Reduction of Controlled Lagrangian Systems with Symmetry

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Abstract

We extend the theory of controlled Lagrangian systems to include systems with symmetry and the Lagrangian reduction theory. This extension is crucial to study examples such as spacecraft control, underwater vehicle control, etc.

1 Introduction

The method of controlled Lagrangian (CL) systems has been successful in designing feedback control laws for mechanical systems;[1], [2], [4], and references therein. In this paper, we develop the reduction theory of controlled Lagrangian systems with symmetry based on the work on Lagrangian reduction in [3]. This will draw a clearer picture of the relation between CL systems with symmetry and the reduced CL systems. This is crucial to study examples such as spacecraft control and underwater vehicle control. We will also show that the Euler-Poincaré matching conditions in [2] is a special case of the results of this paper. In a forthcoming publication, we will present the reduction of controlled *Hamiltonian* systems with symmetry and its relationship with the reduction of CL systems with symmetry.

Notation. We use fairly standard notation. The configuration manifold for the mechanical systems under consideration is denoted Q. We assume that the dimension of Q is n and use $(q^1, \dots, q^n)$ as coordinates on Q. The second-order tangent bundle is denoted $T^{(2)}Q$ and consists of second derivatives of curves in Q. Let G be a Lie group which acts (on the left) on Q freely and properly so that $\pi_G(Q) : Q \to Q/G$ becomes a principal bundle. The tangent lift action of G on TQ is free and proper and $\tau_{/G} : TQ \to TQ/G$ becomes a principal bundle. When M is a manifold on which G acts, we let $[m]_G$ denote the equivalence class of $m \in M$ in the quotient space M/G. Even though we do not explicitly specify the manifold M in this notation, it will be clear in the context.

The Euler-Lagrange operator $\mathcal{EL}$ assigns to a Lagrangian $L : TQ \to \mathbb{R}$, a bundle map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$ which may be written in local coordinates (employing the summation convention) as

$$\mathcal{EL}(L)_i(q, \dot{q}, \ddot{q})dq^i = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q, \dot{q}) - \frac{\partial L}{\partial q^i}(q, \dot{q}) \right) dq^i$$

in which it is understood that one regards the first term on the right hand side as a function on the second-order tangent bundle $T^{(2)}Q$ by formally applying the chain rule and then replacing everywhere dq/dt by $\dot{q}$ and $d\dot{q}/dt$ by $\ddot{q}$.

2 Main Results

2.1 Review of Controlled Lagrangian Systems

We first review the controlled Lagrangian systems of [4].

Definition 2.1. *A* ***controlled Lagrangian (CL) system*** *is a triple* (L, F, W) *where the function* $L : TQ \to \mathbb{R}$ *is the Lagrangian, the fiber-preserving map* $F : TQ \to T^*Q$ *is an external force and* W*, called the* ***control bundle****, is a subbundle of* T^*Q*, representing the* ***actuation directions****.*

When we choose a specific feedback control map $u : TQ \to W$, we call the triple (L, F, u) a ***closed-loop Lagrangian system***. The equation of motion of the closed-loop system (L, F, u) is given by

$$\mathcal{EL}(L)(q, \dot{q}, \ddot{q}) = F(q, \dot{q}) + u(q, \dot{q}). \tag{2.1}$$

A CL system (L, F, W) is called ***simple*** if the Lagrangian L has the form of kinetic minus potential energy: $L(q, \dot{q}) = \frac{1}{2}m(q)(\dot{q}, \dot{q}) - V(q)$. We will use the acronym ***SCL*** for "simple controlled Lagrangian".

We now introduce an equivalence relation by feedback transformations among the CL systems (influenced by [5]).

