Adjoints of Hamiltonian systems and iterative learning control

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Abstract

This paper is concerned with a study on the variational systems and their adjoints of Hamiltonian control systems and its application to iterative learning control, which is applicable to a class of electro-mechanical systems. First of all, the self-adjoint structure of the variational of those systems is clarified. Then a novel iterative learning control scheme is proposed based on it. This method does not require either the knowledge of physical parameters of the target system nor the time derivatives of the output signals. A concrete and effective learning algorithm for mechanical systems is also derived.

1 Introduction

Hamiltonian control systems are the systems described by well known Hamilton's canonical equations with controlled Hamiltonians [2]. They are introduced mainly to characterize variational properties of dynamical systems and is used for optimal control, see also [15]. Those systems were also utilized to describe physical systems, and the related geometric methods of controlling this class of systems supplied fruitful results in control engineering [11, 8, 6]. Furthermore, this control framework was generalized in order to handle electro-mechanical systems as well as conventional mechanical ones [7], and several control methods are proposed for them [7, 4, 12, 9]. Thus a scope of this paper contains control of a class of physical systems such as mechanical and electrical systems.

In this paper, we investigate the self-adjoint structure of Hamiltonian systems. It is revealed that the variational systems of Hamiltonian systems have self-adjoint structures. This fact implies that the input-output mappings of the adjoints of the variational of Hamiltonian systems can be obtained without using precise knowledge of the target system.

This paper also studies iterative learning control of Hamiltonian systems based on the self-adjoint structure of the variational of Hamiltonian systems. A novel framework for iterative learning control of Hamiltonian systems will be proposed. This control scheme is very simple in the sense that it does not require the knowledge of any physical parameters of the target system. Also it does not require any time derivative of the output signal either, whereas existing well-known simple learning scheme by Arimoto [1] does require high order time derivatives. Furthermore, we will show a concrete control system synthesis method for mechanical systems.

2 Self-adjoint structure of Hamiltonian systems

This section discusses the self-adjoint structure of the variational of Hamiltonian systems. Consider an operator $\Sigma : X \times U \to X \times Y$ with Hilbert spaces X, U and Y with a state-space realization

$$(x^1, y) = \Sigma(x^0, u) : \left\{ \begin{array}{lcll} \dot{x} & = & f(x,u,t) & x(t^0) = x^0 \\ y & = & h(x,u,t) & \\ x^1 & = & x(t^1) & \end{array} \right. \tag{2.1}$$

defined on a time interval $[t^0, t^1] \ni t$. Typically, $X = \mathbb{R}^n$, $U = L_2^m[t^0, t^1]$ and $Y = L_2^r[t^0, t^1]$. A simpler notation $\Sigma^{x^0} : U \to Y$ with

$$y = \Sigma^{x^0}(u) : \left\{ \begin{array}{lcll} \dot{x} & = & f(x,u,t) & x(t^0) = x^0 \\ y & = & h(x,u,t) & \end{array} \right.$$

is also employed.

Here let us recall Fréchet derivative of nonlinear operators.

Definition 2.1 Consider an operator $\Sigma : X \to Y$ with Banach spaces X and Y. Σ is said to be *Fréchet differentiable* at $x \in X$ if there exists an operator $\mathrm{d}\Sigma : X \times X \to Y$ such that $\mathrm{d}\Sigma(x, \xi)$ is linear in ξ and that

$$\lim_{\|\xi\|_X \to 0} \frac{\|\Sigma(x+\zeta) - \Sigma(x) - \mathrm{d}\Sigma(x,\xi)\|_Y}{\|\xi\|_X} = 0.$$

Under these circumstances, $\mathrm{d}\Sigma(x, \xi)$ is called the *Fréchet derivative* of Σ at x.

The Fréchet derivative $\mathrm{d}\Sigma^{x^0}(u)(\mathrm{d}u)$ of $\Sigma^{x^0}(x)$ is given by [2, 10, 3]

$$y_v = \mathrm{d}\Sigma^{x^0}((u),(u_v)) : \left\{ \begin{array}{lcll} \dot{x} & = & f(x,u,t), & x(0) = x^0 \\ \begin{pmatrix} \dot{x}_v \\ y_v \end{pmatrix} & = & \dfrac{\partial}{\partial(x,u)} \begin{pmatrix} f(x,u,t) \\ h(x,u,t) \end{pmatrix} \begin{pmatrix} x_v \\ u_v \end{pmatrix}, & x_v(0) = 0 \end{array} \right. .$$

By its construction in Definition 2.1, the Fréchet derivative $\mathrm{d}\Sigma(x, \mathrm{d}x)$ is a locally linear approximation to $\Sigma(x)$, that is

$$\mathrm{d}\Sigma(u, \mathrm{d}u) \approx \Sigma(u + \mathrm{d}u) - \Sigma(u) \tag{2.2}$$

holds when $\mathrm{d}x$ is small.

