{4n}.$$

$\square$

Remark 2.1. When $n \to \infty$, $\mathcal{V}$ will have $O(n^3)$ elements and have diversity product $O(\frac{1}{n})$.

To see this, note that when we pick a k not "too close" to 0 or n, M_k, N_k will be almost the same as n (up to some constant), also the number of k which is not "too close" to 0 or n is almost the same as n (up to some constant), so when we add them up together, we will find that this constellation has $O(n^3)$ elements, the diversity product of this constellation is $O(\frac{1}{n})$ as proved above.

In fact, using Cayley transform, we can prove the above constellation is the best constellation as subsets of $SU(2)$ in sense of asymptotic behavior. Using Cayley transformation, we can prove:

Theorem 2.2. *Given a constellation $\mathcal{V}$ as a subset of $SU(2)$ with cardinality n, there is $C_1 > 0$ and $C_2 > 0$, such that*

$$C_1\left(\frac{1}{n^{\frac{1}{3}}}\right) \le \max_{|\mathcal{V}|=n} \zeta\mathcal{V} \le C_2\left(\frac{1}{n^{\frac{1}{3}}}\right).$$

What if the constellation is a subset of $U(2)$?

In fact, we have:

Theorem 2.3. *Given a constellation $\mathcal{V}$ as a subset of $U(2)$ with cardinality n, there is $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1\left(\frac{1}{n^{\frac{1}{3}}}\right) \le \max_{|\mathcal{V}|=n} \zeta\mathcal{V} \le C_2\left(\frac{1}{n^{\frac{1}{4}}}\right).$$

2.3 Reducible constellation design and analysis

In this section we present how to construct reducible constellations.

Given positive integers m, n and a $m \times m$ matrix A and a $n \times n$ matrix B, we define

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Given positive integers m, n, we call a constellation a reducible constellation if every element in it takes the following form:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where A is a $m \times m$ matrix and B is a $n \times n$ matrix.

Consider the following question:

Given $0 \le a_0, a_1, \cdots, a_n \le 1$ and all a_i distinct, $0 \le b_0, b_1, \cdots, b_n \le 1$ and all b_i distinct, how to make $\min_{i,j} |(a_i - a_j)(b_i - b_j)|$ as large as possible. One natural idea is just to make $a_i{}', b_i{}'s$ have the same relative order, that is, $0 \le a_0 \le a_1 \le \cdots \le a_n \le 1$, $0 \le b_0 \le b_1 \le \cdots \le b_n \le 1$, in this case one verifies that

$$\max_{a_i, b_j, i,j=1,2,\cdots,n} \min_{i,j} |(a_i - a_j)(b_i - b_j)| = \frac{1}{n^2}.$$

But of course, we have a better choice: "permute" the relative order of a_i's, b_i's.

Let $n = m^2$, define the following function ω from $\{0, \frac{1}{n}, \frac{1}{n}, \cdots, \frac{n-1}{n}\}$ to itself. Here for a real number r, we use $\{r\}$ denote $r - \lfloor r \rfloor$.

$$\omega\left(\frac{i}{n}\right) = \begin{cases} \frac{i}{n} & \text{if } m|i \\ \left\{\frac{(k+l)m+k+1}{n}\right\} & \text{if } 0 \le k = \lfloor \frac{i}{m} \rfloor \le (m-1) \text{ and } 0 < l = \{\frac{i}{m}\} < m \\ \frac{(l-1)m+l}{n} & \text{if } \lfloor \frac{i}{m} \rfloor = (m-1) \text{ and } 0 < l = \{\frac{i}{m}\} < m \end{cases}$$

then let $a_i = \frac{i}{n}$, where $i = 0, 1, \cdots, n-1$, and $b_i = \omega(a_i)$, one can check

Lemma 2.3.

$$\min_{i \neq j} |(a_i - a_j)(\omega(a_i) - \omega(a_j))| = \frac{m-1}{n^2} \sim \frac{\sqrt{n}}{n^2}$$