Definition 2.2. *Given the two simple CL systems* (L_1, F_1, W_1) *and* (L_2, F_2, W_2)*, the* ***Euler-Lagrange matching conditions*** *are*

ELM-1 : $W_1 = m_1 m_2^{-1}(W_2)$,

ELM-2 : $\mathrm{Im}\left[(\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2)\right] \subset W_1$,

where m_i *is the mass tensor of* L_i *and* Im *means the pointwise image of the map in brackets.*

We say that the two simple CL Lagrangian systems (L_1, F_1, W_1) *and* (L_2, F_2, W_2) *are* ***CL-equivalent*** *if ELM-1 and ELM-2 hold. We use the symbol* $\overset{L}{\sim}$ *for this equivalence relation.*

The following theorem explains the main property of the CL-equivalence relation.

Theorem 2.1. *Suppose two simple controlled Lagrangian systems* (L_i, F_i, W_i)*,* $i = 1, 2$ *are CL-equivalent. Then, for an arbitrary control law given for one system, there exists a control law for the other system such that the two closed-loop systems produce the same equations of motion. The explicit relation between the two feedback control laws* u_i*,* $i = 1, 2$ *is given by*

$$u_1 = (\mathcal{EL}(L_1) - F_1) - m_1 m_2^{-1}(\mathcal{EL}(L_2) - F_2) + m_1 m_2^{-1} u_2 \tag{2.2}$$

where m_i *is the mass tensor of* L_i*.*

2.2 Reduction of Controlled Lagrangian Systems with Symmetry

Based on the work on the Lagrangian reduction in [3], we develop the reduction theory of controlled Lagrangian systems with symmetry. This will draw a clearer picture of the relation between CL systems with symmetry and the reduced CL systems.

2.2.1 Reduction of CL Systems with Symmetries

We defined the CL system in Definition 2.1. Here, we define G-invariant CL systems on TQ and reduced CL systems on TQ/G where G is a Lie group acting on Q.

Definition 2.3. *Let G be a Lie group acting on Q. A **G-invariant controlled Lagrangian (G-CL) system** is a CL system, (L, F, W), where L is a G-invariant Lagrangian, F is a G-equivariant force map and W is a G-invariant subbundle of T^*Q.*

Definition 2.4. *A **reduced controlled Lagrangian (RCL) system** is a triple (l, f, U) where $l : TQ/G \to \mathbb{R}$ is a smooth function called the reduced Lagrangian, the fiber-preserving map $f : TQ/G \to T^*Q/G$ is called the reduced force map, and U, called the reduced control bundle, is a subbundle of T^*Q/G. A feedback control for the RCL system is a (fiber-preserving) map of TQ/G into U.*

Suppose that we are given a G-CL system (L, F, W). The G-invariance of L induces a reduced Lagrangian l on TQ/G satisfying

$$l \circ \tau_{/G} = L. \tag{2.3}$$

The G-equivariance of F induces a reduced force map $[F]_G : TQ/G \to T^*Q/G$ satisfying

$$[F]_G \circ \tau_{/G} = \pi_{/G} \circ F. \tag{2.4}$$

This leads to the following definition:

Definition 2.5. *The RCL system of a G-CL system (L, F, W) is a triple $(l, [F]_G, W/G)$ where l is the reduced Lagrangian satisfying* (2.3)*, and $[F]_G$ is the reduced force satisfying* (2.4)*.*

One naturally asks if there exists a G-CL system on TQ when one is given a RCL system on TQ/G. The following proposition proves its unique existence.

Proposition 2.1. *Given a RCL system (l, f, U) on TQ/G, there exists a unique G-CL system (L, F, W) on TQ whose RCL system is (l, f, U).*

Proof. Define L by (2.3). Define a force map F on TQ as follows: for $v_q, w_q \in T_qQ$,

$$\langle F(v_q),\, w_q\rangle = \langle f \circ \tau_{/G}(v_q),\, \tau_{/G}(w_q)\rangle. \tag{2.5}$$

One can check the G-equivariance of F. One can also check that relation (2.5) defines a unique *fiber-preserving* map F of TQ to T^*Q. Let $W := \tau_{/G}^{-1}(U)$. By construction, (L, F, W) is the unique G-CL system whose RCL system is (l, f, U). □

By Proposition 2.1, we can, without loss of generality, write an arbitrary RCL system in the form of the RCL system of a G-CL system.

