On the ball and beam problem: regulation with guaranteed transient performance and tracking periodic orbits

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Abstract

Despite the large number of controllers presented in the literature for the stabilization of the ball and beam system, only a few results are available on the transient performance problem. The aim of this paper is to propose an asymptotically stabilizing controller that ensures that, for a well–defined set of initial conditions, the ball remains on the bar during the transient. Tuning the controller, the set can be extended to include any initial position with zero velocity. The controller is a nonlinear static state feedback that is derived using the interconnection and damping assigment energy–shaping controller design methodology. As an extension of the regulation problem we also propose in this paper a controller that forces the ball and beam to oscilate. This is achieved, within the framework of energy–shaping, by assigning an energy function that attains its mimimum along the desired periodic orbit. A key step in our design is the immersion of the oscilator system into the fourth order system dynamics.

1 Introduction

In the past decade an important effort has been made to solve the well–known stabilization problem of the ball and beam. Major results in this directions are those of Hauser through approximate feedback linearization [4], the method of controlled lagrangians [5, 8] and the interconnection and damping assignment passivity based control (IDA–PBC) of [6]. The usual approach is to obtain a nonlinear control law that stabilizes the ball at its rest position at the center of the beam, achieving in some cases global asymptotic stability and in some cases asymptotic tracking [4]. However little effort has been made to analyze the transient performance of the controlled systems. At most, raise time and overshoot have been design parameters of linear controllers based on linear approximations of the models—which are only locally valid. In this paper we propose an asymptotically stabilizing controller that ensures that, for a well–defined set of initial conditions, the ball remains on the bar during the transient. Tuning the controller, the set can be extended to include any initial position with zero velocity. The controller is a nonlinear static state feedback that is derived using the energy–shaping IDA–PBC design methodology of [6].

As an extension of the regulation problem we continue with the research started in [3] and propose a controller that forces the ball and beam to oscillate. This is achieved, within the framework of energy–shaping, by assigning an energy function that attains its minimum along the desired periodic orbit. A key step in our design is the immersion of the oscillator system into the fourth order system dynamics.

This paper is organized as follows. First the system model is presented, and a brief summary of the control calculations via the IDA–PBC method is given. Section 3 presents the main results for transient performance analysis and estimation of the domain of attraction. In Section 3.2 a simulation analysis is made to illustrate the theoretical results. Section 4 contains our results on control of oscillations.

2 System Model and Control Method

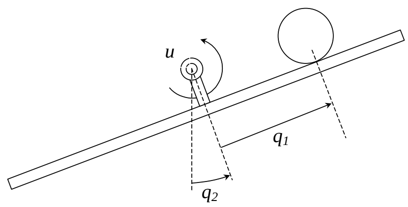

Figure 1: Ball and Beam System.

2.1 Ball and Beam Model

Although there are slight variations in the physical setups and models of the ball and beam, we will focus on the most widely used (see [4]), depicted in Fig. 1. This model is characterized by the fact that the line parallel to the beam starting at the center of gravity of the ball intersects the axis of rotation of the beam. For purposes that will be clear in Section 3, the Euler–Lagrange equations of [4] are scaled in time and torque, in such a way that the length of the bar L appears explicitly in the model. This is possible because L^2 is a factor of the moment of inertia of the bar.

$$\begin{aligned} \ddot{q}_1 + g\sin(q_2) - q_1\dot{q}_2^2 &= 0 \\ (L^2 + q_1^2)\ddot{q}_2 + 2q_1\dot{q}_1\dot{q}_2 + gq_1\cos(q_2) &= u \end{aligned} \tag{2.1}$$

For IDA–PBC it is convenient to provide a Hamiltonian model of the system for which we define the generalized momenta $p = [p_1, p_2]^\top = [\dot{q}_1, (L^2 + q_1^2)\dot{q}_2)]^\top$ and rewrite (2.1) as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0_{2\times 2} & I_2 \\ -I_2 & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0_{2\times 1} \\ G \end{bmatrix} \begin{bmatrix} 0 \\ u \end{bmatrix} \tag{2.2}$$

where I_2 is the identity matrix of order two, and $G = [0\ 1]^\top$. The Hamiltonian function of this model is defined as

$$H \triangleq \underbrace{\frac{1}{2} p^\top M^{-1}(q_1) p}_{T(q,p)} + \underbrace{g q_1 \sin q_2}_{V(q)} \tag{2.3}$$

where $T(q,p)$ and $V(q)$ are the kinetic and potential energy respectively, and M is the mass matrix, namely

$$M(q_1) = \begin{bmatrix} 1 & 0 \\ 0 & L^2 + q_1^2 \end{bmatrix}$$

2.2 Passivity Based Control

What follows is a brief summary of the IDA–PBC presented in [6]. The design procedure in the IDA–PBC method involves two main steps, kinetic energy shaping, for which a system of PDEs must be solved, and potential energy shaping which normally involves the solution of one PDE and the assignment of the desired equilibrium point. Finally a damping term must be added to obtain asymptotic stability. In [6] we also make use of the method proposed in [10] to reduce the PDE's to ODE's.