Corollary 2.1. *Let $n = m^2$, we have*

$$\min_{k \neq l} |(e^{\frac{2\pi k}{n} i} - e^{\frac{2\pi l}{n} i})(e^{\frac{2\pi}{n}\omega(\frac{k}{n})i} - e^{\frac{2\pi}{n}\omega(\frac{l}{n})i})| = 4 \sin\frac{\pi}{n} \sin\frac{(m-1)\pi}{n}$$

In fact, the "permutation" idea can be applied to reducible constellation design.

2.3.1 2×2 reducible constellation design and analysis

Let $n = m^2$, consider the following diagonal constellation as a subset of $U(2)$:

$$\mathcal{V} = \left\{ \begin{pmatrix} e^{i\theta_k} & 0 \\ 0 & e^{2\pi\omega(\frac{\theta_k}{2\pi})i} \end{pmatrix} | \theta_k = \frac{2\pi k}{n}, k = 1, 2, \cdots, n-1 \right\}$$

according to the Corollary 2.1, we have

$$\zeta\mathcal{V} = \sqrt{\sin\frac{\pi}{n} \sin\frac{(m-1)\pi}{n}}$$

2.4 3×3 reducible constellation design and analysis

Firstly we define

$$U(1) \oplus U(2) := \{A \oplus B | A \in U(1), B \in U(2)\}$$

Let $n = m^2$, consider the following constellation as a subset of $U(1) \oplus U(2)$:

$$\mathcal{V} = \left\{ \begin{pmatrix} e^{i(\omega(\frac{\theta_k}{2\pi})2\pi + \frac{\omega(\frac{\varphi_l}{2\pi})}{n}2\pi)} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2}e^{i\theta_k} & \frac{\sqrt{2}}{2}e^{i\varphi_l} \\ 0 & -\frac{\sqrt{2}}{2}e^{-i\varphi_l} & \frac{\sqrt{2}}{2}e^{-i\theta_k} \end{pmatrix} \text{ where } \theta_k = \frac{2\pi}{n}k, \varphi_l = \frac{2\pi}{n}l \right\}$$

then it is a routine to check that the diversity product of this constellation is

$$\zeta\mathcal{V} = \sqrt[3]{\frac{\sin\frac{(\sqrt{n}-1)\pi}{n^2} \sin^2\frac{\pi}{n}}{2}}$$

2.5 4×4 reducible constellation design and analysis

We first define

$$U(2) \oplus U(2) := \{A \oplus B | A \in U(2), B \in U(2)\}$$

Let $n = m^2$, consider the following constellation as subset $U(2) \oplus U(2)$,

$$\mathcal{V} = \{A \oplus B | A = \frac{\sqrt{2}}{2} \begin{pmatrix} e^{i\theta_k} & e^{i\varphi_l} \\ -e^{i\varphi_l} & e^{-i\theta_k} \end{pmatrix} \quad B = \frac{\sqrt{2}}{2} \begin{pmatrix} e^{i\omega(\frac{\theta_k}{2\pi})2\pi} & e^{i\omega(\frac{\varphi_l}{2\pi})2\pi} \\ -e^{-i\omega(\frac{\varphi_l}{2\pi})2\pi} & e^{-i\omega(\frac{\theta_k}{2\pi})2\pi} \end{pmatrix}$$

$$\text{where} \quad \theta_k = \frac{2\pi k}{n}, \varphi_l = \frac{2\pi l}{n}\}$$

so we know

$$|\det(A \oplus B - C \oplus D)| = |\det(A - C)||\det(B - D)|$$

it is a routine to check that the diversity product is

$$\zeta\mathcal{V} = \sqrt{\frac{\sin\frac{\pi}{n}\sin\frac{(m-1)\pi}{n}}{2}}$$

Remark 2.2. As what we did above, similar idea can be applied to any dimensional diagonal constellation construction.