Next we consider a Hamiltonian system Σ_H with a controlled Hamiltonian $H(x,u,t)$ with dissipation

$$(x^1,y)=\Sigma_H(x^0,u):\left\{\begin{array}{lcl}\dot{x} & = & (J-R)\,\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \quad x(t^0)=x^0\\ y & = & -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\\ x^1 & = & x(t^1)\end{array}\right. . \tag{2.3}$$

Here the structure matrices $J\in\mathbb{R}^{n\times n}$ and $R\in\mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite, respectively. The matrix R represents dissipative elements such as friction of mechanical systems and resistance of electric circuits. For this system, the following theorem holds.

Theorem 2.1 *Consider the Hamiltonian system with dissipation Σ_H in (2.3). Suppose that J and R are constant and that there exist nonsingular matrix $T_x\in\mathbb{R}^{n\times n}$ satisfying*

$$\begin{array}{lcl}J & = & -T_xJ\,T_x^{-1}\\ R & = & T_xR\,T_x^{-1}\\ \dfrac{\partial^2H(x,u,t)}{\partial(x,u)^2} & = & \begin{pmatrix}T_x & 0\\ 0 & I\end{pmatrix}\dfrac{\partial^2H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}T_x^{-1} & 0\\ 0 & I\end{pmatrix}.\end{array} \tag{2.4}$$

Then the Fréchet derivative of Σ_H is described by another Hamiltonian system

$$(x_v^1,y_v)=\mathrm{d}\Sigma_H((x^0,u),(x_v^0,u_v)):\left\{\begin{array}{lcl}\dot{x} & = & (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \quad x(t^0)=x^0\\ \dot{x}_v & = & (J-R)\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}, \quad x_v(t^0)=x_v^0\\ y_v & = & -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\\ x_v^1 & = & x_v(t^1)\end{array}\right. \tag{2.5}$$

with a controlled Hamiltonian $H_v(x,u,x_v,u_v,t)$

$$H_v(x,u,x_v,u_v,t)=\frac{1}{2}\begin{pmatrix}x_v\\ u_v\end{pmatrix}^{\mathrm{T}}\frac{\partial^2H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}x_v\\ u_v\end{pmatrix}.$$

Furthermore, the adjoint of the variational system with zero initial state $u_a\mapsto y_a=(\mathrm{d}\Sigma^{x^0}(u))^*(u_a)$ is given by

$$y_a=(\mathrm{d}\Sigma_H^{x^0}(u))^*(u_a):\left\{\begin{array}{lcl}\dot{x} & = & (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, \quad x(t^0)=x^0\\ \dot{x}_v & = & -(J-R)\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial x_v}^{\mathrm{T}}, \quad x_v(t^1)=0\\ y_a & = & -\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial u_a}^{\mathrm{T}}\end{array}\right. . \tag{2.6}$$

Suppose moreover that $J-R$ is nonsingular. Then the adjoint $(x_a^1, u_a) \mapsto (x_a^0, y_a)(\mathrm{d}\Sigma(x^0, u))^*(x_a^1, u_a)$ is given by the same state-space realization

$$\left\{ \begin{array}{rcll} \dot{x} &=& (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0) = x^0 \\ \dot{x}_v &=& -(J-R)\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial x_v}^{\mathrm{T}}, & x_v(t^1) = -(J-R)T_x\ x_a^1 \\ y_a &=& -\dfrac{\partial H_v(x,u,x_v,u_a,t)}{\partial u_a}^{\mathrm{T}} & \\ x_a^0 &=& -T_x^{-1}(J-R)^{-1}x_v(t^0) & \end{array} \right. . \tag{2.7}$$

Proof. First of all, let us calculate the variational system of Σ_H.

$$\left\{ \begin{array}{rcll} \dot{x} &=& (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0) = x^0 \\ \begin{pmatrix} \dot{x}_v \\ y_v \end{pmatrix} &=& \dfrac{\partial}{\partial(x,u)} \begin{pmatrix} (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}} \\ -\dfrac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x_v \\ u_v \end{pmatrix}, & x_v(t^0) = x_v^0 \\ x_v^1 &=& x_v(t^1) & \end{array} \right. .$$

We obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_v \\ y_v \end{pmatrix} &= \begin{pmatrix} J-R & 0 \\ 0 & -I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}} \begin{pmatrix} x_v \\ u_v \end{pmatrix} \\ &= \begin{pmatrix} J-R & 0 \\ 0 & -I \end{pmatrix} \left[\frac{\partial}{\partial(x_v,u_v)} \left\{ \frac{1}{2} \begin{pmatrix} x_v \\ u_v \end{pmatrix}^{\mathrm{T}} \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}} \begin{pmatrix} x_v \\ u_v \end{pmatrix} \right\} \right]^{\mathrm{T}} \\ &= \begin{pmatrix} J-R & 0 \\ 0 & -I \end{pmatrix} \frac{\partial H_v(x,u,x_v,u_v,t)}{\partial(x_v,u_v)}^{\mathrm{T}} = \begin{pmatrix} (J-R)\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}} \\ -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}} \end{pmatrix} \end{aligned}$$

which equals to (2.5). Next we calculate its adjoint as

$$\left\{ \begin{array}{rcll} \dot{x} &=& (J-R)\dfrac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0) = x^0 \\ \begin{pmatrix} \dot{x}_a \\ y_a \end{pmatrix} &=& \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \left(\begin{pmatrix} J-R & 0 \\ 0 & -I \end{pmatrix} \dfrac{\partial^2 H(x,u,t)}{\partial(x,u)^2} \right)^{\mathrm{T}} \begin{pmatrix} x_a \\ u_a \end{pmatrix}, & x_a(t^1) = x_a^1 \\ x_a^0 &=& x_a(t^0) & \end{array} \right. .$$

Here let us define a (possibly singular) coordinate transformation $\bar{x}_a = -(J-R)T_x x_a$, then we obtain

$$
\begin{aligned}
\begin{pmatrix} \dot{\bar{x}}_a \\ y_a \end{pmatrix} &= \begin{pmatrix} -(J-R)T_x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \dot{x}_a \\ y_a \end{pmatrix} \\
&= \begin{pmatrix} -(J-R)T_x & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \left(\begin{pmatrix} J-R & 0 \\ 0 & -I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \right)^{\mathrm{T}} \begin{pmatrix} x_a \\ u_a \end{pmatrix} \\
&= -\begin{pmatrix} -(J-R) & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} T_x & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} -J-R & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} x_a \\ u_a \end{pmatrix} \\
&= \begin{pmatrix} -(J-R) & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} T_x & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} -T_x^{-1}(J-R)T_x x_a \\ u_a \end{pmatrix} \\
&= \begin{pmatrix} -(J-R) & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} T_x & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} T_x^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} -(J-R)T_x x_a \\ u_a \end{pmatrix} \\
&= \begin{pmatrix} -(J-R) & 0 \\ 0 & -I \end{pmatrix} \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} \bar{x}_a \\ u_a \end{pmatrix}.
\end{aligned}
$$

This proves (2.6). Furthermore, if $J-R$ is nonsingular then the behavior of the state $x_a(t)$ can be recovered by $x_a(t) = -T_x^{-1}(J-R)^{-1}\bar{x}_a(t)$. This implies (2.7) and completes the proof. □

Remark 2.1 Note that the dynamics of x_a in (2.6) or (2.7) is the time reversal version of that of x_v in (2.5). Suppose the input u is given such that the time history of the Hessian of the Hamiltonian $\partial^2 H/\partial(x,u)^2$ is symmetrical with respect to the middle of the time interval $t^0 + (t^1 + t^0)/2$, i.e.,

$$\frac{\partial^2 H(x,u,t)}{\partial (x,u)^2}(t-t^0) = \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2}(t^1-t), \quad \forall t \in [t^0, t^1].$$

Then $\mathrm{d}\Sigma_H$ has a self-adjoint state-space realization. This condition often occurs in a PTP control of robot manipulators.

Under the circumstances in Remark 2.1, Theorem 2.1 implies that the time reversal system of the adjoint $(\mathrm{d}\Sigma_H)^*$ coincide with the variational $\mathrm{d}\Sigma_H$, that is,

$$\mathcal{R} \circ (\mathrm{d}\Sigma_H(u))^* \circ \mathcal{R} = \mathrm{d}\Sigma_H(u) \tag{2.8}$$

where $\mathcal{R}$ is a time reversal operator defined by

$$\mathcal{R}(u)(t-t^0) = u(t^1-t), \quad \forall t \in [t^0, t^1]. \tag{2.9}$$

Namely, the variational of the Hamiltonian system (2.3) has self-adjoint structure. Combined with the property of the variational system (2.2), we can calculate the input-output mapping of the adjoint by only using the input-output data of the original system.

