Hybrid Stock Models and Parameter Estimation

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Abstract

In this work, we study a class of hybrid models for the stock market to account for the coexistence of continuous dynamics and discrete events. Different from the original geometric Brownian motion models, both the rate of return and the volatility in the hybrid model depend on a continuous-time Markov chain. This model can deal with random volatility by incorporating market trend with other economic factors. To use the models requires being able to estimate the values of elements of the generator of the underlying Markov chain. We develop a stochastic approximation-based algorithm for the estimation task. The asymptotic properties including convergence and rates of convergence of the algorithm are proved. Using the estimated generator, one can then proceed to make equity liquidation decisions.

1 Introduction

To reflect the coexistence of continuous dynamics and discrete events in a stock market, we model it as a hybrid system. While the celebrated Black-Scholes model, based on geometric Brownian motion (GBM), has been widely used in the analysis of options pricing and portfolio management (see [5, 9]), it has been recognized that there are needs for suitable models to better capture the price movements of the underlying securities. One of the limitations of the GBM model is that the appreciation (or return) rate and the volatility in the model are both deterministic. Therefore they are not responsive to the random environment and are not suitable for a longer horizon. It is desirable to modify the model so as to capture the random parameter changes such as random volatility. A host of researchers have made effort in this direction, see [3, 4, 10] and the references therein.

Built upon the hybrid switching GBM model (HGBM) (a number of GBMs modulated by a finite-state Markov chain) considered in [15] (see also related work [2]), we further our understanding in this paper. In order to use the HGBM, a crucial issue is to be able to estimate the generator of the underlying Markov chain. This brings us to the current work. We propose and develop a class of optimization algorithms to carry out the estimation task. The algorithm is of constrained stochastic approximation type. Some of the recent development on stochastic approximation can be found in [7]; see also [1, 8, 11, 13] and the references therein.

The rest of the paper is organized as follows. Section 2 is devoted to the hybrid model description. Section 3 presents a stochastic optimization algorithm for the estimation task.

Section 4 studies the convergence and rate of convergence of the algorithm. Finally, some remarks are made in Section 5. Due to the page limitation, detailed proofs and numerical results are referred to [14].

2 Formulation

For simplicity, consider a market model of a single stock. Let $[0, T]$ for some $T > 0$ be a finite-time horizon, and $S(t)$ be the price of the stock. Assume that the Markov chain is time homogeneous, and has a finite state space $\mathcal{M} = \{1, \dots, m\}$ and a generator $Q = (q^{ij})$, where $q^{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m} q^{ij} = 0$ for each $i \in \mathcal{M}$. Let $\mu(\cdot)$ and $\sigma(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ be appropriate functions representing the appreciation rate and volatility, respectively. The HGBM model is given by

$$dS(t) = \mu(\alpha(t))S(t)dt + \sigma(\alpha(t))S(t)dw(t), \tag{2.1}$$

where $w(\cdot)$ is a standard Brownian motion independent of the Markov chain $\alpha(\cdot)$. Let $S(0) = S_0$ be the initial price.

Define

$$X(t) = \int_0^t r(\alpha(s))ds + \int_0^t \sigma(\alpha(s))dw(s), \tag{2.2}$$

where

$$r(\alpha(s)) = \mu(\alpha(s)) - \frac{\sigma^2(\alpha(s))}{2}. \tag{2.3}$$

Or write it in a differential form

$$dX(t) = r(\alpha(t))dt + \sigma(\alpha(t))dw(t), \ X(0) = 0. \tag{2.4}$$

Using $X(t)$ defined in (2.2), the solution of the price $S(t)$ can be rewritten as

$$S(t) = S_0 \exp(X(t)), \quad \text{or equivalently,} \quad X(t) = \log\left(\frac{S(t)}{S_0}\right). \tag{2.5}$$

Note that the Markov chain is used to model the market trends as well as other economic factors as was mentioned in the previous section.

3 Algorithms

Estimating Q, the generator of the Markov chain, is a parameter estimation problem. Using the well-known vector operations and piling up the elements of the generator matrix into a vector, we propose an algorithm to estimate the r-vector

$$q = (q^{11}, \dots, q^{m1}, q^{12}, \dots, q^{m2}, \dots, q^{1m}, \dots, q^{mm})' \in \mathbb{R}^r,$$

with $r = m^2$.

