Stability Property of Solutions of Large-Scale Discrete-Time Systems

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Abstract

The aim of this paper is to present one approach to solution of stability problem for discrete-time system based on hierarchical Lyapunov function. The example showing the proposed approach efficiency are given.

1 Hierarchical decomposition of discrete-time system and stability conditions.

We consider a discrete-time system

$$S: \quad x(\tau+1) = f(\tau, x(\tau)), \tag{1.1}$$

where $\tau \in \mathcal{T}_\tau = \{t_0 + k, k = 0, 1, 2, \dots\}$, $t_0 \in R$, $x \in R^n$, $f : \mathcal{T}_\tau \times R^n \to R^n$, the function f is a continuous function on x such that the solution $x(\tau; \tau_0, x_0)$ of system (1.1) exists and is unique for all $\tau \in \mathcal{T}_\tau$ when any $(\tau_0, x_0) \in \mathcal{T}_\tau \times R^n$. Furthermore, assume that $f(\tau, x) = 0$ for all $\tau \in \mathcal{T}_\tau$ if and only if $x = 0$ and the state $x = 0$ is a unique state of equilibrium of system (1.1).

System (1.1) is decomposed into s interconnected subsystems [1]

$$\widetilde{S}_i: \quad x_i(\tau+1) = g_i(\tau, x_i(\tau)) + h_i(\tau, x(\tau)), \;\; i = 1, 2, \dots, s, \tag{1.2}$$

where $x_i \in R^{n_i}$, $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}}, \dots, x_s^{\mathrm{T}})^{\mathrm{T}}$, $R^n = R^{n_1} \times R^{n_2} \times \dots \times R^{n_s}$, $g_i : \mathcal{T}_\tau \times R^{n_i} \to R^{n_i}$, $h_i : \mathcal{T}_\tau \times R^n \to R^{n_i}$.

The equations

$$S_i: \quad x_i(\tau+1) = g_i(\tau, x_i(\tau)), \;\; i = 1, 2, \dots, s, \tag{1.3}$$

describe the dynamics of independent subsystems of system (1.2), the functions h_i represent the interactions of S_i with the rest of the systems (1.1). Equations (1.3) are derived from the equations (1.2) when the connections h_i are equal to zero. Assume that $g_i(\tau, 0) = 0$ for all $\tau \in \mathcal{T}_\tau$ and the states $x_i = 0$ are the unique equilibrium states of subsystems (1.3).

Further each of subsystem (1.3) is decomposed into m_i interconnected components

$$\widetilde{C}_{ij}: \quad x_{ij}(\tau+1) = p_{ij}(\tau, x_{ij}(\tau)) + q_{ij}(\tau, x_i(\tau)), \;\; i = 1, 2, \dots, s, \;\; j = 1, 2, \dots, m_i, \tag{1.4}$$

where $x_{ij} \in R^{n_{ij}}$, $x_i = (x_{i1}^{\mathrm{T}}, x_{i_2}^{\mathrm{T}}, \dots, x_{im_i}^{\mathrm{T}})^{\mathrm{T}}$, $R^{n_i} = R^{n_{i1}} \times R^{n_{i2}} \times \dots \times R^{n_{im_i}}$, $p_{ij} : \mathcal{T}_\tau \times R^{n_{ij}} \to R^{n_{ij}}$, $q_{ij} : \mathcal{T}_\tau \times R^{n_i} \to R^{n_{ij}}$.

The equations

$$C_{ij} : \quad x_{ij}(\tau + 1) = p_{ij}(\tau, x_{ij}(\tau)) \tag{1.5}$$

describe the dynamics of independent components of subsystems (1.3). Equations (1.5) are derived from the equations (1.4) when the connections q_{ij} are equal to zero. Assume that $p_{ij}(\tau, 0) = 0$ for all $\tau \in \mathcal{T}_\tau$ and the states $x_{ij} = 0$ are the unique equilibrium states of the components (1.5).

