MIMO Systems Properties Preservation under SPR Substitutions

G. Fernández-Anaya△ and J. C. Martínez-García◊[1]
△Departamento de Ciencias
Universidad Iberoaméricana
Lomas de Santa Fe
01210 México D.F.
MÉXICO
◊Programa de Investigación en Matemáticas Aplicadas y Computación
Instituto Mexicano del Petróleo
Eje Central Lázaro Cárdenas 152
Col. San Bartolo Atepehuacan
07730 México D.F.
MÉXICO

Abstract

This paper tackle the preservation of positive real properties in Multi-Input Multi-Output transfer functions, when performing substitutions of the Laplace variable s by strictly positive real functions of relative degree equal to zero. We consider also the preservation of stability properties of a class of unforced linear time-invariant systems affected by a memoryless, possibly time-varying nonlinear, input which depends on the system output.

1 Introduction

As is pointed out in [9], the concept of *positive realness* of a transfer function plays a central role in *Stability Theory*. The definition of rational Positive Real functions (*PR* functions) arose in the context of *Circuit Theory*. In fact, the driving point impedance of a passive network is rational and positive real. If the network is *dissipative* (due to the presence of resistors), the driving point impedance of the network is a Strictly Positive Real transfer function (*SPR* function). Thus, positive real systems, also called *passive systems*, are systems that do not generate energy. The celebrated Kalman-Yakubovich-Popov (**KYP**) lemma (see for instance the Lefschetz-Kalman-Yakubovich version of this result in [9]), established the key role that strict positivity realness plays in the obtention of Lyapunov functions associated to the stability analysis of a particular class of nonlinear systems, *i.e.*, Linear Time Invariant systems (**LTI** systems) with a single memoriless nonlinearity. In fact, positive realness has been extensively studied by the Automatic Control community, see for instance the studies concerning: absolutely stability [7], characterization and construction of robust strict positive

[1]Corresponding author, on leave from Departamento de Control Automático, CINVESTAV-IPN, México. E-mail address: martinez@ctrl.cinvestav.mx.

real systems [3], relationships between positivity realness of proper and stable **LTI** systems and stabilizing solutions of Riccati equations [12], stability of adaptive control schemes based on parameter adaptation algorithms [1], and passive filters [2]. As far as the frequency-described continuous **LTI** systems are concerned, the study of control-oriented properties (like stability) resulting from the substitution of the complex Laplace variable s by rational transfer functions have been little studied by the Automatic Control community. However, some interesting results have recently been published:

As far as robust stability of polynomial families is concerned, some results are given in [10] (for a particular class of polynomials), when interpreting substitutions as nonlinearly correlated perturbations on the coefficients. More recently, in [4], some results for proper and stable real rational SISO functions and coprime factorizations were proved, by making substitutions with $\alpha(s) = (as+b)/(cs+d)$, where a, b, c, and d are strictly positive real numbers, and with ad-$bc \neq 0$.

Section 2 is dedicated to some preliminaries, mainly concerning the characterization of several classes of real positive functions. We tackle in Section 3 the preservation of positive real properties in Multi-Input Multi-Output transfer functions, when performing substitutions of the Laplace variable s by strictly positive real functions of relative degree equal to zero. Our Section 4 deals with the preservation of stability properties of a class of unforced linear time-invariant systems affected by a memoryless, possibly time-varying nonlinear, input depending on the system output.

2 Preliminaries

In this section, we give the notation, the basic definitions and some necessary results for the sequel.

Notation : $C^+ = \{\sigma + j\omega \in C \ : \ \sigma > 0\}$, $ImC \equiv \{z \in C \ : \ Re(z) = 0\}$,
$R = (-\infty, \infty)$, $\overline{C}_e^+ \equiv C^+ \cup \{\infty\} \cup ImC$, $\overline{C}^+ \equiv C^+ \cup ImC$, $C_e^+ \equiv C^+ \cup \{\infty\}$.

Definition 2.1. *Let RC be the Euclidean domain of the proper, stable and rational real functions, $R(s)$ the field of real rational functions, $R_p(s)$ the ring of real rational and proper functions and $R[s]$ the ring of the real polinomials.*

Definition 2.2. *Let $p(s) \in R(s)$ be a rational function of complex variable $s = \sigma + j\omega$.*

1. *[7] $p(s)$ is Positive Real (PR) if:*

   (a) $p(s)$ is real for s real; (b) $Re[p(s)] \geq 0$ for all $Re[s] > 0$.