We propose the following form for the desired (closed loop) energy function

$$H_d(q,p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q) \tag{2.4}$$

where $M_d = M_d^\top > 0$ and V_d represent the (to be defined) closed–loop inertia matrix and potential energy function, respectively. We will require that V_d have an isolated minimum at q_*.

In PBC the control input is naturally decomposed into two terms

$$u = u_{es}(q,p) + u_{di}(q,p) \tag{2.5}$$

where the first term is designed to achieve the energy shaping and the second one injects the damping. The desired closed–loop port–controlled Hamiltonian dynamics are taken of the form

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = [J_d(q,p) - R_d(q,p)] \begin{bmatrix} \nabla_q H_d \\ \nabla_p H_d \end{bmatrix} \tag{2.6}$$

where the terms

$$J_d = -J_d^\top = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2(q,p) \end{bmatrix} \; ; \; R_d = R_d^\top = \begin{bmatrix} 0 & 0 \\ 0 & G K_v G^\top \end{bmatrix} \geq 0$$

represent the desired interconnection and damping structures.

For the ball and beam problem we will propose the elements of the closed loop inertia matrix as functions of q_1

$$M_d = \begin{bmatrix} a_1(q_1) & a_2(q_1) \\ a_2(q_1) & a_3(q_1) \end{bmatrix}$$

These elements are obtained as solutions of the underdetermined system of ODEs [10]

$$\begin{aligned}\frac{d}{dq_1}a_1(q_1) &= \frac{2q_1}{(L^2+q_1^2)^2}\frac{a_2^2}{a_1}\\ \frac{d}{dq_1}a_2(q_1) &= \frac{2q_1}{(L^2+q_1^2)^2}\frac{a_2a_3}{a_1}\end{aligned}$$

from which we obtain the globally defined solution

$$M_d = (L^2+q_1^2)\begin{bmatrix}\sqrt{2}(L^2+q_1^2)^{-1/2} & 1\\ 1 & \sqrt{2(L^2+q_1^2)}\end{bmatrix} \tag{2.7}$$

This matrix is globally positive definite, as desired. The kinetic energy shaping is completed evaluating the matrix J_2 which is straightforward as can be read in [6]. For potential energy shaping the following equation must be solved

$$\sqrt{2(L^2+q_1^2)}\frac{\partial V_d}{\partial q_1} + \frac{\partial V_d}{\partial q_2} = g\sin(q_2)$$

Again, after some calculations and with appropriate choice of the degrees of freedom, gives the solution

$$V_d = g[1-\cos(q_2)] + \frac{k_p}{2}\left[q_2 - \frac{1}{\sqrt{2}}\text{arcsinh}\left(\frac{q_1}{L}\right)\right]^2 \tag{2.8}$$

where we have added a constant to shift the minimum to zero. This function has a local minimum at the control objective. The potential energy function obtained is depicted in the contour plot of Fig. 2.

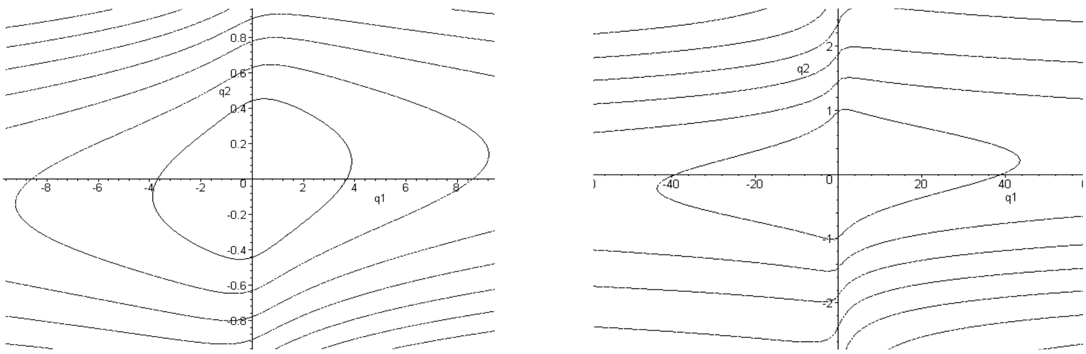

Figure 2: Level curves of $V_d(q)$ around the origin for $k_p = 0.05$ (left) and $k_p = 0.01$ (right).