To study stability of system (1.1) we use two-level construction of the Lyapunov functions [2]. Assume that for components (1.5) there exist the Lyapunov functions $v_{ij}(\tau, x_{ij})$ which establish the asymptotic stability of the equilibrium states $x_{ij} = 0$ of the components (1.5). For subsystems (1.3) we construct auxiliary functions

$$v_i(\tau, x_i) = \sum_{j=1}^{m_i} d_{ij} v_{ij}(\tau, x_{ij}), \tag{1.6}$$

where d_{ij} are positive constants. Similarly for the whole system (1.1) the function

$$V(\tau, x) = \sum_{i=1}^{s} d_i v_i(\tau, x_i) \tag{1.7}$$

is constructed, where d_i are positive constants. Under certain assumptions the function $V(\tau, x)$ constructed by formulas (1.6)–(1.7) is the vector hierarchical Lyapunov function for the system (1.1).

The first difference $\Delta V(\tau, x(\tau))\big|_S$ of the function $V(\tau, x)$ along solutions $x(\tau; \tau_0, x_0)$ of system (1.1) is defined by formula

$$\Delta V(\tau, x(\tau))\big|_S = V(\tau + 1, f(\tau, x(\tau))) - V(\tau, x(\tau)).$$

Below we will need such definitions.

Definition 1.1. [1] *An $n \times n$ matrix $W = (w_{ij})$ is said to be M-matrix, if it has nonpositive off-diagonal elements and all leading principal minors of W are positive, that is,*

$$\begin{vmatrix} w_{11} & w12 & \dots & w_{1k} \\ w_{21} & w22 & \dots & w_{2k} \\ \dots & \dots & \dots & \dots \\ w_{k1} & wk2 & \dots & w_{kk} \end{vmatrix} > 0, \qquad k = 1, 2, \dots, n.$$

Definition 1.2. [3] *A function ψ, $\psi : R_+ \to R_+$, belongs to*

(i) the class K, if and only if it is continuous and strictly increasing and $\psi(0) = 0$;

(ii) the class KR, if and only if $\psi \in K$ and $\lim_{r\to\infty} \psi(r) = +\infty$.

To formulate sufficient stability conditions we require some assumptions.

Assumption 1.1. *There exist:*

(1) open connected neighborhoods $\mathcal{N}_{ij} \subset R^{n_{ij}}$ *of the states* $x_{ij} = 0$ *of components* C_{ij}, $i = 1, 2, \dots, s$, $j = 1, 2, \dots m_i$;

(2) functions $\varphi_{ij}, \phi_{ij}, \psi_{ij} \in K$, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, m_i$;

(3) the functions $v_{ij} : T \times R^{n_{ij}} \to R_+$ *which are continuous functions on* x_{ij} *and satisfy the inequalities:*

(a) $\alpha_{ij}\varphi_{ij}(\|x_{ij}\|) \leqslant v_{ij}(\tau, x_{ij}) \leqslant \beta_{ij}\phi_{ij}(\|x_{ij}\|), \quad \forall\, (\tau, x_{ij}) \in \mathcal{T}_\tau \times \mathcal{N}_{ij}$,

(b) $\Delta v_{ij}(\tau, x_{ij})\big|_{C_{ij}} \leqslant -\pi_{ij}\psi_{ij}(\|x_{ij}\|), \quad \forall (\tau, x_{ij}) \in \mathcal{T}_\tau \times \mathcal{N}_{ij}$,

(c) $\Delta v_{ij}(\tau, x_{ij}(\tau))\big|_{\widetilde{C}_{ij}} - \Delta v_{ij}(\tau, x_{ij}(\tau))\big|_{C_{ij}} \leqslant \sum_{k=1}^{m_i} \xi^i_{jk}\psi_{ik}(\|x_{ik}\|)$, $\forall\, (\tau, x_{ij}) \in \mathcal{T}_\tau \times \mathcal{N}_{ij}$,

where $\alpha_{ij} > 0$, $\beta_{ij} > 0$, $\pi_{ij} > 0$, $\xi^i_{ij} \geqslant 0$ *are real constants,* $\|x\|$ *is the norm of vector* x, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, m_i$.