2. *[7] $p(s)$ is Strictly Positive Real (SPR) if $p(s$-$\varepsilon)$ is PR for some $\varepsilon > 0$.*

3. *[7], [5] $p(s)$ of zero relative degree is SPR (SPR0 function) if and only if:*

   (a) $p(s)$ is analytic in $Re[s] \geq 0$; (b) $Re[p(j\omega)] > 0$ for all $\omega \in R$.

4. *[8] $p(s)$ is Extended Strictly Positive Real (ESPR) if it is SPR and $Re[p(j\infty)] > 0$.*

5. *[8], [6] $p(s)$ is Strongly Strictly Positive Real (SSPR) if it is SPR and $Re[p(\infty)] > 0$.*

Definition 2.3. *[9] $SPR0 := \{G(s) \in RH^\infty \mid G(s)\}$ is SPR0.*

Some properties of SPR0 functions are:

1. If $p(s)$ is a SPR0 function, then $1/p(s)$ is also a SPR0 function.

2. If $p_1(s)$ and $p_2(s)$ are SPR0 functions, then $\alpha p_1(s) + \beta p_2(s)$ is a SPR0 function for α, $\beta \geq 0$ (see [9]).

We can at this level present our:

Lemma 2.1. *[5] Consider a transfer function $p(s)$ be given.*

1. *If $p(s) \in RH^\infty$ with $q(s)$ any SPR0 function, then $p(q(s)) \in RH^\infty$.*

2. *If $p(s), q(s) \in$ SPR0, then $p(q(s))$, $q(p(s)) \in$ SPR0.*

3. *If the function $q(s) \in$ SPR0, then $q(\overline{C}_e^+) \subseteq C^+$.*

4. *If $p(s) \in$ SPR0, then $p(s)$ is ESPR.*

Proof. Items 1, 2 and 3 were proved in [5].

Item 4:

Since $p(s)$ has zero relative degree, then $Re[p(j\omega)]$ has also zero relative degree, while $Im[p(j\omega)]$ has n relative degree, with n denoting of the degree of the denominator of $p(s)$ (which coincides with the degree of its numerator). Then, $\lim_{\omega\to\infty} Re[p(j\omega)]$ is a positive real number and $\lim_{\omega\to\infty} Im[p(j\omega)] = 0$. □

The following is a well-known result:

Lemma 2.2. *If $p(s) \in R(s)$, then:*

$$Re\,[p(j\omega)] = Re\,[p(-j\omega)]$$

and:

$$Im\,[p(j\omega)] = -Im\,[p(-j\omega)]$$

for all $\omega \in R$.

3 Results for MIMO systems

We present in this section some results concerning the preservation of real positivity properties in Multi-Input Multi-Output (MIMO) systems, when performing the substitution of the Laplace variable s by SPR0 functions.

Definition 3.1. *[7] A $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Positive Real (PR) if:*

1. *All elements of $Z(s)$ are analytic for $Re[s] > 0$.*
2. *Any pure imaginary pole of any element of $Z(s)$ is a simple pole and the associated residue matrix of $Z(s)$ is positive semidefinite Hermitian, and:*
3. *For all real ω for which $j\omega$ is not a pole of any element of $Z(s)$, the matrix $Z(j\omega) + Z^T(\text{-}j\omega)$ is positive semidefinite ($Z(j\omega) + Z^T(\text{-}j\omega) \geq 0$).*

Remark 3.1. *Again, a $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Strictly Positive Real (SPR), if $Z(s\text{-}\varepsilon)$ is PR for some $\varepsilon > 0$. Note also, that if $Z(s)$ is SPR, then there exist some $\varepsilon > 0$ such that $Z(s\text{-}\varepsilon)$ is PR.*

Lemma 3.1. *Let $Z(s)$ be a $\rho \times \rho$ proper rational transfer function matrix, and suppose $\det\left[Z(s) + Z^T(-s)\right]$ is not identically zero. Then, $Z(s)$ is SPR if and only if:*