To compute the final control law we first determine the energy–shaping term u_{es} which in this case takes the form

$$u_{es} = \nabla_{q_2}H - (M_dM^{-1})_{(2,1)}\nabla_{q_1}H_d - (M_dM^{-1})_{(2,2)}\nabla_{q_2}H_d + (J_2M_d^{-1})_{(2,1)}p_1 + (J_2M_d^{-1})_{(2,2)}p_2$$

Replacing the functions derived above for M_d and j, and after some straightforward calculations, we obtain the expression

$$u_{es} = \frac{q_1}{\sqrt{2}(L^2+q_1^2)}\left[-\sqrt{L^2+q_1^2}p_1^2 + \sqrt{2}p_1p_2 + \frac{1}{\sqrt{L^2+q_1^2}}p_2^2\right] + \xi(q) \tag{2.9}$$

where

$$\xi(q) \triangleq gq_1\cos q_2 - g\sqrt{2(L^2+q_1^2)}\sin q_2 - k_p\sqrt{\frac{L^2+q_1^2}{2}}\left(q_2 - \frac{1}{\sqrt{2}}\text{arcsinh}\left(\frac{q_1}{L}\right)\right)$$

The controller design is completed with the damping injection term, which yields

$$u_{di} = \frac{k_v}{L^2+q_1^2}\left(p_1 - \sqrt{\frac{2}{L^2+q_1^2}}p_2\right) \tag{2.10}$$

This is an explicit control law with clear physical interpretation, namely: k_p is a *bona fide* proportional gain in position, as it multiplies terms that grow linearly in q, and k_v injects damping along a specified direction of velocities. Commissioning of the controller is simplified by this feature. Asymptotic stability is demonstrated based in Matrosov's Theorem, and it should be read in [6].

3 Transient Performance

In Subsection 2.2 we have designed a controller that makes the origin asymptotically stable with Lyapunov function the desired total energy. This analysis can be refined studying the effect of the tuning parameter k_p on the size of the domain of attraction, and explicitly quantifying a set of initial conditions such that the ball remains all the time in the bar, that is, $|q_1(t)| \leq L$ for all $t \geq 0$.

3.1 Keeping the ball on the bar

First, we note that as H_d decreases and the kinetic energy is non–negative, we have that $V_d(q(t)) \leq H_d(q(0), p(0))$, hence the sub–level sets of V_d are invariant sets for $q(t)$. Further, if we can show that the kinetic energy is bounded, then the bounded sets provide an estimate of the domain of attraction. To study these sets we find convenient to work in the coordinates $(\tilde{q}_1, q_2, p_1, p_2)$, where $\tilde{q}_1$ is defined as[1]

$$\tilde{q}_1 \triangleq q_2 - \frac{1}{\sqrt{2}}\text{arcsinh}\left(\frac{q_1}{L}\right)$$

[1]Note that the coordinate mapping $(q_1, q_2) \mapsto (\tilde{q}_1, q_2)$ defines a global diffeomorphism, and recall that boundedness of sub–level sets is invariant under the action of diffeomorphisms.

In these coordinates the potential energy function becomes

$$\tilde{V}_d(\tilde{q}_1, q_2) \triangleq g(1 - \cos q_2) + \frac{k_p}{2}\tilde{q}_1^2$$

which has the same analytical expression as the total energy of the simple pendulum, and the associated sub–level sets, i.e., $\{(\tilde{q}_1, q_2) \mid \tilde{V}_d(\tilde{q}_1, q_2) \leq c\}$, are of the form shown in Fig 3. We are interested here in the bounded connected components that contain the origin, that we will denote Ξ_c.

The following basic lemma, whose proof may be found in [6], will be instrumental in the sequel.

Lemma 3.1. *The set Ξ_c is bounded if and only if $c < 2g$.*

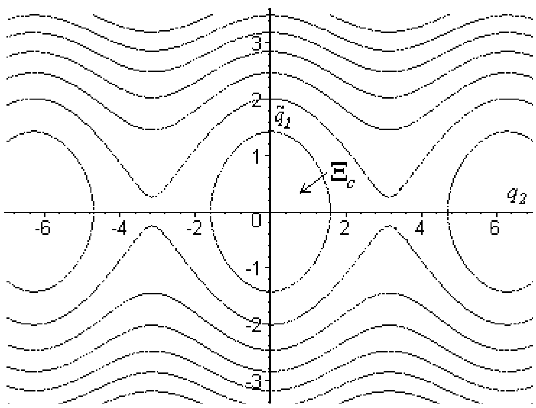

Figure 3: Level curves of $\tilde{V}(\tilde{q}_1, q_2)$.

We are in position to present the main result of this section. To simplify the notation we will use $(\cdot)^o$ to denote the value of the functions at $t = 0$.