Given a G-CL system (L, F, W), the G invariance of L implies the G-equivariance of the map $\mathcal{EL}(L) : T^{(2)}Q \to T^*Q$, which induces a quotient map

$$\mathcal{REL}(l) := [\mathcal{EL}(L)]_G : T^{(2)}Q/G \to T^*Q/G,$$

which depends only on the reduced Lagrangian l on TQ/G induced from L. The operator $\mathcal{REL}$ is called the ***reduced Euler-Lagrange operator***. The equation of motion of a RCL $(l, [F]_G, W/G)$ with a choice of control $[u]_G : TQ/G \to W/G$ is given by

$$\mathcal{REL}(l)([q, \dot{q}, \ddot{q}]_G) = [F]_G([q, \dot{q}]_G) + [u]_G([q, \dot{q}]_G).$$

To write computable equations of $\mathcal{REL}$, one has to choose a principal connection on the principal bundle $Q \to Q/G$ to make the following identifications:

$$TQ/G = T(Q/G) \oplus \tilde{\mathfrak{g}}, \quad T^{(2)}Q/G = T^{(2)}(Q/G) \times_{Q/G} 2\tilde{\mathfrak{g}}, \quad T^*Q/G = T^*(Q/G) \oplus \tilde{\mathfrak{g}}^*$$

where $\tilde{\mathfrak{g}}$ is the adjoint bundle $\mathrm{Ad}(Q)$, $\tilde{\mathfrak{g}}^*$ is the coadjoint bundle $\mathrm{Ad}^*(Q)$, $2\tilde{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$, and $\oplus$ is the Whitney sum(see Lemma 2.4.2 and Lemma 3.2.2 in [3]). With these identifications, $\mathcal{REL}(l)$ induces the ***Lagrange-Poincaré operator***

$$\mathcal{LP}(l) : T^{(2)}(Q/G) \times_{Q/G} 2\tilde{g} \to T^*(Q/G) \oplus \tilde{g}^*. \tag{2.6}$$

Hence, the reduced Euler-Lagrange operator, $\mathcal{REL}$ may be replaced by the Lagrange-Poincaré operator $\mathcal{LP}$ in the following as long as one chooses a connection on $Q \to Q/G$. More details may be found in [3].

We study the relation between trajectories of G-CL systems and trajectories of RCL systems. Let (L, F, W) be a G-CL system and $(l, [F]_G, W/G)$ be its RCL system. Choose an arbitrary G-*equivariant* feedback control law $u : TQ \to W$ for (L, F, W). The control u induces a reduced map $[u]_G : TQ/G \to T^*Q/G$. If $(q(t), \dot{q}(t)) \in TQ$ is a trajectory of the closed-loop system (L, F, u), then $\tau_{/G}(q(t), \dot{q}(t)) \in TQ/G$ is the trajectory of the closed-loop system $(l, [F]_G, [u]_G)$.

2.2.2 Reduced CL-Equivalence

We now define the *reduced* simple controlled Lagrangian system.

Definition 2.6. *A* ***reduced simple controlled Lagrangian (RSCL) system*** *is the reduced CL system $(l, [F]_G, W/G)$ of a G-invariant simple CL system (L, F, W). If the G-invariant simple Lagrangian L is given by $L(q, \dot{q}) = \frac{1}{2}m_q(\dot{q}, \dot{q}) - V(q)$, then its reduced Lagrangian l is denoted by*

$$l([q, \dot{q}]_G) = \frac{1}{2}[m]_G([q, \dot{q}]_G, [q, \dot{q}]_G) - [V]_G([q]_G)$$

where $[m]_G \in \Gamma(Q/G, T^*Q/G \otimes T^*Q/G)$ is the reduced mass tensor induced from the G-invariance of the mass tensor $m \in \Gamma(Q, T^*Q \otimes T^*Q)$ and $[V]_G : Q/G \to \mathbb{R}$ is the reduced potential energy.