1. *$Z(s)$ is Hurwitz i.e., $Z(s) \in \mathbf{RH}^\infty$ where $\mathbf{RH}^\infty$ is the set of matrices with elements in RH^∞,*
2. *$Z(j\omega) + Z^T(-j\omega) > 0$ for all real ω, and*
3. *one of the following three conditions is satisfied:*
   - *(a) $Z(\infty) + Z^T(\infty) > 0$;*
   - *(b) $Z(\infty) + Z^T(\infty) = 0$ and $\lim_{\omega\to\infty} \omega^2 \left[Z(j\omega) + Z^T(-j\omega)\right] > 0$;*
   - *(c) $Z(\infty) + Z^T(\infty) \geq 0$ and there exist positive constants σ_0 and ω_0 such that*

$$\omega^2 \sigma_{\min}\left[Z(j\omega) + Z^T(-j\omega)\right] \geq \sigma_0, \ \forall\, |\omega| \geq \omega_0.$$

Definition 3.2. *A $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ is called Extended Strictly Positive Real (ESPR) if it is SPR and $Z(j\infty) + Z^T(\text{-}j\infty) > 0$.*

Definition 3.3. *[8] Let $Z(s)$ be a $\rho \times \rho$ proper rational transfer function matrix. Then:*

1. *$Z(s)$ is called strongly SPR (SSPR), if $Z(s)$ is SPR and $Z(\infty) + Z^T(\infty) > 0$.*
2. *$Z(s)$ is called weak SPR (WSPR), if $Z(s)$ is SPR and: $Z(j\omega) + Z^T(\text{-}j\omega) > 0$ for all $\omega \in R$.*

3. *$Z(s)$ is called MSPR, if $Z(s)$ is PR and $Z(j\omega) + Z^T(\text{-}j\omega) > 0$ for all $\omega \in R$.*

The following result is evident:

Lemma 3.2. *Let $Z(s)$ be a $\rho \times \rho$ proper rational transfer function matrix. then:*

1. *If $Z(s)$ is a SPR function matrix, then $Z(s+\varepsilon)$ is a SPR $\rho \times \rho$ proper rational transfer function matrix, for each $\varepsilon \geq 0$.*

2. *If $Z(s)$ is a PR function matrix, then $Z(s+\varepsilon)$ is a SPR function matrix for each $\varepsilon > 0$.*

We can at this level present our:

Theorem 3.1. *Consider a transfer function matrix $Z(s) \in \mathbf{RH}^\infty$ be given.*

1. *If $Z(s) \in \mathbf{RH}^\infty$, then $Z(p(s)) \in \mathbf{RH}^\infty$for each $p(s)$ ∈SPR0,*

2. *if $Z(s)$ is a SPR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is a ESPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$ SPR0.*

3. *If $Z(s)$ is a PR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$is a ESPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$ SPR0.*

Proof. We proceed now to prove Theorem 3.1 item by item:

1. Each element $Z_{ij}(s)$ of the matrix $Z(s)$ is in RH^∞, then by Lemma 2.1 $Z_{ij}(p(s)) \in RH^\infty$.

2. For the proof of item 2 we consider two cases:

   (a) When:

   $$\det\left[Z(s) + Z^T(-s)\right] \neq 0,$$

   except for a finite number of $s \in R$. In this case we can use Lemma 3.1 for $Z(p(s))$, and noting that item 1 of Lemma 3.1 corresponds to item 1 of this Theorem 3.1. Therefore, we only need to prove items 2 and 3 of Lemma 3.1.

   - Item 2 in Lemma 3.1: since $Z(s)$ is SPR, then $Z(j\omega) + Z^T(-j\omega) > 0$ for all real ω. In consequence:

     $$\begin{aligned} &Z(p(j\omega)) + Z^T(p(-j\omega)) = \\ &Z(Re\,[p(j\omega)] + jIm\,[p(j\omega)]) + Z^T(Re\,[p(-j\omega)] + jIm\,[p(-j\omega)]) \end{aligned}$$

     for each $p(s)$ ∈SPR0. Now by item 2 in Definition 3.1, $Re\,[p(j\omega)] > 0$ for all real ω. Taking $\eta(\omega) = Re\,[p(j\omega)]\,,\ \delta(\omega) = Im\,[p(j\omega)]\,,$ noting that by Lemma 2.2 $Re\,[p(j\omega)] = Re\,[p(-j\omega)]$ and $Im\,[p(j\omega)] = -Im\,[p(-j\omega)]$ for all $\omega \in R$, and considering that $Re\,[p(j\omega)]$ and $Im\,[p(j\omega)]$ are real numbers for all real ω, we have that:

$$\begin{array}{l} Z(p(j\omega)) + Z^T(p(-j\omega)) = \\ Z(\eta(\omega) + j\delta(\omega)) + Z^T(\eta(\omega) - j\delta(\omega)) \end{array}$$

for all $\omega \in R$. Thus, we can consider $Z(s+\varepsilon)$ instead of $Z(p(s))$, but by item 2 of Lemma 3.1 and item 3 (b) of Definition 2.2:

$$Z(\varepsilon + j\omega) + Z^T(\varepsilon - j\omega) > 0$$

for all real ω. Then, we conclude that:

$$Z(p(j\omega)) + Z^T(p(-j\omega)) > 0$$

for each $p(s) \in$SPR0 and for all real ω.

- Item 3 Lemma 3.1: since $Z(s)$ is SPR, by Lemma 3.2 $Z(s+\varepsilon)$ is also a SPR function for each $\varepsilon > 0$. Thus, by item 2 of Lemma 3.1 we have that:

$$Z(\varepsilon + j\omega) + Z^T(\varepsilon - j\omega) > 0$$

for all real ω and for all $\varepsilon > 0$. In particular for $\omega = 0$:

$$Z(\varepsilon) + Z^T(\varepsilon) > 0,$$

for all real $\varepsilon > 0$. Now taking $\varepsilon_p = \lim_{s\to\infty} p(s)$ and considering that $\varepsilon_p > 0$ for each $p(s) \in$SPR0, we obtain that:

$$Z(\varepsilon_0) + Z^T(\varepsilon_0) > 0$$

i.e.:

$$Z(p(\infty)) + Z^T(p(\infty)) > 0$$

for each $p(s) \in$SPR0.

(b) When:

$$\det\left[Z(s) + Z^T(-s)\right] = 0.$$

In this case, by Definition 3.2 and Remark 3.1 we need to prove that $Z(p(s-\varepsilon_0))$ is PR for some $\varepsilon_0 > 0$. Since $Z(s)$ is SPR, item 1 of the Lemma 3.2 is similar to item 1 of this Theorem 3.1. Moreover:

Since $Z(s)$ is SPR, it has not purely imaginary poles. Now, $Re\,[p(j\omega)] > 0$ (for all real ω) and $Z(p(s)) \in \mathbf{RH}^\infty$ for each $p(s) \in$SPR0, then the elements of $Z(p(s))$ has not purely imaginary poles, for each $p(s) \in$SPR0.

Now, we proceed to prove that:

$$Z(p(j\omega)) + Z^T(p(-j\omega)) \geq 0.$$

Suppose that $p(s-\varepsilon_1)$ is a PR rational function for some $\varepsilon_1 > 0$. Then, by an elemental argument of continuity of the SPR0 function $p(s)$, there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon_1$ and $p(s-\varepsilon_0) \in$SPR0. Then as $Z(s)$ is SPR and $p(s-\varepsilon_0) \in$SPR0, we have for $Z(p(s-\varepsilon_0))$ that:

$$\begin{aligned} &Z(Re\,[p(j\omega-\varepsilon_0)] + jIm\,[p(j\omega-\varepsilon_0)])+ \\ &Z^T(Re\,[p(-j\omega-\varepsilon_0)] + jIm\,[p(-j\omega-\varepsilon_0)]) > 0 \end{aligned}$$

for all real ω, by item 2, case (a) (i.e., det $\left[Z(s) + Z^T(-s)\right] \neq 0$) of this Theorem 3.1. Thus, $Z(p(s-\varepsilon_0))$ is PR and then $Z(p(s))$ is SPR for each $p(s) \in$ SPR0. Now, it only is necessary to prove that:

$$Z(p(j\infty)) + Z^T(p(-j\infty)) > 0,$$

for each $p(s) \in$SPR0. By Lemma 2.1 item 4, it is always true that $\varepsilon_{pr} := Re[p(j\infty)] > 0$ and $Im[p(j\infty)] = 0$, for each $p(s) \in$SPR0. Then:

$$Z(p(j\infty)) + Z^T(p(-j\infty)) = Z(\varepsilon_{pr}) + Z^T(\varepsilon_{pr}).$$

But $Z(\varepsilon_{pr}) + Z^T(\varepsilon_{pr}) > 0$ for all real $\varepsilon_{pr} > 0$, therefore $Z(p(j\infty)) + Z^T(p(-j\infty)) > 0$ for each $p(s) \in$SPR0.