Assumption 1.2. *Assume that:*

(1) there exist open connected neighborhoods $\mathcal{N}_i \subset R^{n_i}$ *of the equilibrium states* $x_i = 0$ *of subsystems (1.3),* $i = 1, 2, \dots, s$;

(2) there exist the functions $\psi_i \in K$, $i = 1, 2, \dots, s$;

(3) the functions $v_i : \mathcal{T}_\tau \times R^{n_{ij}} \to R_+$, *constructed by formula (1.6) satisfy the inequalities:*

(a) $\Delta v_i(\tau, x_i)\big|_{S_i} \leqslant -\pi_i\psi_i(\|x_i\|), \quad \forall (\tau, x_i) \in \mathcal{T}_\tau \times \mathcal{N}_i$,

(b) $\Delta v_i(\tau, x_i(\tau))\big|_{\tilde{S}_i} - \Delta v_i(\tau, x_i(\tau))\big|_{S_i} \leqslant \sum_{j=1}^{s} \xi_{ij}\psi_j(\|x_j\|), \quad \forall\, (\tau, x_i) \in \mathcal{T}_\tau \times \mathcal{N}_i$,

where $\pi_i > 0$, $\xi_{ij} \geqslant 0$ *are real constants,* $i = 1, 2, \dots, s$.

We define the matrices $W_i = (w^i_{jk})$ with the elements

$$w^i_{jk} = \begin{cases} \pi_{ij} - \xi^i_{jj}, & \text{if} \quad j = k, \\ -\xi^i_{jk}, & \text{if} \quad j \neq k \end{cases}$$

and the matrix $W = (w_{jk})$ with elements

$$w_{jk} = \begin{cases} \pi_j - \xi_{jj}, & \text{if} \quad j = k, \\ -\xi_{jk}, & \text{if} \quad j \neq k. \end{cases}$$

Sufficient stability conditions for system (1.1) is founded in the following result.

Theorem 1.1. *Assume that the perturbed motion equation (1.1) admit the decomposition (1.2) – (1.5) and conditions of Assumptions 1.1 and 1.2 are satisfied. Then, if the matrices* $W_1,\ W_2,\dots,W_s$ *and* W *are* M*-matrices, the equilibrium state* $x = 0$ *of system (1.1) is asymptotically stable.*

If all conditions of Assumptions 1.1 and 1.2 are satisfied for $\mathcal{N}_{ij} = R^{n_{ij}}$, $\mathcal{N}_i = R^{n_i}$ *and the functions* $\varphi_{ij} \in KR$*, then asymptotic stability in the whole takes place.*

2 Hierarchical connective stability of a large-scale discrete-time system.

Assume that for system (1.1) the decomposition (1.2) – (1.5) takes place. It is known [1] that the interconnection functions $h_i(\tau, x)$ can be represented

$$h_i(\tau, x) = h_i(\tau, \bar{e}_{i1}x_1, \bar{e}_{i2}x_2, \dots, \bar{e}_{is}x_s), \quad i = 1, 2, \dots, s,$$

where the matrix $\overline{E} = (\bar{e}_{ij})$ is the fundamental matrix of connections of system (1.2) with the elements

$$\bar{e}_{ij} = \begin{cases} 1, & \text{if} \quad x_j \quad \text{is contained in} \quad h_i(\tau, x), \\ 0, & \text{if} \quad x_j \quad \text{is not contained in} \quad h_i(\tau, x). \end{cases}$$

Let the functions of the discrete argument $e_{ij} : \mathcal{T}_\tau \to [0, 1]$ for all $\tau \in \mathcal{T}_\tau$ satisfy the inequalities

$$e_{ij}(\tau) \leqslant \bar{e}_{ij}.$$

The constant $\bar{e}_{ij}$ determine the degree of connection between the independent subsystems (1.3). The matrix $E(\tau) = (e_{ij}(\tau))$ describes the structural perturbations of system (1.1).

If $E(\tau) \equiv 0$, then the system (1.1) is decomposed into s independent subsystems (1.3), each of which is a composition of the interconnected components (1.4). The connection functions between the independent components (1.5) can be written as

$$q_{ij}(\tau, x_i) = q_{ij}(\tau, \bar{\ell}^i_{j1}x_{i1}, \bar{\ell}^i_{j2}x_{i2}, \dots, \bar{\ell}^i_{jm_i}x_{im_i}),$$
$$i = 1, 2, \dots, s, \quad j = 1, 2, \dots, m_i,$$

where

$$\bar{\ell}^i_{jk} = \begin{cases} 1, & x_{ik} \quad \text{is contained in} \quad q_{ij}(\tau, x_i), \\ 0, & x_{ik} \quad \text{is not contained in} \quad q_{ij}(\tau, x_i). \end{cases}$$

Let $\ell^i_{jk} : \mathcal{T}_\tau \to [0, 1]$ and for all $\tau \in \mathcal{T}_\tau$

$$e^i_{jk}(\tau) \leqslant \bar{\ell}^i_{jk}, \quad i = 1, 2, \dots, s, \quad j,\, k = 1, 2, \dots, m_i.$$

The matrices $\overline{L}_i = (\bar{\ell}^i_{jk})$ are the fundamental matrices of the connections for the subsystems (1.3) and describe the initial connections between the independent components (1.5). The matrices $L_i(\tau) = (\ell^i_{jk}(\tau))$ describe structural perturbations of subsystems (1.3).