3 Design criterion for low SNR channel

As we mentioned before, the above design criterion is for high SNR channel design, but in practical world, we are more willing to design codes suitable for low SNR channel so that even the system is working under very noisy situation, the data can still be reliably decoded. Let us recall that under unitary modulation scheme assumption, it is shown that pairwise probability of mistaking Φ_l for $\Phi_{l'}$ using the Maximum Likelihood Decoding satisfies

$$P_{\Phi_l,\Phi_{l'}} \leq \frac{1}{2}\prod_{m=1}^{M}[1 + \frac{(\rho T/M)^2}{4(1+\rho T/M)}(1 - \delta_m^2(\Phi_l^*\Phi_{l'}))]^{-N}$$

At high SNR, the probability primarily depends on $\prod_{i=1}^{M}(1-\delta_i^2)$, but at low SNR, the probability primarily depends on $\sum_{m=1}^{M}\delta_i^2$. To see this, let

$$\rho_1 := \frac{(\rho T/M)^2}{4(1+\rho T/M)}$$

then we will find that

$$\prod_{m=1}^{M}[1+\frac{(\rho T/M)^2}{4(1+\rho T/M)}(1-\delta_m^2(\Phi_l^*\Phi_{l'}))] = \prod_{m=1}^{M}[1+\rho_1-\rho_1\delta_m^2(\Phi_l^*\Phi_{l'})]$$

$$= (1+\rho_1)^M - \rho_1\sum_{m=1}^{M}\delta_m^2(\Phi_l^*\Phi_{l'})(1+\rho_1)^{M-1} + O(\rho_1^2)$$

so when $\rho \to 0$, i.e. $\rho_1 \to 0$, we can omit high order item $O(\rho_1^2)$ and we will see the upper bound of $P_{\Phi_l,\Phi_{l'}}$ really depends on

$$\sum_j \delta_j^2 = \|\Phi_l^*\Phi_{l'}\|_F^2,$$

where $\|\|_F$ denotes the Frobenius norm.

Actually we can define the diversity product to be

$$\zeta\mathcal{V} = \max_{l,l'}\frac{1}{\sqrt{N}}\sqrt{\sum_j \delta_j^2} = \max_{l,l'}\frac{1}{\sqrt{N}}\|\Phi_l^*\Phi_{l'}\|_F$$

So for a low SNR channel, the criterion is as follows:

Find a constellation of matrices $\mathcal{V} = \{v_1, v_2, \cdots, v_L\}$ such that the diversity product

$$\zeta\mathcal{V} = \max_{l,l'}\frac{1}{\sqrt{N}}\|v_l^*v_{l'}\|_F$$

is as small as possible.

Remark 3.1. In order to guarantee our constellation with low SNR design criterion will still work well under high SNR channel, we can further demand that $(v_l|v_{l'})$ is full rank matrix for every l, l', i.e., all the singular values of $v_l^*v_{l'}$ are strictly less than 1. For more detailed reasoning, we refer to [11].

As in [11], a Generalized Non-coherent PSK constellation is presented. Let $T = 2M$ and consider the following constellation as a subset of $\mathbb{R}^{T\times M}$.

$$\mathcal{V}_1 = \left\{ \begin{pmatrix} \cos\theta_k & 0 & \cdots & 0 \\ 0 & \cos\theta_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cos\theta_k \\ \sin\theta_k & 0 & \cdots & 0 \\ 0 & \sin\theta_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sin\theta_k \end{pmatrix} |\theta_k = \frac{2\pi k}{L}, k = 0, 1, \cdots, L-1 \right\}$$

One can routinely check that the above constellation has L elements with diversity product

$$\zeta\mathcal{V}_1 = \cos\frac{2\pi}{L}.$$

Inspired by this constellation, we present the following constellation.