3. We proceed now to prove that if $Z(s)$ is a PR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is a ESPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$ SPR0. Assuming positive realness of $Z(s)$, we first prove that $Z(p(s)) \in \mathbf{RH}^\infty$ and that each element of $Z(p(s))$ has not purely imaginary poles. Finally, we prove that $Z(p(j\omega)) + Z^T(p(-j\omega)) > 0$ for all real ω and that $Z(p(j\infty)) + Z^T(p(-j\infty)) > 0$.

   (a) Each element $Z_{ij}(s)$ of the matrix $Z(s)$ is analytic for $Re[s] > 0$, now by Definition 2.2, item 3 (a) $p(s)$ is analytic for $Re[s] \geq 0$. By Lemma 2.1 item 3 $p(\overline{C}_e^+) \subseteq C^+$, then $Z_{ij}(p(s))$ is analytic for $Re[s] \geq 0$, and in consequence $Z_{ij}(p(s)) \in RH^\infty$, i.e., $Z(p(s)) \in \mathbf{RH}^\infty$ for each $p(s) \in$SPR0.

   (b) Note that any purely imaginary pole of some element of $Z(s)$ is simple pole (its multiplicity order is just equal to one) and it is of the form:

$$\frac{1}{s^2 + k^2},$$

   then substituting s by the SPR0 function $p(s) = \frac{N_p(s)}{D_p(s)}$ with $N_p(s),\, D_p(s) \in R[s]$, we obtain:

$$\frac{1}{p^2(s) + k^2} = \frac{D_p^2(s)}{N_p^2(s) + k^2 D_p^2(s)}.$$

But k^2 is an SPR0 function and by properties 1) and 2) of SPR0 functions, the sum of SPR0 functions is also a SPR0 function, and the multiplicative inverse of an SPR0 function is a SPR0 function, then the function $\frac{1}{p^2(s)+k^2}$ a SPR0 function and has not purely imaginary poles and the same statement is valid for each element of $Z(p(s))$, for each $p(s)$ ∈SPR0.

(c) Now, it is necessary to prove that $Z(p(j\omega)) + Z^T(p(-j\omega)) > 0$ for all real ω and for each $p(s)$ ∈SPR0. Suppose that $p(s-\varepsilon_1)$ is PR rational function for some $\varepsilon_1 > 0$. Then, by an elemental argument of continuity of the SPR0 function $p(s)$, there exists ε_0 such that $0 < \varepsilon_0 < \varepsilon_1$ and such that $p(s-\varepsilon_0)$ ∈SPR0. Now, as $Z(s)$ is PR and $p(s-\varepsilon_0)$ ∈SPR0, by a similar argument to the proof of the item 2 case (a) in this theorem we have that:

$$Z(Re\,[p(j\omega-\varepsilon_0)] + jIm\,[p(j\omega-\varepsilon_0)])+$$
$$Z^T(Re\,[p(-j\omega+\varepsilon_0)] + jIm\,[p(-j\omega+\varepsilon_0)])$$

i.e.:

$$Z(\sigma(\omega) + j\gamma(\omega)) + Z^T(\sigma(\omega) - j\gamma(\omega)),$$

where $\sigma(\omega) = Re\,[p(j\omega-\varepsilon_0)]$ and $\gamma(\omega) = Im\,[p(j\omega-\varepsilon_0)]$, and by Lemma 2.2 we have that:

$$Re\,[p(j\omega-\varepsilon_0)] = Re\,[p(-j\omega+\varepsilon_0)]$$

and:

$$Im\,[p(j\omega-\varepsilon_0)] = -Im\,[p(-j\omega+\varepsilon_0)]\,.$$

We profit at this level from the following fact: if $Z(s)$ is PR, then $Z(s+\varepsilon)$ is SPR for $\varepsilon > 0$. In consequence:

$$Z(\sigma(\omega) + j\gamma(\omega)) + Z^T(\sigma(\omega) - j\gamma(\omega)) > 0.$$

Therefore, $Z(p(j\omega)) + Z^T(p(-j\omega)) > 0$ for all real ω and for each $p(s)$ ∈SPR0. Therefore, since $Z(p(s-\varepsilon_0))$ is PR for some $\varepsilon_0 > 0$, then $Z(p(s))$ is SPR for each $p(s)$ ∈SPR0. The constraint $Z(p(j\infty)) + Z^T(p(-j\infty)) > 0$ is just proved as in item 2 of this proof.