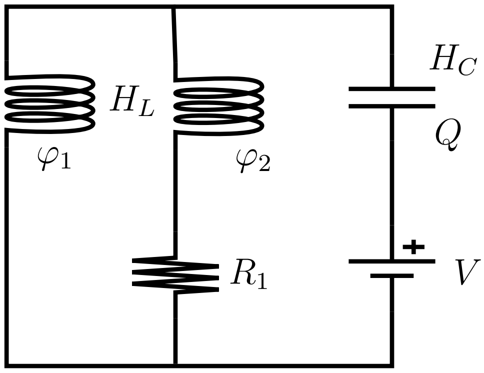

Figure 1: LCR-circuit

Example 2.1 Consider an LCR-circuit depicted in Figure 1. Let φ_1 and φ_2 denote the flux linkages, $H_L(\cdot)$ denote the inductance energy (a nonlinear function of φ_1 and φ_2), R_1 denote the resistance, $H_C(\cdot)$ denote the stored energy of capacitance (a nonlinear function of Q), Q denote the charge, and V denote the input voltage. Let us definite the input $u = V$ and the state $x = (Q, \varphi_1, \varphi_2)$. Then we obtain the Hamiltonian system (2.3) with

$$\begin{aligned} H(Q,\varphi_1,\varphi_2,u) &= H_C(Q) + H_L(\varphi_1,\varphi_2) + Q\ u \\ J &= \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ R &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & R_1 \end{pmatrix}. \end{aligned}$$

This system reduces to a port-controlled Hamiltonian system

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} &=& \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & -R_1 \end{pmatrix} \begin{pmatrix} \frac{\partial (H_C+H_L)}{\partial Q}^{\mathrm{T}} \\ \frac{\partial (H_C+H_L)}{\partial \varphi_1}^{\mathrm{T}} \\ \frac{\partial (H_C+H_L)}{\partial \varphi_2}^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u \\ y &=& -Q \end{array} \right. .$$

This system satisfies the matching condition (2.4) with

$$T_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore, we can calculate the adjoint of the variational system by using the input-output mapping of the original system provided the assumptions in Remark 2.1 hold.

3 Iterative learning control

This section explains how to apply the results in Section 2 to iterative learning control. The simplest problem setting of iterative learning control is as follows. Consider the nonlinear system (2.1) and a prescribed desired output y^d. The main purpose of learning control is to find an input $u = u^d$ which achieves $\Sigma(u^d) = y^d$. To this end, the iteration law

$$u_{(i+1)} = u_{(i)} + k(y^d - y_{(i)})$$

is adopted. Here $u_{(i)}$ and $y_{(i)}$ denote the input and output at the i-th operation. The objective is to find an appropriate law $k(\cdot)$ such that

$$y_{(i)} \to y^d \quad \text{as} \quad i \to \infty.$$

Arimoto et.al [1] adopted the iteration law $k(\cdot)$ in a PD like controller form without using the precise knowledge of the target system (2.1) under mild assumptions. Yamakita and Furuta [14] proposed to use the adjoint of the target system as the iteration law $k(\cdot)$ based on optimization theory, see e.g. [13]. Though this approach brings fast convergence, it needs precise knowledge of the target system. There are some other results adopting in-between approaches, e.g. [5], which give faster convergence and require less information of the target system. The main strategy taken here is similar to the Furuta's approach. But our result does not require the precise knowledge of the target system. Here we are going to utilize qualitative properties of physical systems rather than quantitative ones.

3.1 General framework

Let us consider the system Σ in (2.1) and a cost function $\Gamma : X^2 \times U \times Y \to \mathbb{R}$ such as

$$\Gamma(x^0, u, x^1, y) = \|x^0 - x^{0^d}\|^2_{\Gamma_{x^0}} + \|x^1 - x^{1^d}\|^2_{\Gamma_{x^1}} + \int_{t^0}^{t^1} \left(\|u(t) - u^d(t)\|^2_{\Gamma_u} + \|y(t) - y^d(t)\|^2_{\Gamma_y} \right) \mathrm{d}t$$

with the desired initial and final states x^{0^d} and x^{1^d}, and the desired input and output u^d and y^d. Here $\|x\|_{\Gamma_x}$ with $\Gamma_x \in \mathbb{R}^{n\times n}$ denotes $\sqrt{x^{\mathrm{T}}\Gamma_x x}$. The objective is to find the optimal input $(x^0_\star, u_\star)$ minimizing the cost function Γ, that is,

$$(x^0_\star, u_\star) := \arg \min_{(x^0,u)\in X_1\times U_1} \Gamma(x^0, u, x^1, y) \tag{3.10}$$