We defined the Euler-Lagrange matching conditions and the CL-equivalence relation in Definition 2.2. We now define an equivalence relation among RCL systems on TQ/G.

Definition 2.7. *Two RSCL systems $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$ are said to be* ***reduced-CL-equivalent (RCL-equivalent)*** *if the following* ***reduced Euler-Lagrange matching conditions*** *hold:*

RELM-1 : $W_1/G = [m_1]_G[m_2]_G^{-1}$,

RELM-2 : $\mathrm{Im}\,[\mathcal{REL}(l_1) - [F_1]_G - [m_1]_G[m_2]_G^{-1}(\mathcal{REL}(l_2) - [F_2]_G)] \subset W_1/G$

where $[m_i]_G$ is the reduced mass tensor of l_i, $i = 1, 2$.

The following proposition explains the relationship between the CL-equivalence relation among G-SCL's and the RCL-equivalence relation among RSCL's.

Proposition 2.2. *Two G-SCL systems are CL-equivalent if and only if their associated RSCL systems are RCL-equivalent.*

Proof. Let (L, F, W) be a G-SCL system, and $(l, [F]_G, W/G)$ be its associated RSCL system. Then, the proposition follows from the G-invariance of W and the following relations:

$$\mathcal{REL}(l) \circ \tau^{(2)}_{/G} = \pi_{/G} \circ \mathcal{EL}(L), \qquad [F]_G \circ \tau_{/G} = \pi_{/G} \circ F$$

where $\tau^{(2)}_{/G} : T^{(2)}Q \to T^{(2)}Q/G$ is the G quotient map. □

Hence, one can check the RCL-equivalence of two RSCL's in two ways: one is to directly check it, and the other is to check CL-equivalence of their associated *unreduced* G-SCL's.

The following theorem explains the main property of the RCL-equivalence relation:

Theorem 2.2. *Suppose that two RSCL systems $(l_i, [F_i]_G, W_i/G)$, $i = 1, 2$ are RCL-equivalent. Then, for an arbitrary control law for one system, there exists a control law for the other system such that the two closed-loop RSCL systems produce the same equations of motion. The explicit relation between the two feedback control laws $[u_i]_G$, $i = 1, 2$ is given by*

$$[u_1]_G = \mathcal{REL}(l_1) - [F_1]_G - [m_1]_G[m_2]_G^{-1}(\mathcal{REL}(l_2) - [F_2]_G) + [m_1]_G[m_2]_G^{-1}[u_2]_G \tag{2.7}$$

where m_i is the mass tensor of L_i, $i = 1, 2$.

Proof. Let $[u_i]_G$ be a feedback control for $(l_i.[F_i]_G, W_i/G)$, $i = 1, 2$. Let (L_i, F_i, W_i) be the unreduced G-SCL system of $(l_i.[F_i]_G, W_i/G)$, $i = 1, 2$. By Proposition 2.2, the two G-SCL are CL-equivalent. By Theorem 2.1, the two closed-loop G-SCL systems (L_i, F_i, u_i), $i = 1, 2$ produce the same equations of motion when u_1 and u_2 satisfy (2.2). Hence, the two closed-loop RSCL systems $(l_i.[F_i]_G, [u_i]_G)$, $i = 1, 2$ produce the same equations of motion when $[u_1]_G$ and $[u_2]_G$ satisfy (2.7) because each term in (2.2) is G-equivariant. In addition,notice that for any choice of $[u_i]_G$, one can choose the other $[u_j]_G$ such that (2.7) holds. □