□

Please remark that by the definition of SSPR, if $Z(s)$ is either SPR or PR, then $Z(p(s))$ is SSPR for each $p(s) \in$ SPR0, too. In what follows we apply the results corresponding to Theorem 3.1 to the matrix function classes introduced in Definition 3.2.

Definition 3.4. *Let $Z(s)$ be a $\rho \times \rho$ proper rational transfer function matrix.*

1. *[6] $Z(s)$ is called bounded real (BR) if:*

   (a) *All elements of $Z(s)$ are analytic for $Re[s] \geq 0$, and:*

   (b) *I - $Z^T(\text{-}j\omega)Z(j\omega) \geq 0$ for all $\omega \in R$. Equivalently, the condition b) can be replaced by:*

   (c) $\|Z(s)\|_\infty \leq 1$.

2. *[6] $Z(s)$ is called strictly bounded real (SBR) if:*

   (a) *All elements of $Z(s)$ are analytic for $Re[s] \geq 0$, and:*

   (b) *I - $Z^T(\text{-}j\omega)Z(j\omega) > 0$ for all $\omega \in R$. Again, the condition b) can be replaced by:*

   (c) $\|Z(s)\|_\infty < 1$.

Remark 3.2. *If the transfer function matrix $Z(s)$ is SBR, then I - $D^TD > 0$, where $D := Z(\infty)$.*

The following corollary follows:

Corollary 3.1. *If $Z(s)$ is ESPR, SSPR, WSPR or MSPR $\rho \times \rho$ proper rational transfer function matrix, then $Z(p(s))$ is ESPR and SSPR $\rho \times \rho$ proper rational transfer function matrix, for each $p(s) \in$SPR0.*

Lemma 3.3. *[1], [11] Consider a $\rho \times \rho$ proper rational transfer function matrix $Z(s)$ be given. Then:*

1. *Suppose that $Z(s)$ satisfies $\det(Z(s)+I) \neq 0$ for $Re[s] \geq 0$, then $Z(s)$ is ESPR if and only if:*

$$H(s) = (I - Z(s))\,(I + Z(s))^{-1}$$

   is SBR. Also, $Z(s)$ is PR if and only if $H(s)$ is BR.

   Equivalently, item 1 can be replaced by:

2. *Consider a $\rho\times\rho$ proper rational transfer function matrix $H(s)$ satisfying $\det(I+H(s)) \neq 0$, for $Re[s] \geq 0$. Then, $H(s)$ is SBR if and only if:*

$$Z(s) = (I + H(s))^{-1}\,(I - H(s))$$

   is ESPR. Also, $H(s)$ is BR if and only if $Z(s)$ is PR.

Proposition 3.1. *Suppose that $Z(s)$ and $H(s)$ are $\rho \times \rho$ proper rational transfer function matrices, such that $\det(Z(s)+I) \neq 0$ and $\det(I+H(s)) \neq 0$ for $Re[s] \geq 0$, then:*

1. *If $Z(s)$ is PR, SPR, ESPR, SSPR, WSPR or MSPR $\rho \times \rho$ proper rational transfer function matrix, then $H(p(s))$ is SBR for each $p(s) \in$SPR0.*

2. *If $H(s)$ is either SBR or BR $\rho\times\rho$ proper rational transfer function matrix, then $Z(p(s))$ is ESPR for each $p(s) \in$SPR0.*

3. *If $H(s)$ is either SBR or BR $\rho\times\rho$ proper rational transfer function matrix, then $H(p(s))$ is SBR for each $p(s) \in$SPR0.*