with $X_1 \times U_1 \subset X \times U$. In general, however, it is difficult to obtain a global minimum since the cost function Γ is not *convex*. Hence we try to obtain a local minimum here, i.e., $X_1 \times U_1 \subsetneq X \times U$. Note that the Fréchet derivative of Γ is

$$\mathrm{d}\Gamma(x^0, u, x^1, y)(\mathrm{d}x^0, \mathrm{d}u, \mathrm{d}x^1, \mathrm{d}y)$$

where

$$\mathrm{d}\Gamma(x^0, u, x^1, y) \in (X^2 \times U \times Y)^*.$$

It follows from well-known Riesz's representation theorem and the linearity of Fréchet derivative that there exists an operator $\Gamma' : X^2 \times U \times Y \to X^2 \times U \times Y$ such that

$$\mathrm{d}\Gamma(x^0, u, x^1, y)(\mathrm{d}x^0, \mathrm{d}u, \mathrm{d}x^1, \mathrm{d}y) = \langle\ \Gamma'(x^0, u, x^1, y),\ (\mathrm{d}x^0, \mathrm{d}u, \mathrm{d}x^1, \mathrm{d}y)\ \rangle_{X^2\times U\times Y}. \tag{3.11}$$

Since $(x^1, y) = \Sigma(x^0, u)$, the cost function Γ is described by

$$\Gamma(x^0, u, x^1, y) = \Gamma((x^0, u), \Sigma(x^0, u)).$$

Hence a necessary condition for the optimality (3.10) is characterized via its Fréchet derivative as

$$\mathrm{d}\left(\Gamma((x^0_\star, u_\star), \Sigma(x^0_\star, u_\star))\right)(\mathrm{d}x^0, \mathrm{d}u) = 0, \quad \forall(\mathrm{d}x^0, \mathrm{d}u).$$

Here we can calculate

$$\begin{aligned}
&\mathrm{d}\left(\Gamma((x^0, u), \Sigma(x^0, u))\right)(\mathrm{d}x^0, \mathrm{d}u)\\
&= \mathrm{d}\Gamma((x^0, u), \Sigma(x^0, u))\left((\mathrm{d}x^0, \mathrm{d}u), \mathrm{d}\Sigma(x^0, u)(\mathrm{d}x^0, \mathrm{d}u)\right)\\
&= \langle \Gamma'((x^0, u), \Sigma(x^0, u)), \begin{pmatrix} \mathrm{id}_{X\times U} \\ \mathrm{d}\Sigma(x^0, u) \end{pmatrix} (\mathrm{d}x^0, \mathrm{d}u)\rangle_{X^2\times U\times Y}\\
&= \langle \left(\mathrm{id}_{X\times U},\ (\mathrm{d}\Sigma(x^0, u))^*\right)\Gamma'((x^0, u), \Sigma(x^0, u)),\ (\mathrm{d}x^0, \mathrm{d}u)\rangle_{X\times U}\\
&= \langle \left(\mathrm{id}_{X\times U},\ (\mathrm{d}\Sigma(x^0, u))^*\right)\Gamma'(x^0, u, x^1, y),\ (\mathrm{d}x^0, \mathrm{d}u)\rangle_{X\times U}.
\end{aligned}$$

Therefore, if the adjoint $(\mathrm{d}\Sigma(x^0, u))^*$ is available, we can reduce the cost function Γ down at least to a local minimum by an iteration law

$$(x^0_{(i+1)}, u_{(i+1)}) = (x^0_{(i)}, u_{(i)}) - K_{(i)}\left(\mathrm{id}_{X\times U},\ (\mathrm{d}\Sigma(x^0_{(i)}, u_{(i)}))^*\right)\Gamma'(x^0_{(i)}, u_{(i)}, x^1_{(i)}, y_{(i)}) \tag{3.12}$$

or, in the case x^0 is fixed, by another one

$$u_{(i+1)} = u_{(i)} - K_{(i)}\,(0_{UX}, \mathrm{id}_U)\left(\mathrm{id}_{X\times U},\ (\mathrm{d}\Sigma(x^0_{(i)}, u_{(i)}))^*\right)\Gamma'(x^0_{(i)}, u_{(i)}, x^1_{(i)}, y_{(i)}) \tag{3.13}$$

with a small $K_{(i)} > 0$.

The results in Section 2, especially the relation (2.8), enable us to execute this procedure without using the parameters of the original operator Σ, provided Σ is a Hamiltonian system Σ_H. More precise discussion in the case of a special class of cost functions will be made in the following subsection.