The notion of hierarchical connective stability of the discrete-time system (1.1) is defined similarly to the continuous case [2].

Definition 2.1. *Discrete-time system (1.1) is called hierarchically connective stable, if:*

(i) for $E(\tau) \equiv 0$ the equilibrium states $x_i = 0$ of subsystems (1.3) are asymptotically stable in the whole for any structural matrices $L_i(\tau)$, $i = 1, 2, \dots, s$;

(ii) for $L_i(\tau) \equiv \overline{L}_i$ the equilibrium states $x = 0$ of system (1.1) is asymptotically stable in the whole for any structural matrix $E(\tau)$.

In order that to formulate sufficient conditions for the hierarchical connective stability of the system (1.1) we introduce some assumptions.

Assumption 2.1. *Assume that:*

(1) conditions (1) – (3)(b) of Assumption 1.1 are satisfied for $\mathcal{N}_{ij} = R^{n_{ij}}$ and functions φ_{ij} are of class KR, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, m_i$;

(2) the first differences of functions v_{ij} satisfy the inequalities

$$\Delta v_{ij}(\tau, x_{ij}(\tau))\big|_{\widetilde{C}_{ij}} - \Delta v_{ij}(\tau, x_{ij}(\tau))\big|_{C_{ij}} \leqslant \sum_{k=1}^{m_i} \ell^i_{jk}(\tau)\xi^i_{jk}\psi_{ik}(\|x_{ik}\|)$$

for all $(\tau, x_{ij}) \in \mathcal{T}_\tau \times R^{n_{ij}}$, where $\xi^i_{jk} \geqslant 0$ are real constants, $i = 1, 2, \dots, s$, $j = 1, 2, \dots, m_i$.

Assumption 2.2. *Assume that:*

(1) conditions (1) – (3)(a) of Assumption 1.2 are satisfied for $\mathcal{N}_i = R^{n_i}$, $i = 1, 2, \dots, s$;

(2) the first differences of functions v_i satisfy the inequalities

$$\Delta v_i(\tau, x_i(\tau))\big|_{\widetilde{S}_{ij}} - \Delta v_i(\tau, x_i(\tau))\big|_{S_i} \leqslant \sum_{j=1}^{s} e_{ij}(\tau)\xi_{ij}\psi_j(\|x_j\|)$$

for all $(\tau, x_i) \in \mathcal{T}_\tau \times R^{n_i}$, where $\xi_{ij} \geqslant 0$ are real constants, $i = 1, 2, \dots, s$;

In this case the elements of matrices $W_i(\tau) = (w^i_{jk}(\tau))$ and $W(\tau) = (w_{ij}(\tau))$ depend on discrete time,

$$w^i_{jk}(\tau) = \begin{cases} \pi_{ij} - \ell^i_{jj}(\tau)\,\xi^i_{jj}, & \text{if} \quad j = k, \\ -\ell^i_{jk}(\tau)\,\xi^i_{jk}, & \text{if} \quad j \neq k, \end{cases}$$

$$w_{jk}(\tau) = \begin{cases} \pi_j - e_{jj}(\tau)\,\xi_{jj}, & \text{if} \quad j = k, \\ -e_{ij}(\tau)\,\xi_{jk}, & \text{if} \quad j \neq k. \end{cases}$$

Now we designate by $\overline{W}_1, \overline{W}_2, \dots, \overline{W}_s$ and $\overline{W}$ the matrices corresponding to the fundamental matrices of connections $\overline{L}_1, \overline{L}_2, \dots, \overline{L}$ and $\overline{E}$. We formulate the following test for connective stability of system (1.1).