4. *If $G(s) \in \mathbf{RH}^{\infty}$ is such that either $\|G(s)\|_{\infty} < \gamma$ or $\|G(s)\|_{\infty} \leq \gamma$, then $\|G(p(s))\|_{\infty} < \gamma$ for each $p(s) \in$SPR0.*

Proof. 1. By Theorem 3.1, and Corollary , if $Z(s)$ is PR, SPR, ESPR, SSPR, WSPR or MSPR, then $Z(p(s))$ is ESPR and SSPR for each $p(s) \in$SPR0. Now by Lemma 3.3 item 1, $Z(p(s))$ is ESPR if and only if the transfer function matrix:

$$H(p(s)) = (I - Z(p(s)))\,(I + Z(p(s)))^{-1}$$

is SBR for each $p(s) \in$SPR0.

2. By Lemma 3.3 item 2, if $H(s)$ is either SBR or BR, then $H(s)$ is SBR if and only if:

$$Z(s) = (I + H(s))^{-1}\,(I - H(s))$$

is ESPR or PR, respectively. Now, by Theorem 3.1 $Z(p(s))$ is ESPR for each $p(s) \in$SPR0.

3. This item is easily proved just applying item 2 of this theorem and as a consequence of Lemma 3.3 item 1.

4. From Definition 3.4 items 1 (c) and 2 (c), item 3 of this Proposition 3.1 can be written as follows:

   If $M(s) \in \mathbf{RH}^{\infty}$ is such that either $\|M(s)\|_{\infty} < 1$ or $\|M(s)\|_{\infty} \leq 1$, then $\|M(p(s))\|_{\infty} < 1$ for each $p(s) \in$SPR0. Now define $M(s) := \gamma^{-1}G(s)$. Remark that $\gamma > 0$ implies $\|\gamma^{-1}G(s)\|_{\infty} = \gamma^{-1}\|G(s)\|_{\infty}$, by item 3 of this Proposition 3.1, we obtain the result.

□

4 Results on absolute stability

Consider the unforced system described by:

$$\begin{cases} \dot{x} = Ax(t) + Bu(t), \\ y = C(t)\,x, \\ u(t) = -\varphi(t, y(t)), \end{cases} \tag{4.1}$$

where: $x \in R^n$, $u, y \in R^\rho$, (A, B) is controllable, (A, C) is observable, and $\varphi : [0, \infty) \times R^\rho \rightarrow R^\rho$ is a memoryless, possibly time-varying nonlinearity which is piecewise continuous in t and locally Lipschitz in y. A, B and C are real constant matrices The so-called sector condition for φ is given by:

$$[\varphi(t, y)\text{-}K_{\min}y]^T[\varphi(t, y)\text{-}K_{\max}y] \leq 0, \ \forall\ t \geq 0, \ \forall\ y \in \Gamma \subset R^\rho \tag{4.2}$$

For some real matrices $K_{\min}$ and $K_{\max}$, where $K = K_{\max}\text{-}K_{\min}$ is a positive definite symmetric matrix and the interior of Γ is connected and contains the origin. If $\Gamma = R^\rho$, then φ satisfies the sector condition globally, in which case it is said that φ belongs to a sector $[K_{\min}, K_{\max}]$. We define now the following systems:

$$\begin{cases} G_T(s) = G(s)[I + K_{\min}G(s)]^{\text{-}1}, \\ Z_T(s) = [I + K_{\max}G(s)][I + K_{\min}G(s)]^{\text{-}1}, \\ G(s) = C(sI\text{-}A)^{\text{-}1}B. \end{cases}$$

Definition 4.1. *Consider the system (4.1), where φ satisfies a sector condition (4.2). The system is absolutely stable if the origin is globally uniformly asymptotically stable for any nonlinearly in the given sector.*

We can now present the main result of this section:

Proposition 4.1. *Consider the system (4.1), where (A, B) is controllable, (A, C) is observable, and $\varphi(t, y)$ satisfies the sector condition (4.2) globally. If $G_T(s) \in \mathbf{RH}^\infty$ and $Z_T(s)$ is a PR function matrix (i.e., the system is uniformly asymptotically stable). Then:*

$$\begin{cases} \dot{x}\,(t) = A_p x\,(t) + B_p u\,(t)\,, \\ y\,(t) = C_p x\,(t) + D_p u\,(t)\,, \\ u\,(t) = \text{-}\varphi(t, y\,(t)), \end{cases} \tag{4.3}$$

is absolutely stable for each $p(s)$ ∈SPR0, where (A_p, B_p, C_p, D_p) is a minimal realization of the following $\rho \times \rho$ proper rational transfer function matrix: $G(p(s)) = C_p(sI\text{-}A_p)^{\text{-}1}B_p + D_p$.