Proposition 3.1. *Consider the ball and beam model (2.1) in closed-loop with the static state feedback IDA–PBC $u = u_{es} + u_{di}$, with (2.9)–(2.10), and $k_v > 0$.*

(i) We can compute a constant $k_p^M > 0$, function of the initial conditions (q^o, p^o), such that for all $k_p \leq k_p^M$, the set

$$\{(q,p) \mid \frac{1}{2}p^\top M_d^{-1}(q)p + g(1 - \cos q_2) < 2g\} \tag{3.11}$$

is an estimate of the domain of attraction of the zero equilibrium. In particular, all trajectories starting with zero velocity, and $q_2^o \in (-\pi, \pi)$ will asymptotically converge to the origin.

(ii) Fix $k_p \leq k_p^M$ and assume $|q_1^o| < L$. Then,

$$\{(q,p) \mid \frac{1}{2}p^\top M_d^{-1}(q)p + g(1 - \cos q_2) + \frac{k_p}{2}\left(q_2 - \frac{1}{\sqrt{2}}\text{arcsinh}\left(\frac{q_1}{L}\right)\right) < \frac{1}{8}\frac{k_p g}{2k_p + g}\} \tag{3.12}$$

is a domain of attraction of the zero equilibrium, such that all trajectories starting in this set satisfy $|q_1(t)| < L$ *for all* $t \geq 0$.

Proof. We have shown above that the sub–level sets of $V_d(q)$ are invariant sets for $q(t)$. Further, Lemma 3.1 establishes that the connected component of the sub–level sets of $V_d(q)$ containing the origin is bounded if and only if $c < 2g$. Hence, setting $c = H_d^o$ in Lemma 3.1 it follows that this set is bounded if and only if

$$H_d^o = \frac{1}{2}(p^o)^\top M_d^{-1}(q^o)p^o + g(1 - \cos(q_2^o)) + \frac{k_p}{2}\tilde{q}_1^2 < 2g$$

It is clear that, if the first two terms are strictly smaller than g, we can always find and upperbound on k_p such that the inequality holds. To complete the proof of point (i) of the proposition we remark that, for all trajectories starting in the set (3.11), $q(t)$ is bounded. Hence, from (2.2), we conclude that $M_d^{-1}(q(t)) > \epsilon I$ from some constant $\epsilon > 0$. This, together with the fact that $\frac{1}{2}p^\top(t)M_d^{-1}(q(t))p(t) < H_d^o$, establishes that the corresponding $p(t)$ is also bounded and the set (3.11) is an estimate of the domain of attraction.

The proof of (ii) proceeds as follows. From (3.1) we have that

$$|q_1| \leq L \Leftrightarrow |q_2 - \tilde{q}_1| \leq \frac{1}{\sqrt{2}}\text{arcsinh}(1) \tag{3.13}$$

where we have used the fact that arcsinh$(\cdot)$ is odd and monotonic. The region defined by (3.13) is depicted in Fig. 4 together with two sets Ξ_c. Our problem is then to compute the largest c such that the set Ξ_c does not intersect the lines $\tilde{q}_1 = q_2 \pm \frac{1}{\sqrt{2}}\text{arcsinh}(1)$. To simplify the expressions we note that $\frac{1}{\sqrt{2}}\text{arcsinh}(1) > \frac{1}{2}$, and check the intersection with the "closer" lines $\tilde{q}_1 = q_2 \pm \frac{1}{2}$ in the band $q_2 \in [-\frac{1}{2}, \frac{1}{2}]$. For completeness, we substitute $\tilde{q}_1 = q_2 + \frac{1}{2}$ in the boundary equation of Ξ_c, and use the bound $\cos q_2 < 1 - \frac{q_2^2}{4}$, which is valid in the aforementioned band, to get the first inequality below

$$g(1 - \cos q_2) + \frac{k_p}{2}\left(q_2 + \frac{1}{2}\right) > g\frac{q_2^2}{4} + \frac{k_p}{2}\left(q_2 + \frac{1}{2}\right) > c$$

The second inequality holds for all $c < \frac{1}{8}\frac{k_p g}{2k_p + g}$ and all q_2 in the band. This proves that the boundary of Ξ_c does not intersect the limit lines within the band. They cannot intersect outside the interval either because $c > g(1 - \cos(\frac{1}{2}))$ implies that, in Ξ_c, $|q_2| < \frac{1}{2}$, and this bound on c is less strict than $c < \frac{1}{8}\frac{k_p g}{2k_p + g}$. This completes the proof.