2.2.3 Euler-Poincaré Matching.

Here we briefly sketch the proof that the set of Euler-Poincaré matching conditions in [2] is a special case of the reduced Euler-Lagrange matching conditions. This set of matching conditions can handle such examples as a spacecraft with a rotor and underwater vehicles with internal rotors. Let $Q = G \times H$ be the configuration space where G is a Lie group acting trivially on H, and H is an Abelian Lie group [1]. We choose the trivial connection on $Q \to H$ to write down the Lagrange-Poincaré equation on $TQ/G \simeq \mathfrak{g} \times TH$ with the Lie algebra $\mathfrak{g}$ of the Lie group G. We use $\eta = (\eta^\alpha)$

[1] In [2], they used H for the symmetry group. For the sake of consistency, we use G for the symmetry group in this paper.

as coordinates on $\mathfrak{g}$ and $(\theta, \dot\theta) = (\theta^a, \dot\theta^a)$ as coordinates on TH. The Lagrange-Poincaré operator $\mathcal{LP}$ with respect to the trivial connection is given by

$$\mathcal{LP}(l) = \begin{pmatrix} \frac{d}{dt}\frac{\partial l}{\partial \eta^\alpha} - c^\beta_{\alpha\gamma}\eta^\gamma \frac{\partial l}{\partial \eta^\beta} \\ \frac{d}{dt}\frac{\partial l}{\partial \dot\theta^a} - \frac{\partial l}{\partial \theta^a} \end{pmatrix} \tag{2.8}$$

for any reduced Lagrangian $l = l(\eta^\alpha, \dot\theta^a, \theta^a)$, where $c^\beta_{\alpha\delta}$ are the structure coefficients of the Lie algebra $\mathfrak{g}$. See [3] for the derivation of (2.8).

Let $(l, 0, T^*H)$ be the given RCL system with the reduced Lagrangian,

$$l(\eta^\alpha, \dot\theta^a) = \frac{1}{2} g_{\alpha\beta}\eta^\alpha\eta^\beta + g_{\alpha a}\eta^\alpha\dot\theta^a + \frac{1}{2} g_{ab}\dot\theta^a\dot\theta^b,$$

where $g_{\alpha\beta}, g_{\alpha a}, g_{ab}$ are constant functions on TQ/G. Notice that this Lagrangian is cyclic in the Abelian variables θ^a and the controls act only on the cyclic variables. Let $(l_{\tau,\sigma,\rho}, 0, T^*H)$ be an another RCL system with the reduced Lagrangian of the following form:

$$\begin{aligned} l_{\tau,\sigma,\rho} &= l(\eta^\alpha, \dot\theta^a + \tau^a_\alpha\eta^\alpha) + \frac{1}{2}\sigma_{ab}\tau^a_\alpha\tau^b_\beta\eta^\alpha\eta^\beta \\ &\quad + \frac{1}{2}(\rho_{ab} - g_{ab})(\dot\theta^a + g^{ac}g_{c\alpha}\eta^\alpha + \tau^a_\alpha\eta^\alpha)(\dot\theta^b + g^{bc}g_{c\beta}\eta^\beta + \tau^b_\beta\eta^\beta) \end{aligned} \tag{2.9}$$

which is exactly the equation (11) in [2]. See also [2] for the motivation of this choice of the form for the Lagrangian. The paper [2] assumes the following so-called Euler-Poincaré matching conditions:

EP-1 : $\tau^a_\alpha = -\sigma^{ab}g_{b\alpha}$,

EP-2 : $\sigma^{ab} + \rho^{ab} = g^{ab}$.

Then, one can show that the two assumptions of **EP-1** and **EP-2** imply the RCL-equivalence of the two reduced CL systems $(l, 0, T^*H)$ and $(l_{\tau,\sigma,\rho}, 0, T^*H)$. By Theorem 2.2, one can equivalently work with the second system . See [2] for some applications.

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