The estimation that we propose is a constrained optimization procedure using stochastic approximation methods. It can be formulated as:

$$\begin{aligned} &\text{minimize } \ \overline{f}(q) = Ef(q,\xi), \quad \text{where } \ f(q,\xi) = [\widetilde{X}(q,\xi) - \widetilde{X}(q^*,\xi)]^2 \\ &\text{subject to } \ q^{ii} \geq 0, \ \sum_{j=1}^{m} q^{ij} = 0, \quad \text{for each } \ i = 1,\ldots,m. \end{aligned} \tag{3.1}$$

Using e_i to denote the standard unit vector with the ith component being 1 and all other components being 0, let $\delta_n > 0$ be the finite difference intervals. We use the finite difference approximation given by

$$G_{n,i} = -\frac{f(q_n + \delta_n e_i, \xi_n^+) - f(q_n - \delta_n e_i, \xi_n^-)}{2\delta_n}, \ i = 1,\ldots,r, \tag{3.2}$$

and write $G_n = (G_{n,1},\ldots,G_{n,r})'$, where $\{\xi_n^{\pm}\}$ are two sequences of the observation noise.

As was pointed out in [7, p. 87], an important issue in application of stochastic approximation concerns if the iterates become too large. To make sure that the iterates remain in a bounded region, we require that the iterates be confined to a bounded domain. To take into consideration of both the constraints given in (3.1) and the requirement of the iterates being in a bounded region, we propose a projection procedure

$$q_{n+1} = \pi_H[q_n + \varepsilon_n G_n], \tag{3.3}$$

where π_H denotes the projection operator onto the constraint set H that includes the constraints in (3.1) as well as the boundedness of the iterates. The algorithm (3.3) can be rewritten as

$$q_{n+1} = q_n + \varepsilon_n G_n + \varepsilon_n Z_n, \tag{3.4}$$

where Z_n is the vector having the shortest Euclidean length necessary to bring $q_n + \varepsilon_n G_n$ back to H if it escapes from H. Note that $\{\varepsilon_n\}$ is the step size sequence satisfying $\varepsilon_n \to 0$, $\varepsilon_n/\delta_n \to 0$, $\sum_n \varepsilon_n = \infty$. For example, one may use $\varepsilon_n = K/n^\gamma$ and $\delta_n = K_0/n^{\gamma/6}$, for some $K > 0$, $K_0 > 0$, and $0 < \gamma < 1$ are some positive constants.

4 Asymptotic Properties

To proceed, we state the conditions needed in the convergence of the algorithm.

(A1) The observed or simulated solution $\widetilde{X}(q,\xi)$ is twice continuously differentiable with respect to the parameter q.

(A2) $f(q,\widetilde{\xi}) = f_0(q,\xi) + \widehat{\xi}$ such that $\{\xi_n^{\pm}\}$ are sequences of bounded and stationary ϕ-mixing processes with mixing measure $\phi(k)$ satisfying $Ef_0(q,\xi_n^{\pm}) = \overline{f}(q)$ for each q and $\sum_k \phi^{1/2}(k) < \infty$; $\{\widehat{\xi}_n^{\pm}\}$ are stationary martingale difference sequences satisfying $E|\widehat{\xi}_n^{\pm}|^2 < \infty$.

Define

$$t_0 = 0, \quad \text{and} \quad t_n = \sum_{i=0}^{n-1} \varepsilon_i.$$

For $t \geq 0$, let $m(t)$ be the unique value of n such that $t_n \leq t < t_{n+1}$; for $t < 0$, let $m(t) = 0$. Define the continuous-time interpolation $q^0(\cdot)$ of the iterates as

$$q^0(t) = q_n, \quad t_n \leq t < t_{n+1}, \; q^n(t) = q^0(t_n + t), \; t \in (-\infty, \infty). \tag{4.1}$$