□

3.2 Simulations

A set of simulations of the ball and beam system with $g = 9.8$, $k_p = 1$ has been made. The results are shown in Fig. 5. The graphs on the upper row depict the ball position q_1 and the beam angle q_2 for zero initial velocity, and varying initial positions and parameters. Under

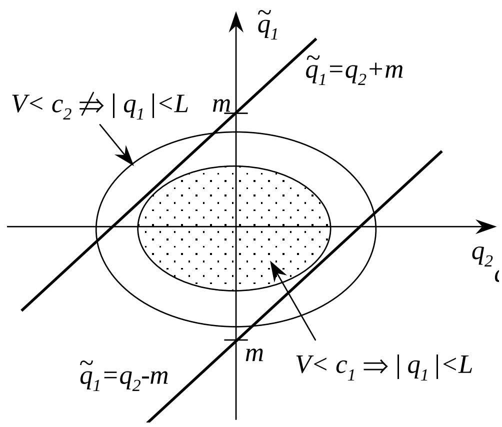

Figure 4: Graphical interpretation of $|q_1| < L$, with $m \triangleq \frac{1}{\sqrt{2}}\text{arcsinh}(1)$.

each of these, the corresponding graphs with the desired hamiltonian H_d and potential energy V_d are shown. Each column in the graph array belongs to a single simulation. From the first two simulations we see the effect of increasing the damping constant k_v starting with the bar in vertical position. Note that the convergence is not always accelerated with higher values of k_v, as new oscillations come into play. The third simulation starts at rest with the bar in horizontal position and the ball on the edge of the bar. Apparently, the controller works best starting from $q_2(0) = 0$. To ensure that the initial condition is within the domain of attraction, k_p has been chosen smaller than k_p^M according to Proposition 3.1. Figure 5 also illustrates the monotonic nature of H_d together with the fact that $V_d(t) < H_d(t) < H_d(0)$ for all t.

The effect of the limited bar length and the use of the last part of Proposition 3.1 is illustrated by simulation in Fig. 6. The parameters are $g = 9.8$, $k_p = 1$, $k_v = 50$, and $L = 10$. Hence the condition for keeping the ball within the limits of the bar is $H_d(0) < 0.1038$. The first simulation starts at $(q^0, p^0) = (8, 0, 1, 1)$ with $H_d(0) = 0.1837$. Due to the initial velocity the controller is unable to *catch* the ball before it trespasses the limit of the bar ($L = 10$). In the second simulation we have $(q^0, p^0) = (6, 0, 0.5, 1)$ and $H_d(0) = 0.0928$, thus the bound $|q_1| < L$ is guaranteed and the ball remains within the limits of the bar.

4 Stabilization of oscillations

In this section a different problem is addressed: the objective is to stabilize an oscillatory movement in the ball and beam. Usually the control objective is either to stabilize an operating point or to track a reference. Nevertheless, in some systems—including AC converters and walking robots—the objective is to obtain stable and robust oscillations. The problem can, of course, be posed as a tracking problem but this leads to non–robust solution with high dependence on initial conditions and noise. In [3] the problem of obtaining oscillations with a constant energy is addressed. Here, we use a different approach: we define a two-

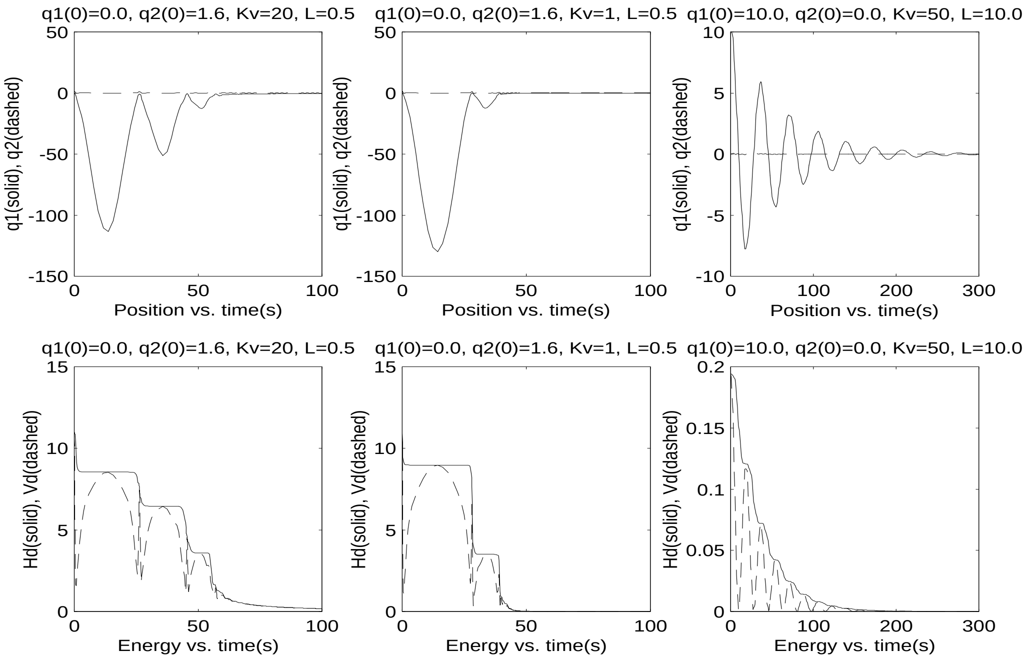

Figure 5: Simulations of the ball and beam starting from rest.