Proof. First note that $p(s)$ is a zero relative degree rational transfer function, then in general for realization of $G(p(s))$, $D_p \neq 0_{\rho\times\rho}$. The multivariable circle criterion (Theorem 10.1 in [7]) establishes that under the conditions of this theorem, absolutely stability of the system (4.1) is obtained, if $G_T(s) \in \mathbf{RH}^\infty$ and $Z_T(s)$ is SPR. Observe that:

$$\begin{aligned} G_T(p(s)) &= G(p(s))[I + K_{\min}G(p(s))]^{-1} \\ Z_T(p(s)) &= [I + K_{\max}G(p(s))][I + K_{\min}G(p(s))]^{-1}. \end{aligned}$$

Now, by Theorem 3.1 item 1: if $G_T(s) \in \mathbf{RH}^\infty$, then $G_T(p(s)) \in \mathbf{RH}^\infty$ for each $p(s)$ ∈SPR0. Moreover, by Theorem 3.1 item 3: if $Z_T(s)$ is PR, then $Z_T(p(s))$ is SPR for each $p(s)$ ∈SPR0. Therefore, by the multivariable circle criterion, the system (4.3) is absolutely stable. □

Remark 4.1. *This result is a generalization of the multivariable circle criterion. Under the same hypothesis of the multivariable circle criterion, absolutely stability of the system (4.3) is obtained for each* $p(s) \in$*SPR0.*

Corollary 4.1. *Consider the sistem (4.1) where* A *is Hurwitz,* (A, B) *is controllable,* (A, C) *is observable, and* $\varphi(y)$ *is a time-invariant nonlinearly that satisfies the sector condition (4.2) globally with* $K_{\min} = 0$ *and a positive definite symmetric* $K_{\max}$*. Suppose that* $K_{\max}\varphi(y)$ *is the gradient of a scalar function and* $\int_0^y \varphi^T(\tau)K_{\max}d\tau \geq 0$, $\forall\, y \in \Gamma \subset R^\rho$ *, or is satisfied globally. Then, the system:*

$$\begin{cases} \dot{x}(t) = \overline{A}_p x(t) + \overline{B}_p u(t), \\ y(t) = \overline{C}_p x(t) + \overline{D}_p u(t), \\ u(t) = -\varphi(y(t)) \end{cases} \tag{4.4}$$

is absolutely stable, where $(\overline{A}_p, \overline{B}_p, \overline{C}_p, \overline{D}_p)$ *is a minimal realization of the* $\rho \times \rho$ *proper rational transfer function matrix* $Z(p(s)) = (1 + \eta p(s))G(p(s))$*, if there exists* $\eta \geq 0$ *(with* $-\frac{1}{\eta}$ *not an eigenvalue of* $\overline{A}_p$ *for* $p(s) \in$*SPR0) such that* $Z_1(s) = I + (1 + \eta s)G(s)$ *is PR.*

Proof. By Theorem 3.1 item 3: $Z_1(p(s))$ is SPR if $Z_1(s)$ is PR for each $p(s) \in$SPR0 such that $\eta \geq 0$, where $-\frac{1}{\eta}$ is not an eigenvalue of $\overline{A}_p$. Now, since $I + K_{\max}(1 + \eta p(s))G(p(s))$ is SPR, using the Lemma 10.3 in [7] for the system $Z(p(s))$, absolute stability of the system (4.4) is obtained for each $p(s) \in$ SPR0 such that $\eta \geq 0$, where $-\frac{1}{\eta}$ is not an eigenvalue of $\overline{A}_p$. □

Remark 4.2. *The multivariable Popov criterion (Theorem 10.3 in [7]), establishes that under the conditions of this corollary, absolute stability of the system (4.1) with* $u = -\varphi(y)$ *is obtained if* $Z_1(s) = I + (1 + \eta s)K_{\max}G(s)$ *is SPR. This corollary is a partial extension of the multivariable Popov criterion and the Lemma 10.3 in [7].*

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