Let $Z_n = 0$ for $n < 0$ and define

$$\begin{aligned} Z^0(t) &= \sum_{k=0}^{m(t)-1} \varepsilon_k Z_k, \; t \geq 0, \\ Z^n(t) &= Z^0(t_n + t) - Z^0(t_n), \; t \geq 0, \\ Z^n(t) &= \sum_{k=m(t_n+t)}^{n-1} \varepsilon_k Z_k, \; t < 0. \end{aligned}$$

It then follows that

$$\begin{aligned} q^n(t) &= q_n + \sum_{k=n}^{m(t_n+t)-1} \varepsilon_k \left[G_k + Z_k \right] \\ &= q_n + G^n(t) + Z^n(t), \; t \geq 0, \\ q^n(t) &= q_n - \sum_{k=m(t_n+t)}^{n-1} \varepsilon_k \left[G_k + Z_k \right] \\ &= q_n + G^n(t) + Z^n(t), \; t < 0. \end{aligned}$$

Theorem 4.1 *Assume* (A1) *and* (A2) *are satisfied. Then there is a null set N such that for all $\omega \notin N$, the $\{q^n(\cdot), Z^n(\cdot)\}$ is equicontinuous in the extended sense. Let $(q(\cdot), Z(\cdot))$ denote the limit of a convergent subsequence. Then it satisfies the projected ordinary differential equation*

$$\dot{q} = \overline{f}_q(q) + z, \quad z \in C(q), \tag{4.2}$$

where $z(\cdot)$ is the projection or the constraint term that is the minimum force needed to keep $q(\cdot)$ in H. If q^* is an asymptotically stable point of (4.2) *and q_n is in some compact set in the domain of attraction of q^* w.p.1, then $q_n \to q^*$ w.p.1.*

To study the rate of convergence, take $\varepsilon_n = O(1/n^\gamma)$, $\delta_n = \delta/n^{\gamma_1}$ with $0 < \gamma_1 < \gamma \leq 1$. Define $u_n = n^{\gamma_2}(q_n - q^*)$ for $\gamma_2 > 0$. The rate of convergence of the stochastic approximation algorithm is concerned with the choice of γ_2 that leads to a nontrivial limit in the sense of distribution of u_n. The scaling factor together with the limit covariance gives the desired rate of convergence. Following the approach in KW type of algorithm, choose $\gamma_2 = 2\gamma_1$ and $\gamma_2 + \gamma_1 - \gamma/2 = 0$. This yields that $\gamma_1 = \gamma/6$. In fact, other choices of (γ_1, γ) lead to slower rate of convergence. Let $u^n(\cdot)$ be the right continuous piecewise constant interpolation of $\{u_k : k \geq n\}$ (more detail to follow). We shall derive the limit of $u^n(\cdot)$. We need another condition for the rate of convergence study.

(A3) $q_n \to q^* \in H^0$, interior of H such that q^* is a globally asymptotically stable point of the ODE (4.2) w.p.1. The set $\{n^{\gamma/3}(q_n - q^*)\}$ is tight.

Theorem 4.2. *Consider $u_n = n^{\gamma/3}(q_n - q^*)$ and $u^n(\cdot)$ is the continuous-time interpolation of q_n for some $0 < \gamma \le 1$. Suppose that* (A1)–(A3) *are satisfied and $u^n(0) \to u_0$.*

(i) *If $\gamma = 1$ and all eigenvalues of $(I/3) - f_{qq}(q^*)$ have negative real parts, then $u^n(\cdot)$ converges weakly to $u(\cdot)$, which is a solution of the stochastic differential equation*

$$du(t) = \left[\left(\frac{I}{3} - f_{qq}(q^*)\right) u(t) - \delta^2 B(q^*)\right] dt + \frac{1}{2\delta} dw,\ u(0) = u_0, \tag{4.3}$$

where $w(\cdot)$ is the Brownian motion obtained from the weak limit of the rescaled noise process, and $B(q^*)$ is the bias term.