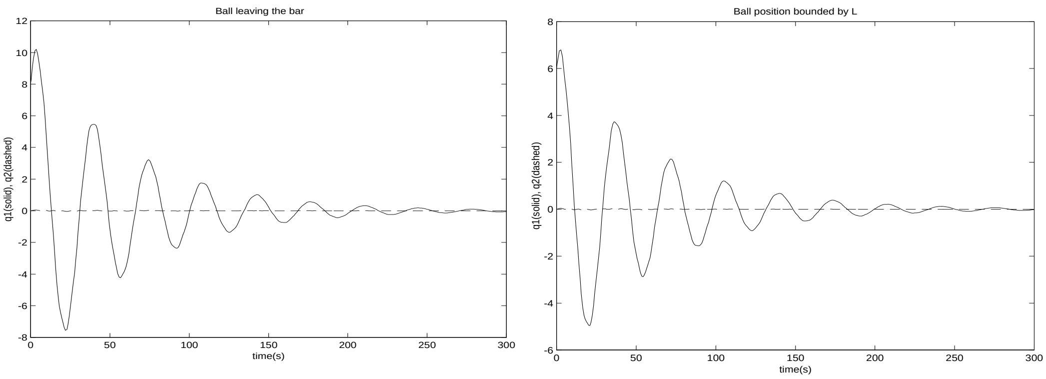

Figure 6: Ball and beam starting with initial velocity. Effect of the finite bar length.

dimensional system that exhibits stable and robust oscillations and immerse it in the whole system using the technique of Immersion and Invariance of [1]. For, we recall the main result of this technique as applied to stabilization of equilibria—with slight modifications the result applies as well to stabilization of periodic orbits of interest here.

Proposition 4.1. *Consider the system*

$$\dot{x} = f(x) + g(x)u, \tag{4.14}$$

with state $x \in I\!R^n$ *and control* $u \in I\!R^m$, *with an equilibrium point* $x_* \in I\!R^n$ *to be stabilized. Let* $p < n$ *and assume we can find mappings* $\alpha(\cdot) : I\!R^p \rightarrow I\!R^p$, $\pi(\cdot) : I\!R^p \rightarrow I\!R^n$, $c(\cdot) : I\!R^p \rightarrow I\!R^m$, $\phi(\cdot) : I\!R^n \rightarrow I\!R^{n-p}$, $\psi(\cdot, \cdot) : I\!R^{n \times (n-p)} \rightarrow I\!R^m$ *such that:*

(H1) (Target system) *The system*

$$\dot{\xi} = \alpha(\xi), \tag{4.15}$$

with $\xi \in I\!\!R^p$*, has a globally asymptotically stable equilibrium at* $\xi_* \in I\!\!R^p$ *and* $x_* = \pi(\xi_*)$*.*

(H2) (Immersion condition) *For all* $\xi \in I\!\!R^p$

$$f(\pi(\xi)) + g(\pi(\xi))c(\xi) = \frac{\partial \pi}{\partial \xi}\alpha(\xi) \tag{4.16}$$

(H3) (Implicit manifold) *The following set identity holds*

$$\{x \in I\!\!R^n \mid \phi(x) = 0\} = \{x \in I\!\!R^n \mid x = \pi(\xi), \xi \in I\!\!R^p\} \tag{4.17}$$

(H4) (Manifold attractivity and trajectory boundedness) *The system*

$$\dot{z} = \frac{\partial \phi}{\partial x}\left(f(x) + g(x)\psi(x,z)\right) \tag{4.18}$$

with state z*, has a globally asymptotically stable equilibrium at zero uniformly in* x*. Further, the trajectories of the system*

$$\dot{x} = f(x) + g(x)\psi(x,z) \tag{4.19}$$

are bounded for all $t \in [0, \infty)$*.*

Then, x_* *is a globally asymptotically stable equilibrium of the closed loop system* $\dot{x} = f(x) + g(x)\psi(x, \phi(x))$*.*

4.1 Target system

We now define a two-dimensional target system $\dot{\xi} = f(\xi)$, $\xi = (\xi_1, \xi_2) \in I\!\!R^2$, that presents a limit cycle and, in order to underscore the role of the damping injection, express it in port–controlled Hamiltonian form [2]. To this end, consider the function $H_a = \frac{1}{4}P^2(\xi)$ with $P(\xi) \stackrel{\triangle}{=} \omega_c^2\xi_1^2 + \xi_2^2 - \mu$. Depending on the values of the parameter μ, the function H_a has two different shapes. For $\mu < 0$ it has a single minimum at the origin of the space (ξ_1, ξ_2), as shown in Fig. 7-a; but for $\mu > 0$ the minima of H_a are reached in the closed curve $\omega_c^2\xi_1^2 + \xi_2^2 = \mu$ (Fig. 7-b).