3.2 Iterative learning control

In this subsection, we consider the Hamiltonian system $\Sigma = \Sigma_H$ in (2.3) and execute the iterative learning procedure (3.12) with respect to a typical cost function. A typical problem of iterative learning control is to produce an input u^d letting the output y track a given desired trajectory y^d, that is, to reduce the cost function

$$\Gamma(y) = \int_{t^0}^{t^1} (y(t) - y^d(t))^{\mathrm{T}}\Gamma_y(y(t) - y^d(t))\mathrm{d}t \tag{3.14}$$

with a positive definite matrix $\Gamma_y \in \mathbb{R}^{m\times m}$. In this case, Γ' in (3.11) is given by

$$\Gamma'(y) = 2\left(0,\ 0,\ 0,\ \Gamma_y(y-y^d)\right).$$

Hence the iteration law (3.13) reduces to

$$u_{(i+1)} = u_{(i)} + K_{(i)}(\mathrm{d}\Sigma_H^{x^0}(u_{(i)}))^* \Gamma_y(y^d - y_{(i)}).$$

The input-output mapping of the adjoint $(\mathrm{d}\Sigma_H^{x^0}(u_{(i)}))^*$ can be obtained by that of the original operator Σ_H using (2.2) and (2.8).

Thus iterative learning control with respect to the cost function (3.14) can be executed. Of course this procedure can be performed with any cost function $\Gamma(x^0, u, x^1, y)$, provided $\Sigma = \Sigma_H$ as in (2.3) (under the circumstances in Remark 2.1). Here we formally adopt the following assumptions according to Remark 2.1 in order to use the self-adjoint property (2.8).

Assumption A1 It is assumed that the desired trajectory $x^d(t)$ and input $u^d(t)$ satisfy

$$\left.\frac{\partial^2 H(x,u)}{\partial(x,u)^2}\right|_{\substack{x=x^d(t-t^0)\\ u=u^d(t-t^0)}} = \left.\frac{\partial^2 H(x,u)}{\partial(x,u)^2}\right|_{\substack{x=x^d(t^1-t)\\ u=u^d(t^1-t)}}, \quad \forall t\in[t^0,t^1].$$

Procedure 3.1 Consider the Hamiltonian system (2.3) with a given desired trajectory $x^d(t)$. Suppose the assumptions in Theorem 2.1 and Assumption A1 hold. Then the iterative learning control law is given by

$$u_{(2i+1)} = u_{(2i)} + \mathcal{R}\left(\kappa_{(i)}\Gamma_y(y^d - y_{(2i)})\right) \tag{3.15}$$

$$u_{(2i+2)} = u_{(2i)} + K_{(i)}\,\mathcal{R}\left(y_{(2i+1)} - y_{(2i)}\right) \tag{3.16}$$

for $i = 0, 1, 2, \cdots$. Here Γ_y defines the cost function Γ in (3.14) and T_u is the parameter defined in Theorem 2.1. The parameters $\kappa_{(i)} > 0 \in \mathbb{R}$ and $K_{(i)} > 0 \in \mathbb{R}^{m\times m}$ are small enough design parameters. $\mathcal{R}$ denotes the time reversal operator defined in (2.9).

This result will provide a basis of a new iterative learning control for a class of physical systems. Unfortunately, this iteration procedure only guarantees the convergence to a local minimum of the cost function (3.14), that is, the convergence to an optimal input u^d is not ensured in general.

4 Iterative learning control of mechanical systems

A typical mechanical system can be described by a Hamiltonian system

$$\Sigma_H : \begin{cases} \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & -R_p \end{pmatrix} \begin{pmatrix} \frac{\partial H(q,p,u)}{\partial q}^{\mathrm{T}} \\ \frac{\partial H(q,p,u)}{\partial p}^{\mathrm{T}} \end{pmatrix} \\ y = \frac{\partial H(q,p,u)}{\partial u}^{\mathrm{T}} = q \end{cases} \tag{4.17}$$

with the Hamiltonian

$$H(q,p,u) = H_0(q,p) - u^{\mathrm{T}} q = \underbrace{\frac{1}{2} p^{\mathrm{T}} M(q)^{-1} p + V(q)}_{H_0(q,p)} - u^{\mathrm{T}} q$$

where a positive definite matrix $M(q) > 0 \in \mathbb{R}^{m\times m}$ denotes the inertia matrix, a scalar function $V(q)$ denotes the potential energy of the system and H_0 denotes the total physical energy.

Unfortunately, however, this system does not satisfy the assumptions in Theorem 2.1 since there does not exist the matrix T_x satisfying the matching condition (2.4). The procedure in the sequel enables the system to satisfy this condition approximately.