(ii) *If $0 < \gamma < 1$ and all eigenvalues of $-f_{qq}(q^*)$ have negative real parts, then $u^n(\cdot)$ converges weakly to $u(\cdot)$ with* (4.3) *replaced by*

$$du(t) = \left[(-f_{qq}(q^*))\, u(t) - \delta^2 B(q^*)\right] dt + \frac{1}{2\delta} dw,\ u(0) = u_0. \tag{4.4}$$

Remark 4.3. Loosely, Theorem 4.2 tells us that $(q_n - q^*)$ is asymptotically normal with mean $n^{-1/3}((I/3) - f_{qq}(q^*))^{-1}\delta^2 B(q^*)$ and covariance $n^{-2/3}\widetilde{\Sigma}$, where

$$\widetilde{\Sigma} = \int_0^t e^{\{(I/3) - f_{qq}(q^*)\}t} \Sigma e^{\{(I/3) - f'_{qq}(q^*)\}t} dt.$$

Since we are considering random processes, the rate of convergence will consist of not only the scaling factor, but also the variation of the iterates. Since a normal random variable is completed specified by its mean and covariance, the asymptotic normality enables us to completely characterize the movements of the iterates. The scaling factor $n^{1/3}$ indicates how the iterates vary with respect to the iteration number n, and the covariance $\widetilde{\Sigma}$ provides us with the information of the amount of variation.

5 Further Remarks

This paper concerns estimation problems associated with a class of hybrid geometric Brownian motion models for the stock market. Note that in lieu of (3.4), one viable alternative is

$$q^\varepsilon_{n+1} = q^\varepsilon_n + \varepsilon G^\delta_n + \varepsilon Z_n, \tag{5.5}$$

where

$$G^\delta_{n,i} = -\frac{f(q_n + \delta e_i, \xi^+_n) - f(q_n - \delta e_i, \xi^-_n)}{2\delta},\ i = 1, \ldots, r, \tag{5.6}$$

where both the step size of the iteration and the finite difference step size are taken to be constant. This is particularly suitable for taking account of slight parameter variations. The asymptotic properties of such an algorithm can be studied, in which we will need $\varepsilon \to 0$ and $\delta = \delta_\varepsilon \to 0$ and $\varepsilon/\delta_\varepsilon \to 0$.

Once the estimate of the generator of the Markov is obtained, one can proceed to the liquidation decision of an equity [15]. In a related work, another approach using stochastic approximation for stock liquidation is presented [13]. It is conceivable that the approach using stochastic optimization algorithms will play a more important role in financial engineering.

References

[1] H.F. Chen and Y.M. Zhu, *Stochastic Approximation*, Shanghai Scientific and Technical Publishers, Shanghai, 1996.

[2] G.B. Di Masi, Y.M. Kabanov and W.J. Runggaldier, Mean variance hedging of options on stocks with Markov volatility, *Theory Prob. Appl.* **39** (1994), 173-181.

[3] J.P. Fouque, G. Papanicolaou, and K.R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, 2000.

[4] J.C. Hull, *Options, Futures, and Other Derivatives*, 3rd Ed., Prentice-Hall, Upper Saddle River, NJ, 1997.

[5] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, Springer, New York, 1998.

[6] H. J. Kushner, *Approximation and Weak Convergence Methods for Random Processes, with applications to Stochastic Systems Theory*, MIT Press, Cambridge, MA, 1984.

[7] H.J. Kushner and G. Yin, *Stochastic Approximation Algorithms and Applications*, Springer-Verlag, New York, 1997.

[8] P. L'Ecuyer and G. Yin, Budget-dependent convergence rate of stochastic approximation, *SIAM J. Optim.* **8** (1998), 217-247.

[9] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time case, *Rev. Econom. Statist.*, vol. 51, pp. 247-257, (1969).

[10] M. Musiela and M. Rutkowski, *Martingale Methods in Financial Modelling*, Springer-Verlag, New York, 1997.

[11] G. Yin, Convergence of a global stochastic optimization algorithm with partial step size restarting, *Adv. Appl. Probab.*, **32** (2000), 480-498.

[12] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer-Verlag, New York, 1998.

[13] G. Yin, R.H. Liu, and Q. Zhang, Recursive algorithms for stock liquidation: A stochastic optimization approach, to appear in *SIAM J. Optim.*

[14] G. Yin, Q. Zhang, and K. Yin, Stochastic optimization and simulation of a class of hybrid market models, preprint, 2002.

[15] Q. Zhang, Stock trading: An optimal selling rule, *SIAM J. Control. Optim.*, **40**, (2001), 64-87.