Define the generalized hamiltonian system with damping [9]

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\omega_c^2\xi_1^2+\xi_2^2-\mu} \\ -\frac{1}{\omega_c^2\xi_1^2+\xi_2^2-\mu} & -k_a \end{bmatrix} \begin{bmatrix} \frac{\partial H_a}{\partial \xi_1} \\ \frac{\partial H_a}{\partial \xi_2} \end{bmatrix}. \tag{4.20}$$

being $k_a > 0$ a damping coefficient. Clearly, this system can be written as

$$\dot{\xi}_1 = \xi_2 \tag{4.21}$$

$$\dot{\xi}_2 = -\omega_c^2\xi_1 - k_aP\xi_2. \tag{4.22}$$

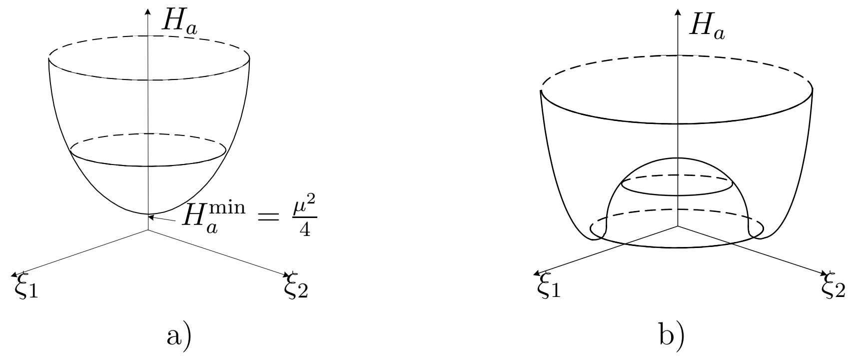

Figure 7: Function H_a for a) $\mu < 0$ and b) $\mu > 0$.

Proposition 4.2. *Consider system (4.21)-(4.22). If $\mu < 0$ the origin is globally asymptotically stable. If $\mu > 0$, for any initial condition, except the origin, the trajectories tend to the limit cycle $P = 0$ with period $2\pi/\omega_c$.*

Proof. It is easy to see that the radially unbounded function H_a fulfills $\dot{H}_a = -k_a P^2 \xi_2^2 \leq 0$. Let us compute the invariant sets for which $\dot{H}_a = 0$.

- A first possibility is that $\xi_2(t) \equiv 0$ with $\dot{\xi}_2 = 0$. Using (4.22) this means that $\omega_c \xi_1^2 = 0 \Rightarrow \xi_1 = 0$. Therefore, this case corresponds to the point $(\xi_1, \xi_2) = (0, 0)$. By linearization it can be seen that this point is locally stable for $\mu < 0$ and locally unstable for $\mu > 0$.

- The second possibility is that $P = 0$. This set does not exist for $\mu < 0$.

Thus, applying LaSalle's theorem the first part of the proposition is proven. For the second one, we have still to prove that the curve $P = 0$ is a true limit cycle, that is, that there are no equilibrium points in $P = 0$. We proceed by contradiction: assume that an equilibrium exists for $P = 0$. From (4.21), $\xi_2 = 0$ and from (4.22) with $P = 0$ results $\xi_1 = 0$. But $\xi_1 = \xi_2 = P = 0$ is only possible if $\mu = 0$.

Finally, for $P = 0$, it is easy to see that system (4.21)–(4.22) is reduced to an harmonic oscillator with a oscillation period equal to $2\pi/\omega_c$. □

4.2 Immersion and invariance

We will now apply Proposition 4.1 of the Immersion and Invariance method of [1] to a simplified model of the ball and beam system (2.1) with the target dynamics described above. First, we introduce the partial-linearization control law $u = (-2q_1\dot{q}_1\dot{q}_2 - gq_1\cos(q_2))/(L^2 + q_1^2) + v$ and denoting the state vector as $[x_1, x_2, x_3, x_4] = [q_1, \dot{q}_1, q_2, \dot{q}_2]$ the system can be written as:

$$\dot{x}_1 = x_2 \quad (4.23)$$
$$\dot{x}_2 = x_1 x_4^2 - \sin x_3 \quad (4.24)$$
$$\dot{x}_3 = x_4 \quad (4.25)$$
$$\dot{x}_4 = v \quad (4.26)$$

Second, an approximated model—which is suggested in [4]—is obtained neglecting the centrifugal acceleration term $x_1 x_4^2$, assuming a small angular velocity of the bar yielding

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_3 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} v. \quad (4.27)$$