Typically, feedback controllers are employed to control the system (4.17) even when the iterative learning control is applied, since it is marginally stable. This subsection discusses feedback system design for iterative learning control. It was shown in [11] that a simple PD feedback preserves the structure of the Hamiltonian system (4.17). Further discussions on controller design preserving the structure of general Hamiltonian systems can be found in [12, 4, 9]. Let us consider a PD controller

$$u = \bar{u} - K_q\, q - K_p\, \dot{q} \tag{4.18}$$

where $\bar{u}$ is a new input and $K_q, K_p > 0 \in \mathbb{R}^{m\times m}$ are symmetric positive definite matrices. Applying a coordinate transformation

$$q = \varepsilon \bar{q}$$

with a positive constant $\varepsilon > 0$ converts the system into another Hamiltonian system

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot{\bar{q}} \\ \dot{p} \end{pmatrix} & = & \begin{pmatrix} 0 & \frac{1}{\varepsilon} I \\ -\frac{1}{\varepsilon} I & -(R_p + K_p) \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}(\bar{q},p,\bar{u})}{\partial \bar{q}}^{\mathrm{T}} \\ \frac{\partial \bar{H}(\bar{q},p,\bar{u})}{\partial p}^{\mathrm{T}} \end{pmatrix} \\ y & = & -\frac{\partial \bar{H}(\bar{q},p,\bar{u})}{\partial \bar{u}}^{\mathrm{T}} = \varepsilon \bar{q} = q \end{array} \right. \tag{4.19}$$

with a new Hamiltonian

$$\bar{H}(\bar{q},p,\bar{u}) = \underbrace{\frac{1}{2} p^{\mathrm{T}} M(\varepsilon\bar{q})^{-1} p + V(\varepsilon\bar{q})}_{H_0(\varepsilon\bar{q},p)} + \frac{\varepsilon^2}{2} \bar{q}^{\mathrm{T}} K_q\, \bar{q} - \varepsilon\, \bar{u}^{\mathrm{T}} \bar{q}.$$

Let us choose the parameter matrices in Theorem 2.1 as

$$T_x = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{4.20}$$

and check the matching condition (2.4). The former two equations hold straightforwardly and the left and right hands of the last equation become

$$\frac{\partial^2 \bar{H}(\bar{q},p,\bar{u})}{\partial(\bar{q},p,\bar{u})^2} = \begin{pmatrix} \varepsilon^2\left(\frac{\partial^2 H_0(q,p)}{\partial q^2}+K_q\right) & \varepsilon\frac{\partial M(q)^{-1}p}{\partial q}^{\mathrm{T}} & -I \\ \varepsilon\frac{\partial M(q)^{-1}p}{\partial q} & M(q)^{-1} & 0 \\ -I & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} T_x & 0 \\ 0 & I \end{pmatrix}\frac{\partial^2 \bar{H}(\bar{q},p,\bar{u})}{\partial(\bar{q},p,\bar{u})^2}\begin{pmatrix} T_x^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \varepsilon^2\left(\frac{\partial^2 H_0(q,p)}{\partial q^2}+K_q\right) & -\varepsilon\frac{\partial M(q)^{-1}p}{\partial q}^{\mathrm{T}} & -I \\ -\varepsilon\frac{\partial M(q)^{-1}p}{\partial q} & M(q)^{-1} & 0 \\ -I & 0 & 0 \end{pmatrix}.$$

Hence, if the "P gain" K_q is chosen large enough and the parameter ε is taken small enough accordingly, then the relation

$$\frac{\partial^2 \bar{H}(\bar{q},p,\bar{u})}{\partial(\bar{q},p,\bar{u})^2} \approx \begin{pmatrix} T_x & 0 \\ 0 & T_u \end{pmatrix}\frac{\partial^2 \bar{H}(\bar{q},p,\bar{u})}{\partial(\bar{q},p,\bar{u})^2}\begin{pmatrix} T_x & 0 \\ 0 & T_u \end{pmatrix}^{-1}$$

holds, that is, the assumption (2.4) in Theorem 2.1 is satisfied approximately. Note that the "D gain" K_p should also be chosen large enough to let the matrix $R_p + K_p$, which describes the dissipation behavior of the system (4.19) in the coordinate $(\bar{q}, p)$, sufficiently large compared with the matrix I/ε, which denotes the oscillation behavior. This should be done for numerical stability of the iterative learning procedure. Here we adopt the following assumptions corresponding to Assumption A1.