Theorem 2.1. *Assume that the perturbed motion equation (1.1) admit decomposition (1.2)–(1.5) and all conditions of Assumptions 2.1 and 2.2 are satisfied. Then, if the matrices* $\overline{W}_1, \overline{W}_2, \dots, \overline{W}_s$ *and* $\overline{W}$ *are M-matrices, then the equilibrium state* $x = 0$ *of system (1.1) is hierarchically connective stable.*

3 Example

Consider the system

$$S: \quad x(\tau+1) = \begin{pmatrix} 0.99 & 0.001 & 0 \\ 0.002 & 0.5 & 1 \\ 0.2 & 0.2 & 0.56 \end{pmatrix} x(\tau), \tag{3.1}$$

where $\tau \in \mathcal{T}_\tau,\ x \in R^3$. Decompose system (3.1) and arrive at two independent subsystems

$$\begin{aligned} S_1: &\quad x_1(\tau+1) = \begin{pmatrix} 0.99 & 0.001 \\ 0.002 & 0.5 \end{pmatrix} x_1(\tau), \\ S_2: &\quad x_2(\tau+1) = 0.56\, x_2(\tau), \end{aligned}$$

where $x_1 \in R^2,\ x_2 \in R$. We decompose the subsystem S_1 and distinguish two independent components

$$\begin{aligned} C_{11}: &\quad x_{11}(\tau+1) = 0.99\, x_{11}(\tau), \\ C_{12}: &\quad x_{12}(\tau+1) = 0.5\, x_{12}(\tau), \end{aligned}$$

where $x_{11}, x_{12} \in R$. Choosing functions $v_{11} = |x_{11}|,\ v_{12} = |x_{12}|,\ \psi_{11} = |x_{11}|,\ \psi_{12} = |x_{12}|$, we compute the constants $\pi_{11} = 0.01,\ \pi_{12} = 0.5,\ \xi^1_{11} = 0,\ \xi^1_{12} = 0.001,\ \xi^1_{21} = 0.002,\ \xi^1_{22} = 0$ and the matrix

$$W_1 = \begin{pmatrix} 0.01 & -0.001 \\ -0.002 & 0.5 \end{pmatrix},$$

which is the M-matrix, because $\Delta_1 = 0.01 > 0$ and $\Delta_2 = 0.004998 > 0$. We take $d_{11} = 45$ and $d_{12} = 1$. Then

$$v_1(x_1) = 45\,|x_{11}| + |x_{12}|.$$

Choosing functions $v_2(x_2) = |x_2|,\ \psi_1 = |x_{11}| + |x_{12}|,\ \psi_2 = |x_2|$, we calculate the constants $\pi_1 = 0.455,\ \pi_2 = 0.44,\ \xi_{11} = 0,\ \xi_{12} = 1,\ \xi_{21} = 0.2,\ \xi_{22} = 0$. The matrix

$$W = \begin{pmatrix} 0.455 & -1 \\ -0.2 & 0.44 \end{pmatrix}$$

is the M-matrix, because $\Delta_1 = 0.455 > 0$ and $\Delta_2 = 0.0002 > 0$. We take $d_1 = 128$ and $d_2 = 291$. The function

$$V(x) = 128(45|x_{11}| + |x_{12}|) + 291|x_2|)$$

is the hierarchical Lyapunov function establishing asymptotic stability of system (3.1).

Now we investigate system (3.1) by means of one-level construction of function $V(x)$ [4]. Decompose system (3.1) and distinguish three independent subsystems

$$\begin{aligned} S_1: &\quad x_1(\tau+1) = 0.99\,x_1(\tau),\\ S_2: &\quad x_2(\tau+1) = 0.5\,x_2(\tau),\\ S_3: &\quad x_3(\tau+1) = 0.56\,x_3(\tau). \end{aligned}$$

We choose the functions $v_i = |x_i|,\ \psi_i = |x_i|,\ i = 1, 2, 3$, and obtain the matrix

$$\widetilde{W} = \begin{pmatrix} 0.01 & -0.001 & 0 \\ -0.002 & 0.5 & -1 \\ -0.2 & -0.2 & 0.44 \end{pmatrix},$$

Which is not the M-matrix, because $\Delta_1 = 0.01 > 0,\ \Delta_2 = 0.04998 > 0,\ \Delta_3 = -0.00000088 < 0$.

Using matrix $\widetilde{W}$ we cannot reach a conclusion on stability of system (3.1); however matrices W_1 and W allow the conclusion that system (3.1) is asymptotically stable.

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