Let us determine the map $\pi(\xi)$. To enforce the oscillations we choose $\pi_1 = \xi_1$ and $\pi_2 = \xi_2$. π_3 and π_4 can be determined imposing the invariance condition, that yields

$$\pi_3(\xi) = \sin^{-1}(\xi_1 + k_a \xi_2 P(\xi)) \quad (4.28)$$
$$\pi_4(\xi) = \frac{\partial \pi_3}{\partial \xi_1}\xi_2 + \frac{\partial \pi_3}{\partial \xi_2}(-\xi_1 - k_a \xi_2 P(\xi)) \quad (4.29)$$

We choose as the first element of the implicit manifold, ϕ_1, the obvious choice $\phi_1(x) = x_3 - \sin^{-1}(x_1 + k_a x_2 P(x_1, x_2))$. Let ϕ_2 be

$$\phi_2(x) = x_4 - \frac{\partial \pi_3|_x}{\partial x_1} x_2 - \frac{\partial \pi_3|_x}{\partial x_2}(-\sin x_3)$$

Notice that the implicit manifold identity holds since

$$\{\phi_1 = 0, \phi_2 = 0\} \Rightarrow x_4 = \frac{\partial \pi_3}{\partial \xi_1}\xi_2 + \frac{\partial \pi_3}{\partial \xi_2}(-\xi_1 - k_a \xi_2 P(\xi))$$

With this choice, it is obvious that $\dot{\phi}_1 = \phi_2$. Then, defining $S(x_1, x_2, x_3) \triangleq -\frac{\partial \pi_3|_x}{\partial x_1} x_2 + \frac{\partial \pi_3|_x}{\partial x_2} \sin x_3$, we obtain the off–the–manifold system (4.18) like

$$\dot{z}_1 = z_2 \quad (4.30)$$
$$\dot{z}_2 = v + \frac{\partial S}{\partial x_1} x_2 - \frac{\partial S}{\partial x_2} \sin x_3 + \frac{\partial S}{\partial x_3} x_4 \quad (4.31)$$

Then, a possible stabilizing control law is

$$v = -\frac{\partial S}{\partial x_1} x_2 + \frac{\partial S}{\partial x_2} \sin x_3 - \frac{\partial S}{\partial x_3} x_4 - k_1 \phi_1(x) - k_2 \phi_2(x)$$

with $k_1, k_2 > 0$.

4.3 Simulation results

In order to show the behaviour of the obtained control law a simulation is presented for $\mu = 0.1, k_a = 1, k_1 = 1$ and $k_2 = 1$. The initial conditions are $x(0) = [0.9, 0.1, 0, 0]$. Figure 8 shows the evolution of the off–the–manifold coordinates z_1 and z_2 tending to zero, while Fig. 9 shows that, after a transient period, the variables x_1 and x_2 indeed exhibit sinusoidal oscillations of the desired amplitude $\sqrt{\mu}$. Finally, Fig. 10 shows the phase portrait projection on $(x_1 - x_2)$.

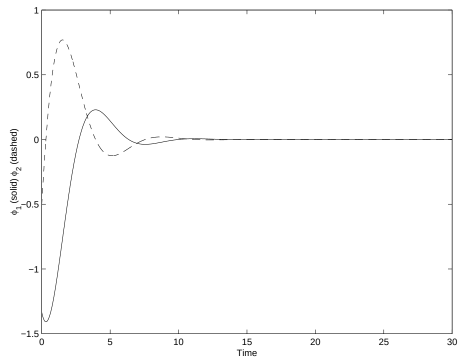

Figure 8: Evolution of z_1 and z_2

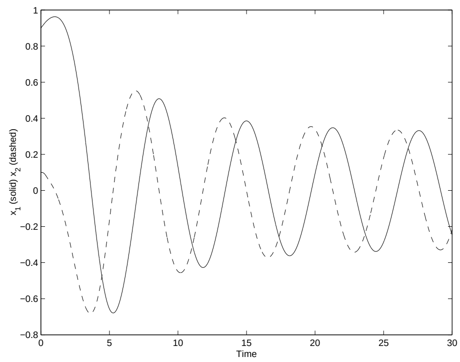

Figure 9: Evolution of x_1 and x_2

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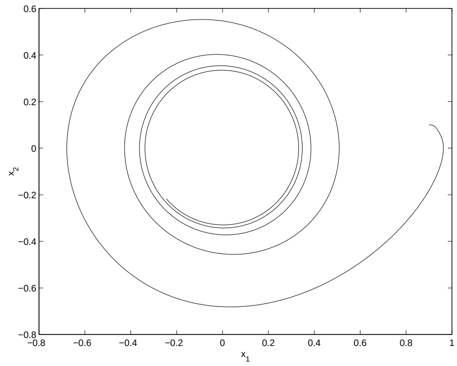

Figure 10: Phase portrait projection on $(x_1 - x_2)$

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