Assumption B1 It is assumed that the desired trajectory $x^d(t) = (q^d(t), p^d(t))$ satisfies

$$\left.\frac{\partial^2 H_0(q,p)}{\partial(q,p)^2}\right|_{x=x^d(t-t^0)} = \left.\frac{\partial^2 H_0(q,p)}{\partial(q,p)^2}\right|_{x=x^d(t^1-t)}, \quad \forall t \in [t^0, t^1].$$

Assumption B2 PD gains K_q and K_p are large enough.

Remark 4.1 When the desired trajectory $x^d(t)$, $t \in [t^0, t^1]$ does not satisfy Assumption B1, we can produce a desired trajectory fulfilling B1 by simply reproducing the same trajectory in the time domain $t \in [t^1, 2t^1 - t^0]$ as

$$x^d_{\mathrm{new}}(t) = \begin{cases} x^d(t) & t \in [t^0, t^1] \\ x^d(2t^1 - t^0 - t) & t \in [t^1, 2t^1 - t^0] \end{cases}.$$

The iterative learning procedure is given below on the assumptions B1 and B2.

Procedure 4.1 Consider the mechanical Hamiltonian system (4.17) with the PD feedback (4.18) and a prescribed desired trajectory $q^d(t)$. Suppose Assumptions B1 and B2 hold. Then the iterative learning control law is given by

$$\begin{aligned} \bar{u}_{(2i+1)} &= \bar{u}_{(2i)} + \mathcal{R}\,(\kappa_{(i)}\Gamma_y(q^d - q_{(2i)})) \\ \bar{u}_{(2i+2)} &= \bar{u}_{(2i)} + K_{(i)}\,\mathcal{R}\,(q_{(2i+1)} - q_{(2i)}) \end{aligned} \qquad (4.21)$$

for $i = 0, 1, 2, \cdots$. Here Γ_y defines the cost function Γ in (3.14). The parameters $\kappa_{(i)} > 0 \in \mathbb{R}$ and $K_{(i)} > 0 \in \mathbb{R}^{m \times m}$ are small enough design parameters. $\mathcal{R}$ denotes the time reversal operator defined in (2.9).

This iterative learning control scheme is very simple in the sense that it does not employ any physical parameters of the target system. Compared with Arimoto's method [1] which is also simple, the proposed method is expected to be numerically more stable because our approach does not employ time derivative of the output signal whereas Arimoto's method requires second order time derivative of q.

Furthermore, we can prove the convergence to the global minimum, i.e., the convergence to the optimal input $\bar{u}^d$, of this iteration procedure, though the general version of this procedure given in Procedure 3.1 only guarantees the convergence to a local minimum.

Proposition 4.1 *Consider the mechanical Hamiltonian system (4.17). Suppose Assumptions B1 and B2 hold and there exists a positive constant ϵ satisfying*

$$\kappa_{(i)} K_{(i)} \geq \epsilon I > 0, \quad \forall i. \tag{4.22}$$

Then, for any initial input $\bar{u}_{(0)}$, the iterative learning control law (4.21) in Procedure 4.1 converges to an optimal input $\bar{u}^d$.

Proof. The variational system $\mathrm{d}\Sigma_H^{x^0}$ of the mechanical Hamiltonian system (4.17) can be described by

$$\left\{ \begin{array}{lcl} \begin{pmatrix} \dot{q}_v \\ \dot{p}_v \end{pmatrix} & = & \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} q_v \\ p_v \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u_v \\ y_v & = & q_v \end{array} \right.$$

with appropriate matrices A_{ij}'s. Let us now calculate the zero-dynamics of this system. Take $y_v \equiv 0$. Then it follows that

$$0 \equiv \dot{q}_v = A_{11} q_v + A_{12} p_v = A_{12} p_v = M(q)^{-1} p_v.$$

Therefore we prove $p_v \equiv 0$. Finally we obtain

$$0 \equiv \dot{p}_v = A_{21} p_v + A_{22} q_v + u_v = u_v.$$

This suggests that the variational system has no zero-dynamics. Therefore, the iteration law (4.21) and the assumption (4.22) imply

$$\bar{u}_{(2i)} \to \bar{u}_{(2i+2)} \quad \Rightarrow \quad q_{(2i)} \to q^d,$$

that is, the control law converges to an optimal input $\bar{u}^d$. This completes the proof. □

5 Conclusion

This paper has discussed the self-adjoint properties of the adjoints of the variational systems of Hamiltonian control systems. A novel iterative learning control scheme has been proposed based on these properties. This method does not require either the knowledge of physical parameters of the target system nor the time derivatives of output signals. A concrete and effective learning algorithm for mechanical systems is also derived.

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