Let $T = 2M$, consider the following constellation as a subset of $\mathbb{C}^{T\times M}$, given $0 < \theta_0 < \frac{\pi}{4}$, L_1, L_2, M, N positive integers,

$$\mathcal{V}_2 = \left\{ \left(\begin{array}{cccc} \cos\theta_k & 0 & \cdots & 0 \\ 0 & \cos\theta_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cos\theta_k \\ \sin\theta_k e^{i\varphi_j} & 0 & \cdots & 0 \\ 0 & \sin\theta_k e^{i\varphi_j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \sin\theta_k e^{i\varphi_j} \end{array} \right) \middle| \begin{array}{ll} \theta_k = \theta_0 + \frac{k(\frac{\pi}{2}-2\theta_0)}{L_1}, & k = 0, 1, \cdots, L_1 \\ \varphi_j = \frac{2\pi j}{L_2}, & j = 0, 1, \cdots, L_2 - 1 \end{array} \right\}$$

So one can check routinely this is a constellation with $L_2(L_1+1)$ elements and

$$\begin{aligned} \zeta\mathcal{V}_2 &= \max\left\{ \cos\frac{\frac{\pi}{2}-2\theta_0}{L_1}, |\cos^2\theta_0 + \sin^2\theta_0 e^{\frac{2i\pi}{L_2}}| \right\} \\ &= \max\left\{ \cos\frac{\frac{\pi}{2}-2\theta_0}{L_1}, \sqrt{1-\sin^2(2\theta_0)\sin^2\frac{\pi}{L_2}} \right\} \end{aligned}$$

In fact, we are going to prove that with the same or even larger diversity product, our constellation has more elements compared to the Generalized Non-coherent PSK constellation.

Let $\cos\frac{\frac{\pi}{2}-2\theta_0}{L_1} = \cos\frac{2\pi}{L}$, we have

$$2\theta_0 = \frac{\pi}{2} - \frac{2L_1\pi}{L}$$

if we carefully choose L_1, L_2 s.t.

$$1 - \sin^2(2\theta_0)\sin^2\frac{\pi}{L_2} \leq \cos^2\frac{2\pi}{L}$$

that is

$$\sin\frac{2\pi}{L} \leq \cos\frac{2L_1\pi}{L}\sin\frac{\pi}{L_2}$$

then we will have a constellation with the same or larger diversity product while having $O(L^2)$ elements.

The following table illustrates how many more elements a constellation $\mathcal{V}_2$ has than a constellation $\mathcal{V}_1$ with the same diversity product.

For simplicity, we assume $M = 1$, and $T = 2M = 2$.

$\zeta\mathcal{V}_1 = \zeta\mathcal{V}_2$	$\mathcal{V}_1$	$\mathcal{V}_2$
0.97815	$\lvert\mathcal{V}_1\rvert = L = 30$	$\lvert\mathcal{V}_2\rvert = L_2(L_1+1) = (3+1)12 = 48$
0.99803	$\lvert\mathcal{V}_1\rvert = L = 100$	$\lvert\mathcal{V}_2\rvert = L_2(L_1+1) = (13+1)34 = 476$
0.99951	$\lvert\mathcal{V}_1\rvert = L = 200$	$\lvert\mathcal{V}_2\rvert = L_2(L_1+1) = (27+1)66 = 1848$
0.99992	$\lvert\mathcal{V}_1\rvert = L = 500$	$\lvert\mathcal{V}_2\rvert = L_2(L_1+1) = (65+1)171 = 11286$
0.99998	$\lvert\mathcal{V}_1\rvert = L = 1000$	$\lvert\mathcal{V}_2\rvert = L_2(L_1+1) = (140+1)318 = 44838$

Remark 3.2. As described in [11], Generalized Non-coherent PSK constellation has a very simple decoding algorithm, since our constellation basically keeps the algebraic structure, it is easy to see that our constellation has also very small decoding complexity.

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